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TAKETA'S THEOREM FOR RELATIVE CHARACTER DEGREES

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ABSTRACT. It has been conjectured by Isaacs that, for finite group G, the inequality $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$ holds for all normal solvable subgroups N of G. We show that this conjecture holds for M-groups. Also, we prove that, if G is solvable and the common-divisor graph $\Gamma(G|N)$ is disconnected, then $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$, which is a generalization of [**5**, Theorem A].

1. Introduction. Let G be a finite group. The idea of this paper is to focus on the relative degree sets, which are certain subsets of $\mathrm{cd}\,(G)$. If $N \lhd G$, we define $\mathrm{cd}\,(G \mid N) = \{\chi(1) \mid \chi \in \mathrm{Irr}\,(G \mid N)\}$, where $\mathrm{Irr}\,(G \mid N) = \{\chi \in \mathrm{Irr}\,(G) \mid N \not\subseteq \ker\,(\chi)\}$. In other words, the relative degree set $\mathrm{cd}(G \mid N)$ is the set of degrees of those irreducible characters of G whose kernels do not contain N.

Isaacs conjectured that, if G is any finite group, then the inequality $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$ holds for all normal solvable subgroups N of G, where $\mathrm{dl}\,(N)$ is the derived length of N. In [**4**], it has been proved by Isaacs and Knutson that this conjecture holds if G is solvable and $|\mathrm{cd}\,(G \mid N)| \leq 3$. Also, in the same paper, they proved that if N is a nilpotent normal subgroup of the finite group G, then $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$, and this conjecture holds.

In this paper, we verify this conjecture in two different cases. First, we prove that Isaacs' conjecture holds for M-groups. In other words, we show that, if G is an M-group, then the inequality $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$ holds for all normal subgroups N of G.

Theorem 1. *Let G be an M-group. If N is a normal subgroup of G, then* $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$.

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Note that, if G is any finite group, then $\operatorname{Irr}(G \mid G')$ is exactly the set of nonlinear irreducible characters of G, and hence $\operatorname{cd}(G \mid G') = \operatorname{cd}(G) - \{1\}$. It follows from an old result of Taketa when G is an M-group that $\operatorname{dl}(G) \leq |\operatorname{cd}(G)|$. Thus, if G is an M-group, then the inequality $\operatorname{dl}(N) \leq |\operatorname{cd}(G \mid N)|$ holds for $N = G'$ as $\operatorname{dl}(G') = \operatorname{dl}(G) - 1$. Theorem 1 above implies that the inequality $\operatorname{dl}(N) \leq |\operatorname{cd}(G \mid N)|$ holds not only for $N = G'$ but also for all normal subgroups N of G if G is an M-group.

Second, in the last decade, the study of $\operatorname{cd}(G \mid N)$ has been assisted by attaching a graph to $\operatorname{cd}(G \mid N)$. In fact, there are two graphs connected with this set. Let $\rho(\operatorname{cd}(G \mid N))$ be the set of all primes dividing elements of $\operatorname{cd}(G \mid N)$. The *prime graph* of $\operatorname{cd}(G \mid N)$ is the graph $\Delta(G \mid N)$ with $\rho(\operatorname{cd}(G \mid N))$ as vertices, and there is an edge between p and q if pq divides some element of $\operatorname{cd}(G \mid N)$. The *common-divisor graph* of $\operatorname{cd}(G \mid N)$ is the graph $\Gamma(G \mid N)$ with $\operatorname{cd}(G \mid N) - \{1\}$ as vertices, and distinct vertices m and n are joined if they have a nontrivial common divisor. It is not difficult to see that these graphs have the same number of connected components. In particular, $\Delta(G \mid N)$ is disconnected if and only if $\Gamma(G \mid N)$ is disconnected.

