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# THE BOUNDARY BEHAVIOR OF THE KOBAYASHI METRIC

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**1. Introduction.** Let $\Omega$ be a *domain*, that is, a connected open set, in $\mathbf{C}^n$. If $P \in \Omega$ and $\xi \in \mathbf{C}^n$, then define $\mathcal{M}(P,\xi)$ to be the collection of holomorphic mappings $\phi$ of the unit disc $D$ into $\Omega$ such that $\phi(0) = P$ and $\phi'(0)$ is a scalar multiple of $\xi$. The *Kobayashi* (or *Kobayashi/Royden*) *length* of $\xi$ at the point $P$ is defined to be

$$F_K^\Omega(P,\xi) \equiv \inf\{\alpha : \alpha > 0, \exists \phi \in \mathcal{M}(P,\xi) \text{ with } \phi'(0) = \xi/\alpha\}.$$

See [**6**] and [**5**] for more on the Kobayashi metric.

This metric is becoming increasingly important in the function theory of several complex variables (see [**7, 8, 9, 10, 11, 12**]). In particular, it is important to calculate and estimate the metric on a variety of domains. An interesting conjecture is that any smoothly bounded pseudoconvex domain is complete in the Kobayashi metric.

If $P \in \Omega$, then let $\delta(P) = \delta_\Omega(P)$ denote the distance of $P$ to $\partial\Omega$. An important step in determining the validity of the last conjecture would be to prove that, for $P$ near the boundary and $\xi = \nu_P$ (the unit outward normal vector to $\partial\Omega$ at $P$) it holds that

$$F_K^\Omega(P,\xi) \approx \frac{1}{\delta(P)}.$$

Here the notation $A \approx B$ means that the quotient $A/B$ is bounded above and below by absolute constants.

The mapping

$$\phi(\zeta) = P + \delta(P)\zeta\nu_P,$$

together with the definition of the metric, shows that

$$F_K^\Omega(P,\xi) \leq C \cdot \frac{1}{\delta(P)}.$$

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A recent unpublished example of J. Fornæss and the author shows that, in general, the opposite inequality fails for smoothly bounded pseudoconvex domains. The conjecture, however, still remains open.

If $r_2 > r_1 > 0$, then let

$$\mathcal{A}(0, r_1, r_2) = \mathcal{A} = \{z \in \mathbf{C}^2 : r_1 < |z - 0| < r_2\}.$$

Let $\delta > 0$ be small. If $P_\delta = (-r_1 - \delta, 0)$ and $\nu_\delta = \nu_{P_\delta} = (1, 0)$, then any estimate from below for $F_K^{\mathcal{A}}$ gives an estimate from below for arbitrary *smoothly bounded domains* (pseudoconvexity is not assumed). This is so because if $\Omega$ is such a domain, $P \in \Omega$ is near the boundary, and $P' \in \partial\Omega$ is the (unique) nearest boundary point to $P$, then there are $r_1, r_2 > 0$, uniform in $P'$, such that

$$\mathcal{A} = \mathcal{A}(P' + r_1 \nu_{P'}, r_1, r_2) \supseteq \Omega.$$

Here $P' + r_1 \nu_{p'}$ is the *center* of the annulus.

It then follows from elementary considerations that

$$F_K^{\Omega}(P, \nu) \geq F_K^{\mathcal{A}}(P, \nu).$$

In this paper, we prove the following result:

**Theorem 1.1.** *For any $\mathcal{A} = \mathcal{A}(Q, r_1, r_2)$ and $P \in \mathcal{A}$ near the inner boundary of $\mathcal{A}$, it holds that*

$$F_K^{\mathcal{A}}(P, \nu) \approx \delta_{\mathcal{A}}(P)^{-3/4}.$$

(*The constants of comparison depend, of course, on $r_1$ and $r_2$.*) *More generally, for any $3/4 \leq \lambda \leq 1$, there is a bounded domain $\Omega_\lambda \subseteq \mathbf{C}^2$ with twice continuously differentiable boundary such that, for a continuum of points $P \in \Omega_\lambda$ tending to $\partial\Omega_\lambda$, it holds that*

$$F_K^{\Omega_\lambda}(P, \nu) \approx \delta(P)^{-\lambda}.$$

In the statement of the theorem it should be noted that, for $3/4 \leq \lambda < 1$, the domain $\Omega_\lambda$ is *not* pseudoconvex, while for $\lambda = 1$ it is strongly pseudoconvex.

