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HARMONIC MAPPINGS RELATED TO SCHERK'S SADDLE-TOWER MINIMAL SURFACES

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1. Introduction. A *harmonic mapping* is a complex-valued univalent harmonic function defined in some domain of the complex plane. Harmonic mappings are of interest in differential geometry because they provide isothermal coordinates for nonparametric minimal surfaces, leading to the classical Weierstrass-Enneper representation in terms of analytic functions. (See, for instance, [**8**], [**9**], [**2**], [**6**], [**4**].) More recently, harmonic mappings have been studied from the viewpoint of complex analysis, as generalizations of conformal mappings, see [**1**], [**3**].

The purpose of this note is to investigate a family of harmonic mappings that arise in connection with Scherk's classical "saddle-tower" minimal surface (see [**2**] or [**9**]) and its generalizations recently found by Karcher [**7**]. The mappings in question are defined on the unit disk $\mathbf{D}$ by

$$F_n(z) = -\frac{2}{n}\sum_{k=1}^{n} \alpha^k \log|z-\alpha^k|, \quad n = 3, 4, \dots , \tag{1}$$

where $\alpha = e^{2\pi i/n}$ is a primitive nth root of unity. Each function F_n is clearly harmonic in $\mathbf{D}$, but its univalence is not so obvious *a priori*. A direct proof of the univalence will be given in Section 2. Meanwhile, some *Mathematica*-produced images of the disk under F_n are displayed in Figure 1 for $n = 3$, 4, 6, and 10. The figure shows the images of equally spaced concentric circles and radial segments, giving in particular a graphical demonstration of the univalence. The infinite spires correspond to the nth roots of unity: $1, \alpha, \alpha^2, \dots , \alpha^{n-1}$. It is clear from the formula (1) that F_n maps each radial segment from 0 to α^k onto the radial half-line in the same direction. In fact, essentially the same geometric argument (pairing symmetric terms of the sum) shows that F_n maps each intermediate segment from 0 to $\alpha^{k-1/2}$ onto a radial segment in the same direction.

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After studying the mappings F_n more closely, we explain in Section 3 how they arise in canonical representations of Scherk's surfaces. The final section discusses some asymptotic properties of F_n as n tends to infinity.

2. Geometric properties of the harmonic mappings. The univalence of F_n in $\mathbf{D}$ will now be proved by showing that each of the mappings actually has the starlikeness property visually apparent in Figure 1. In other words, the argument of $F_n(e^{i\theta})$ is strictly increasing.

Theorem 1. *For each integer $n \geq 3$, the harmonic function F_n defined by* (1) *maps the unit disk univalently onto a strictly starlike region with n-fold rotational symmetry.*

Proof. Further inspection of the formula reveals that $F_n(0) = 0$ and that F_n has the symmetries

$$F_n(\alpha z) = \alpha F_n(z) \quad \text{and} \quad F_n(\bar{z}) = \overline{F_n(z)}.$$

We have already noted that $F_n(z) \to \infty$ and $\arg\{F_n(z)\} \to 2k\pi/n$ as $z \to \alpha^k$, for each $k = 0, 1, \ldots, n-1$. Thus it will suffice to show that the argument of $F_n(e^{i\theta})$ increases from 0 to π/n as θ increases from 0 to π/n. It will then follow from the above symmetries that $\arg\{F_n(z)\}$ increases by exactly 2π as z moves once around the unit circle in the counterclockwise direction. Since, as will be shown, F_n is locally an orientation-preserving map, the argument principle for harmonic functions [**5**] can then be invoked to conclude that F_n maps the unit disk univalently onto the region bounded by the curve that is the image of the unit circle.