Isaacs in [**3**] studied the graph $\Gamma(G \mid N)$. He proved in Theorem A of that paper that, if $N' < N \subseteq G'$, then $\Gamma(G \mid N)$ has at most two connected components. In [**7**], Lewis removed the hypotheses on N, and he proved that $\Gamma(G \mid N)$ has at most three connected components. Also, in [**7**], he defined a family of groups which he called *groups of disconnected type* and he could show that a finite group G has a normal subgroup N so that $\Delta(G \mid N)$ is disconnected (and so $\Gamma(G \mid N)$ is disconnected) if and only if either $\Delta(G)$ is disconnected or G is of disconnected type.

Since $\operatorname{cd}(G \mid G') = \operatorname{cd}(G) - \{1\}$, we see that the graph $\Gamma(G \mid G')$ has vertex $\operatorname{cd}(G) - \{1\}$. It follows from [**5**, Theorem A] that, if G is solvable and the graph $\Gamma(G \mid G')$ is disconnected, then $\operatorname{dl}(G) \leq |\operatorname{cd}(G)|$. As $\operatorname{dl}(G') = \operatorname{dl}(G) - 1$, we deduce that, if the graph $\Gamma(G \mid G')$ is disconnected, then $\operatorname{dl}(G') \leq |\operatorname{cd}(G \mid G')|$.

In the following theorem, we let the group G be solvable and we show that the inequality $\operatorname{dl}(N) \leq |\operatorname{cd}(G \mid N)|$ actually holds for all normal subgroups N of G with the property that the graph $\Gamma(G \mid N)$

is disconnected. In other words, we prove that, if N is any normal subgroup of G such that $\Gamma(G \mid N)$ is disconnected, then $\mathrm{dl}\,(N) \leq |\mathrm{cd}(G \mid N)|$. Not only does this result extend Theorem A of [**5**], but also we believe that this result will be a tool for studying the long-standing conjecture of Seitz and Isaacs that, if G is a finite solvable group G, then $\mathrm{dl}\,(G) \leq |\mathrm{cd}\,(G)|$, where $\mathrm{dl}\,(G)$ is the derived length of G.

Theorem 2. *Let G be solvable, and let $N \lhd G$ be such that the graph $\Gamma(G|N)$ is disconnected. Then* $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$.

2. Proof of Theorem 1. In this section, we prove Theorem 1. Let G be a finite group. Since $\mathrm{Irr}\,(G \mid G')$ is exactly the set of non-linear irreducible characters of G, we have $\mathrm{cd}\,(G \mid G') = \mathrm{cd}\,(G) - \{1\}$. It follows from an old result of Taketa, when G is an M-group, that $\mathrm{dl}\,(G') \leq |\mathrm{cd}\,(G \mid G')|$. This result is Theorem 5.12 of [**1**]. In Theorem 1, we prove that the inequality $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$ holds not only for $N = G'$ but also for all normal subgroups N of G, if G is an M-group. To do this, we use a similar argument to the standard proof of Taketa's theorem ([**1**, Theorem 5.12]).

Proof of Theorem 1. Let $f_1 < f_2 < \cdots < f_r$ be the members of $\mathrm{cd}\,(G \mid N)$. We argue that the $N^{(k)} \subseteq \ker \chi$ if $\chi \in \mathrm{Irr}\,(G \mid N)$ with $\chi(1) \leq f_k$. This will show that $N^{(r)}$ is contained in the kernel of every member of $\mathrm{Irr}\,(G \mid N)$. On the other hand, since $N^{(r)} \subseteq N$, this subgroup is also contained in the kernel of every irreducible character of G whose kernel contains N. Hence, $N^{(r)}$ is contained in the kernels of all irreducible characters of G, and so $N^{(r)} = 1$, and N has derived length at most r.