For $\lambda = 3/4$, the example presented here was discovered by the author and Erik Løw in 1985. It was discovered independently, and somewhat earlier, by Halsey Royden and by Bedford/Fornæss. The example for $\lambda = 3/4$ can be found in the unpublished manuscript [**3**]. Function theoretic consequences of the example (in particular, a new way to view the Hartogs extension phenomenon) are explored in [**8, 9, 10**].

It is easy to see that the proof of the theorem also shows that if $\Omega$ is any bounded domain and $P \in \partial\Omega$ has Levi form with a negative eigenvalue, then at points $z \in \Omega$ near $P$ the Kobayashi metric in the normal direction blows up like $1/\mathrm{dist}(z, \partial\Omega)^{3/4}$.

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**2. The estimate from above.** Notice that for the unit ball $B \subseteq \mathbf{C}^n$ and $P = P_\delta = (1 - \delta, 0)$, $\delta = (1, 0)$, it is well known (see [**6**]) that

$$F_K^B(P, \delta) \approx \frac{1}{\delta(P)}.$$

This takes care of the case $\lambda = 1$. Thus, we henceforth consider only $3/4 \leq \lambda < 1$.

Fix a $3/4 \leq \lambda < 1$. Set $m = 1/(2 - 2\lambda)$, and define

$$U_\lambda = \{(z_1, z_2) \in \mathbf{C}^2 : 1 < \rho(z) \equiv |z_1|^2 + |z_2|^m < 4\}.$$

Clearly, this domain has twice continuously differentiable boundary since $3/4 \leq \lambda < 1$. Let $\delta > 0$ be small and set $P = P_\delta = (-1 - \delta, 0)$ and $\xi = (1, 0)$. In order to obtain the desired estimate for $F_K^{U_\lambda}(P, \xi)$ from above, it is enough to exhibit a function $\phi = \phi_\lambda \in \mathcal{M}(P, \xi)$ such that

$$|\phi'(0)| \geq C \cdot \delta(P)^\lambda.$$

We define such a $\phi$ by the formula

$$\phi(\zeta) = (-1 - \delta + (\delta^\lambda/10)\zeta, \zeta^2).$$

Clearly, $\phi(0) = P$ and $\phi'(0) = (\delta^\lambda/10, 0)$. If we can show that $\phi(D) \subseteq U_\lambda$, then this $\phi$ lies in $\mathcal{M}(P, \xi)$ and we will have established that
$$F_K^{U_\lambda} \leq C \cdot \delta^{-\lambda}.$$

Notice that the estimate $|\varphi(z)| < 4$ is trivial. Now
$$|\varphi(\zeta)| > 1$$
if and only if
$$2\delta + \delta^2 + \frac{1}{100}\delta^{2\lambda}|\zeta|^2 - \frac{1}{5}(1+\delta)\delta^\lambda \Re\zeta + |\zeta|^{2m} > 0.$$

There are two cases to consider:

a) If $|\zeta| < 5\delta^{1-\lambda}$, then
$$\left|\frac{1}{5}(1+\delta)\delta^\lambda \Re\zeta\right| < (1+\delta)\delta^\lambda \delta^{1-\lambda} < 2\delta.$$

Thus, $\phi(\zeta) \in U_\lambda$ for these values of $\zeta$, as desired.

b) If $|\zeta| \geq 5\delta^{1-\lambda}$, then
$$\left|\frac{1}{5}(1+\delta)\delta^\lambda \Re\zeta\right| \leq \left|\frac{1}{5}(1+\delta)(|\zeta|/5)^{\lambda/(1-\lambda)} \Re\zeta\right|$$
which in turn is majorized by
$$|\zeta|^{2m}.$$

Thus, $\phi(\zeta) \in U_\lambda$ for these values of $\zeta$, as desired.