In order to show that $\arg\{F_n(e^{i\theta})\}$ increases from 0 to π/n as θ goes from 0 to π/n, we will need an expression for the derivative. For this purpose the two partial-fraction expansions

$$\frac{1 + z^{n-2}}{z^n - 1} = \frac{2}{n} \sum_{k=1}^{n} \frac{\cos(2k\pi/n)}{z - \alpha^k} \tag{2}$$

and

$$\frac{1 - z^{n-2}}{z^n - 1} = \frac{2i}{n} \sum_{k=1}^{n} \frac{\sin(2k\pi/n)}{z - \alpha^k} \tag{3}$$

will be useful. With the notation

$$(4) \qquad u = \mathrm{Re}\left\{ \int_0^z \frac{1+z^{n-2}}{1-z^n}\, dz \right\}, \qquad v = \mathrm{Im}\left\{ \int_0^z \frac{1-z^{n-2}}{1-z^n}\, dz \right\},$$

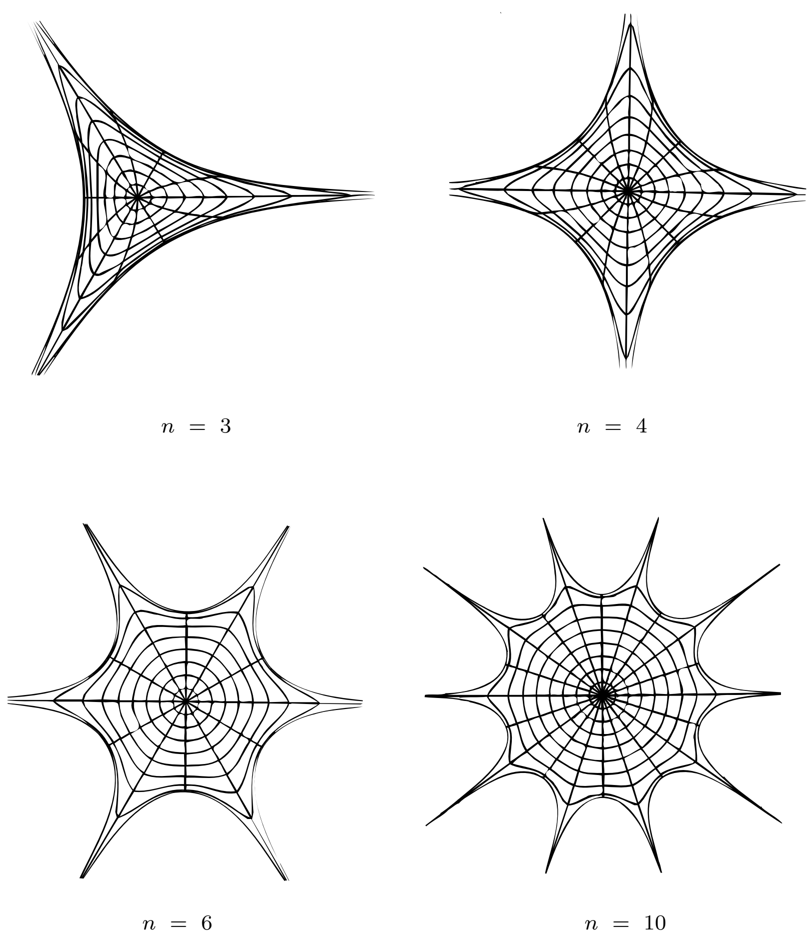

FIGURE 1. Image of the disk under the mapping F_n.

we therefore find

$$u = -\frac{2}{n}\sum_{k=1}^{n}\cos\left(\frac{2k\pi}{n}\right)\log|z-\alpha^k|$$

and

$$v = -\frac{2}{n}\sum_{k=1}^{n}\sin\left(\frac{2k\pi}{n}\right)\log|z-\alpha^k|,$$

so that $F_n(z) = u + iv$.

On the other hand, it can be seen from (1) by direct geometric estimates that $\mathrm{Re}\,\{F_n(e^{i\theta})\} > 0$ for $0 < \theta \le \pi/n$. Thus in particular $F_n(e^{i\theta}) \neq 0$ and

$$\arg\{F_n(e^{i\theta})\} = \mathrm{Im}\,\{\log F_n(e^{i\theta})\},$$

so that

$$\frac{d}{d\theta}\arg\{F_n(e^{i\theta})\} = |F_n(e^{i\theta})|^{-2}\ \mathrm{Im}\left\{\overline{F_n(e^{i\theta})}\frac{d}{d\theta}F_n(e^{i\theta})\right\}. \tag{5}$$