To see this, we use induction on k. Let $\chi \in \mathrm{Irr}\,(G \mid N)$ with $\chi(1) \leq f_k$. Write $\chi = \lambda^G$ for some linear character λ of some subgroup $U \leq G$. Non-principal constituents of $(1_U)^G$ have degree less than $\chi(1)$, so each constituent ψ of $(1_U)^G$ either has N in its kernel, or else it lies in $\mathrm{Irr}\,(G \mid N)$ and has degree at most f_{k-1}. In either case, we have $N^{(k-1)} \subseteq \ker \psi$, and thus $N^{(k-1)} \subseteq \ker\,(1_U)^G \subseteq U$. Then $N^{(k)} \subseteq U' \subseteq \ker \lambda$, and since $N^{(k)}$ is a normal subgroup of G, we have $N^{(k)} \subseteq \ker \chi$. This proves the result. (This argument also establishes the base case of the induction, where $k = 1$, because then no constituent of $(1_U)^G$ lies in $\mathrm{Irr}\,(G \mid N)$.) $\square$

3. Proof of Theorem 2. In this section, we prove Theorem 2. To do this, we need the following lemma which is a consequence of the main theorem of [**6**].

Lemma 3.1. *Let G be a solvable group such that the prime graph $\Delta(G)$ is disconnected. Then either* $\mathrm{dl}\,(G) \leq 5$ *or G satisfies the hypothesis of Example* 2.6 *of* [**6**].

Proof. We suppose that G does not satisfy the hypothesis of Example 2.6 of [**6**], and we show that $\mathrm{dl}\,(G) \leq 5$. It follows from the main theorem of [**6**] that G is one of the groups included in Examples 2.1–2.5 of [**6**].

If G satisfies the hypothesis of Example 2.1 of [**6**], then G has a normal nonabelian Sylow p-subgroup P and an abelian p-complement K for some prime p. It follows from [**6**, Lemma 3.1] that $\Delta(G)$ has two connected components, $\{p\}$ and $\pi(|G : F|)$, where F is the Fitting subgroup of G.

We claim that K fixes every nonlinear irreducible character of P. To do this, let θ be a nonlinear irreducible character of P such that $K \nsubseteq \mathrm{Stab}_G(\theta)$. This implies that some prime divisor l of $|K|$ divides $|G : \mathrm{Stab}_G(\theta)|$. By applying the Clifford correspondence ([**1**, Theorem 6.11]), we obtain that pl divides some character degree of G as $\theta(1)$ is a p-power. Since $p \neq l$, this is a contradiction because we know that the singleton $\{p\}$ is one of the connected components of $\Delta(G)$. We conclude that K fixes every nonlinear irreducible character of P as claimed.

Since K acts coprimely on P and K fixes every nonlinear irreducible character of P, it follows from [**2**, Theorem 3.3] that $P' = [P, K]'$ and either $[P, K]$ is a p-group of class 2 or $[P, K]$ is a Frobenius group with kernel $[P, K]'$. As $[P, K]$ is a p-group, we deduce that $[P, K]$ is not a Frobenius group. Hence, $[P, K]$ is a p-group of class 2 and $\mathrm{dl}\,([P, K]) = 2$. We conclude that $\mathrm{dl}\,(P) = 2$ as $P' = [P, K]'$. Thus, $\mathrm{dl}\,(G) \leq 3$ as G/P is abelian, which is the desired conclusion.

Suppose that G satisfies the hypothesis of Example 2.2 of [**6**]. It follows from [**6**, Lemma 3.2] that $\mathrm{cd}\, G = \{1, 2, 3, 8\}$, and so $\mathrm{dl}\,(G) \leq 4$ which is the desired conclusion.

Assume that G satisfies the hypothesis of Example 2.3 of [**6**]. Then G is the semi-direct product of a subgroup H acting on a subgroup P_1 where P_1 is an elementary abelian group of order 9 and $\mathrm{cd}\, H = \{1, 2, 3, 4\}$. This implies that $\mathrm{dl}\,(H) \leq 4$, and hence $\mathrm{dl}\,(G) \leq 5$ as G/P_1 is isomorphic to H. This yields the desired conclusion in this case.

Now, suppose that F is the Fitting subgroup of G and E/F is the Fitting subgroup of G/F. If G satisfies the hypothesis of Example 2.4 of [**6**], then it follows from [**6**, Lemma 3.4] that G/F is metacyclic and $F = V \times Z$, where V is an elementary abelian group and Z is the central subgroup of G. We obtain that $\mathrm{dl}\,(G/F) \leq 2$ and $\mathrm{dl}\,(F) = 1$. Thus, $\mathrm{dl}\,(G) \leq 3$, which is the desired conclusion.