This concludes the proof that
$$F_K^{U_\lambda} \leq C \cdot \delta^{-\lambda}.$$

**3. The estimate from below.** For this part of the proof, we must consider all elements $\phi \in \mathcal{M}(P_\delta, \xi)$ and prove that their derivatives are bounded above in absolute value by $C \cdot \delta^\lambda$. We begin by introducing a little notation.

If $\delta > 0$ is small, we let $\mathcal{R}_\delta = \{w \in \mathbf{C} : 1 - \delta < |w| < 4\}$. Elementary conformal mapping arguments show that if $Q \in \mathcal{R}_\delta$ and $\xi \in \mathbf{C}$ is any Euclidean unit vector, then

$$F_K^{\mathcal{R}_\delta}(Q, \xi) \approx \operatorname{dist}(Q, \partial\mathcal{R}_\delta)^{-1},$$

*with the constants of comparison being independent of* $\delta$.

Now let $\phi \in \mathcal{M}(P_\delta, \xi)$ for the domain $U_\lambda$. We immediately have that $|\phi_2(\zeta)| \leq C|\zeta|^2$ and a moment's reflection shows that we may take $C = 2$. If $|\zeta| \leq \delta^{1-\lambda}/\sqrt{2}$, then

$$|\phi_2(\zeta)| \leq 2\delta^{2-2\lambda}/2 = \delta^{1/m}.$$

Therefore, for such $\zeta$,

$$|\phi_1(\zeta)| \geq \sqrt{1 - |\phi_2(\zeta)|^m} \geq \sqrt{1 - \delta} > 1 - \delta.$$

Therefore, the function $g(\zeta) \equiv \phi_1(\delta^{1-\lambda}\zeta/\sqrt{2})$ maps the disc $D$ to $\mathcal{R}_\delta$, with $g(0) = -1 - \delta$. By our remarks about uniform estimates for the Kobayashi metric on the domains $\mathcal{R}_\delta$, we may conclude that

$$\frac{\delta^{1-\lambda}}{\sqrt{2}}|\phi_1'(0)| = |g'(0)| \leq C \cdot \delta.$$

Therefore,

$$|\phi_1'(0)| \leq C \cdot \delta^\lambda.$$

Since $\phi$ was an arbitrary element of $\mathcal{M}(P_\delta, \xi)$, this gives the desired estimate from below:

$$F_K^{U_\lambda}(P, \xi) \geq C' \cdot \delta^{-\lambda}.$$

**4. Concluding remarks.** Conformal mapping techniques in one variable indicate that, for domains in $\mathbf{C}^n$ which are not smoothly bounded, the estimates for the Kobayashi metric will be different from the ones presented here. For domains $\Omega$ which have $C^2$ boundary, the internally tangent ball which each boundary point possesses shows that

$$F_K^\Omega(P, \nu) \leq C \cdot \delta(P)^{-1}.$$

Work in [**1, 4**] shows that the same estimate holds from below on strongly pseudoconvex domains and on domains of finite type in $\mathbf{C}^2$. For general domains, we cannot expect this estimate to be sharp.

The remarks in Section 1 about the domains $\mathcal{A}(Q, r_1, r_2)$ give a lower bound of

$$F_K^\Omega(P, \nu) \geq C \cdot \delta(P)^{-3/4}$$

provided that $\Omega$ is bounded with $C^2$ boundary. For general domains, we cannot expect this estimate to be sharp.

*We cannot compare arbitrary domains with twice continuously differentiable boundary with homothetes of the domains* $U_\lambda$, $3/4 < \lambda < 1$, because the comparison arguments in Section 1 (particularly, the external tangency) fail.

The examples in this paper yield little information about the Kobayashi metric in tangential directions. Clearly, there is much work left to be done in estimating the Kobayashi metric.

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