Now, letting $f_n(z)$ and $g_n(z)$ denote the two integrals in (4), we can write

$$F_n(z) = \mathrm{Re}\,\{f_n(z)\} + i\ \mathrm{Im}\,\{g_n(z)\}, \tag{6}$$

and

$$\frac{d}{d\theta}F_n(e^{i\theta}) = \mathrm{Re}\,\{ie^{i\theta}f_n'(e^{i\theta})\} + i\ \mathrm{Im}\,\{ie^{i\theta}g_n'(e^{i\theta})\}.$$

But simple calculations show that

$$ie^{i\theta}f_n'(e^{i\theta}) = -\frac{\cos((n/2)-1)\theta}{\sin(n\theta/2)},$$
$$ie^{i\theta}g_n'(e^{i\theta}) = i\frac{\sin((n/2)-1)\theta}{\sin(n\theta/2)}.$$

Thus, with the notation $\mathrm{sgn}\,\{F_n(e^{i\theta})\} = e^{i\varphi_n}$, we find from (5) that

$$\frac{d}{d\theta}\arg\{F_n(e^{i\theta})\} = |F_n(e^{i\theta})|^{-1}\sin\left[\left(\frac{n}{2}-1\right)\theta + \varphi_n\right] > 0 \tag{7}$$

whenever $0 < \theta < (\pi/n)$ and $0 < \varphi_n < (\pi/n) + (\pi/2)$. As already observed, however, $\arg\{F_n(e^{i\theta})\}$ moves from 0 to π/n as θ goes from 0 to π/n. Because (7) shows that $\varphi_n = \arg\{F_n(e^{i\theta})\}$ is increasing whenever it lies in a certain two-sided neighborhood of π/n, the conclusion is that it must *increase* to π/n as θ increases from 0 to π/n. (Recall that $\operatorname{Re}\{F_n(e^{i\theta})\} > 0$ in that interval, so that $F_n(e^{i\theta})$ cannot wind around the origin.) As previously noted, this shows by symmetry that $\arg\{F_n(e^{i\theta})\}$ increases by exactly 2π as θ goes from 0 to 2π.

In order to complete the proof of univalence, we must now compute the dilatation of the mapping F_n. This is the quantity

$$\omega_n(z) = \overline{\partial F_n/\partial \bar{z}} \Big/ \partial F_n/\partial z.$$

In making the calculation it is again convenient to use the representation (6), or

$$F_n(z) = \frac{1}{2}[f_n(z) + \overline{f_n(z)}] + \frac{1}{2}[g_n(z) - \overline{g_n(z)}].$$

This shows that

$$\partial F_n/\partial z = \frac{1}{2}[f_n'(z) + g_n'(z)] = \frac{1}{1-z^n} \tag{8}$$

and

$$\overline{\partial F_n/\partial \bar{z}} = \frac{1}{2}[f_n'(z) - g_n'(z)] = \frac{z^{n-2}}{1-z^n}; \tag{9}$$

so the dilatation is $\omega_n(z) = z^{n-2}$. Since $|\omega_n(z)| < 1$ in $\mathbf{D}$, the mapping F_n is everywhere orientation-preserving. It then follows from the argument principle for harmonic functions [**5**] that F_n is univalent in $\mathbf{D}$ and it maps $\mathbf{D}$ onto the region inside the curve $w = F_n(e^{i\theta})$. This completes the proof of the theorem.

3. The corresponding minimal surfaces. We now turn to the connection between the harmonic mappings just discussed and Scherk's saddle-tower minimal surfaces. Scherk's classical surface is defined by the equation

$$\sin t = \sinh u \sinh v$$

in rectangular coordinates (u, v, t), and is depicted in Figure 2. It turns out that the harmonic mapping F_4 provides isothermal parameters for Scherk's saddle-tower surface; and for each even integer $n \geq 6$ the mappings F_n lift to the generalized versions of Scherk's surface recently discovered by Karcher [**7**]. These connections are based on the Weierstrass-Enneper representation of a minimal surface, which will now be briefly reviewed.

Let $p(z)$ be an analytic function and $q(z)$ a meromorphic function in the unit disk $\mathbf{D}$, with p vanishing only at the poles (if any) of q and having a zero of precise order $2m$ wherever q has a pole of order m. Then the formulas

$$u = \mathrm{Re}\left\{ \int_0^z p(1+q^2)\, dz \right\}$$
$$v = \mathrm{Im}\left\{ \int_0^z p(1-q^2)\, dz \right\}$$
$$t = 2\mathrm{Im}\left\{ \int_0^z pq\, dz \right\}$$

define an isothermal (*i.e.*, an angle-preserving) parametric representation of a regular minimal surface known as its *Weierstrass-Enneper representation*. Conversely, every regular minimal surface has locally an isothermal representation of this form. In particular, the projection of any such representation onto the (u, v)-plane defines a harmonic mapping $w = u + iv = f(z)$ of $\mathbf{D}$ whose dilatation can be calculated as $\omega = q^2$. Thus in order for a sense-preserving harmonic mapping to be the projection of an isothermally represented minimal surface, its dilatation must necessarily be the square of an analytic function.

Conversely, it can be shown (see [**6**] or [**4**]) that the condition is also sufficient. In other words, a sense-preserving harmonic mapping f lifts to a minimal surface represented by isothermal parameters, if and only if the dilatation of f has no zero of odd order.

One simple example is to take $p(z) = 1$ and $q(z) = z$. The resulting harmonic mapping of $\mathbf{D}$ is $f(z) = z + (1/3)\bar{z}^3$, which lifts to a well-known minimal surface called *Enneper's surface*. More generally, the choices $p(z) = 1$ and $q(z) = z^n$ produce the harmonic mapping $f(z) = z + (1/2n+1)\bar{z}^{2n+1}$ and a generalization of Enneper's surface. For another example, the choices $p(z) = 2/(1 - z^4)$ and $q(z) = iz$

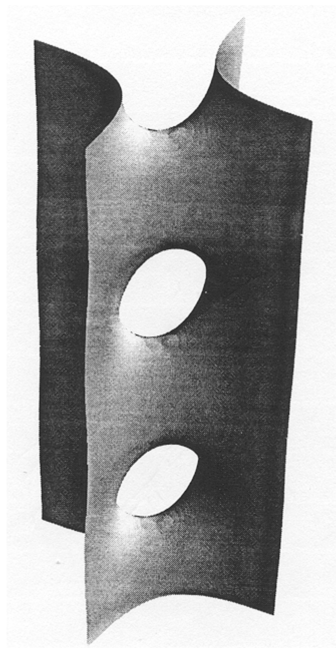

FIGURE 2. Scherk's classical saddle-tower surface.

lead to a standard harmonic mapping of $\mathbf{D}$ onto a square inscribed in the unit disk. It lifts to a saddle-surface with vertical walls known as *Scherk's first surface*, described by the equation $t = \log(\cos v/\cos u)$. Corresponding harmonic mappings onto inscribed regular $2n$-gons lift to multiple saddle-surfaces which may be viewed as generalizations of Scherk's first surface.

Scherk's saddle-tower surface, also known variously as *Scherk's second surface* or *Scherk's fifth surface*, results from the choices $p(z) = 1/(1-z^4)$ and $q(z) = z$. The associated harmonic mapping is $w = F_4(z)$ as defined by (1), which lifts to a surface of height

$$t = \frac{1}{2}\arg\left\{\frac{1+z^2}{1-z^2}\right\}.$$

More generally, the partial-fraction expansions (2) and (3) show that, for any integer $m \geq 2$, the choices $p(z) = 1/(1-z^{2m})$ and $q(z) = z^{m-1}$ produce the harmonic mapping $w = F_{2m}(z)$. An elementary integration gives the corresponding minimal surface with height

$$t = \frac{1}{m} \arg \left\{ \frac{1+z^m}{1-z^m} \right\}.$$

These "multiple saddle-towers" are Karcher's generalizations of Scherk's surfaces. Note that, because the harmonic mapping F_n has dilatation $\omega(z) = z^{n-2}$, it lifts to a minimal surface if and only if n is an even integer. The procedure suggests a natural framework in which Karcher's surfaces might have been discovered, since the mappings F_{2m} are obvious candidates for generalizations of F_4. The strategy may perhaps prove fruitful in discovering generalizations of other minimal surfaces: calculate the underlying harmonic mapping and attempt to generalize it.

4. Further properties of the mappings. We conclude with two more remarks about the harmonic mappings F_n. First we show that F_n approaches the identity as n tends to infinity. Curiously, the two other classes of harmonic mappings, offered above as examples in generalizing Enneper's surface and Scherk's first surface, are easily seen to have the same property.

Theorem 2. *The harmonic mappings $F_n(z)$, as defined by* (1), *converge to z as n tends to infinity, uniformly on each compact subset of* $\mathbf{D}$.

Proof. We shall outline two proofs. The first is longer but more elegant. With z held fixed, the expression (1) can be viewed as a Riemann sum for the integral

$$-\frac{1}{\pi} \int_0^{2\pi} e^{i\varphi} \log |z - e^{i\varphi}| \, d\varphi = \frac{i}{\pi} \int_{\mathbf{T}} \log |z - \zeta| \, d\zeta = z,$$

where $\mathbf{T}$ denotes the unit circle. The integral is easily calculated through an integration by parts and an appeal to the Poisson formula.

The second proof is more prosaic. According to (2) and (3), the mapping has the form $F_n(z) = u + iv$, where u and v are given by (4). But the integrands in (4) both tend to 1 as $n \to \infty$, uniformly on compact subsets of $\mathbf{D}$, so it is clear that $F_n(z) \to z$ as $n \to \infty$, uniformly on compact subsets of $\mathbf{D}$.

Finally we remark that, although the mappings F_n are unbounded, they map the disk onto regions of finite area. In fact, the area A_n is easily calculated and is seen to remain bounded as $n \to \infty$. The Jacobian of F_n is found from (8) and (9), so that

$$\begin{aligned} A_n &= \iint_{\mathbf{D}} \{|\partial F_n/\partial z|^2 - |\partial F_n/\partial \bar{z}|^2\}\, dx\, dy \\ &= \int_0^1 \int_0^{2\pi} \frac{1-r^{2n-4}}{|1-r^n e^{in\theta}|^2} r\, dr\, d\theta \\ &= \int_0^1 r(1-r^{2n-4})(1-r^{2n})\, dr \\ &= \frac{2n^2(n-2)}{(2n-1)(n^2-1)}\pi < \pi, \quad n \geq 3. \end{aligned}$$

Note that $A_n \to \pi$ as $n \to \infty$, a property consistent with Theorem 2.

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REFERENCES

1. J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. **9** (1984), 3–25.

2. U. Dierkes, S. Hildebrandt, A. Küster and O. Wohlrab, *Minimal surfaces* I, Springer-Verlag, New York, 1991.

3. P. Duren, *A survey of harmonic mappings in the plane*, Texas Tech. University, Mathematics Series, Visiting Scholars' Lectures 1990–1992, vol. 18 (1992), 1–15.

4. ———, *Harmonic mappings in the plane*, Cambridge University Press, to appear.

5. P. Duren, W. Hengartner and R.S. Laugesen, *The argument principle for harmonic functions*, Amer. Math. Monthly **103** (1996), 411–415.

6. W. Hengartner and G. Schober, *Curvature estimates for some minimal surfaces*, in *Complex analysis*: *Articles dedicated to Albert Pfluger on the occasion of his 80th birthday* (J. Hersch and A. Huber, eds.), Birkhäuser Verlag, Basel, 1988, 87–100.

7. H. Karcher, *Embedded minimal surfaces derived from Scherk's examples*, Manuscripta Math. **62** (1988), 83–114.

8. R. Osserman, *A survey of minimal surfaces*, Dover Publications, 2nd ed., New York, 1986.

9. J.C.C. Nitsche, *Lectures on minimal surfaces*, Vol. 1, Cambridge University Press, Cambridge, 1989.

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