Finally, assume that G satisfies the hypothesis of Example 2.5 of [**6**]. By applying [**6**, Lemma 3.5], we deduce that E satisfies the hypotheses of Example 2.1 of [**6**]. By the second paragraph of the proof, we have that $\mathrm{dl}\,(E) \leq 3$, and so $\mathrm{dl}\,(G) \leq 4$ as G/E is cyclic. This completes the proof of the lemma. $\square$

Now, we are ready to prove Theorem 2 as a corollary.

Corollary 3.2. *Let G be solvable, and let $N \lhd G$ be such that the graph $\Gamma(G \mid N)$ is disconnected. Then* $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$.

Proof. Since $N \lhd G$ is such that the graph $\Gamma(G \mid N)$ is disconnected, it follows from [**7**, Theorem B] that either $\Gamma(G)$ is disconnected or G is of disconnected type (for the definition of groups of disconnected type, see [**7**, Definition 4.1]). Suppose that G is of disconnected type but $\Gamma(G)$ is connected. By applying Lemma 4.3 of [**7**], we determine that $N = [K, P]$, where K is some $2'$-subgroup and P is a normal non abelian 2-subgroup of G. Hence, we observe that $N \subseteq P$ as P is normal in G. We obtain that N is a normal 2-subgroup of G, and so N is a normal nilpotent subgroup of G. Corollary 3.3 of [**3**] implies that $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$, which is the desired conclusion.

Therefore, we can assume that $\Gamma(G)$ is also disconnected. Note that, if $N \not\subseteq G'$, then it follows from [**7**, Lemma 2.1] that $\mathrm{cd}\,(G) = \mathrm{cd}\,(G \mid N)$.

We deduce that $\mathrm{dl}\,(N) \leq \mathrm{dl}\,(G) \leq |\mathrm{cd}\,(G)| = |\mathrm{cd}\,(G \mid N)|$, where the inequality $\mathrm{dl}\,(G) \leq |\mathrm{cd}\,(G)|$ holds because of Theorem A of [**5**]. We conclude that $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$. Thus, we assume that $N \subseteq G'$.

On the other hand, Theorems B and C of [**4**] imply that we may assume that $|\mathrm{cd}\,(G \mid N)| \geq 4$. Since $\Gamma(G)$ is disconnected, it follows from Lemma 3.1 that either $\mathrm{dl}\,(G) \leq 5$ or G is the group included in Example 2.6 of [**6**].

Suppose that $\mathrm{dl}\,(G) \leq 5$. Since $N \subseteq G'$, this implies that $\mathrm{dl}\,(N) \leq 4$. We conclude that $\mathrm{dl}\,(N) \leq |\mathrm{cd}\,(G \mid N)|$ as $|\mathrm{cd}\,(G \mid N)| \geq 4$. This is the desired conclusion.

Thus, we assume that G satisfies the hypotheses of Example 2.6 of [**6**]. Then [**6**, Lemma 3.6] implies that G/E and E/F are both cyclic, G/F' satisfies the hypotheses of Example 2.4 of [**6**], and $\mathrm{cd}\,(G \mid F')$ consists of degrees that divide $|P \parallel E : F|$ and are divisible by $p|B|$, where P is a normal Sylow p-subgroup of G for some prime p and $|B|$ is some divisor of $|E : F|$. As G/F' satisfies the hypotheses of [**6**, Example 2.4], it follows from Lemma 3.4 of [**6**] that $\mathrm{cd}\,(G/F' \mid E'/F') = \{|E : F|\}$ as F/F' is the Fitting subgroup of G/F'.

On the other hand, since $N \subseteq G'$ and the groups G/E and E/F are both cyclic, we see that $N \subseteq E$ and $N' \subseteq E' \subseteq F$. This implies that $\mathrm{cd}\,(G \mid N') \subseteq \mathrm{cd}\,(G \mid E')$. Observe that $\mathrm{cd}\,(G \mid E') = \mathrm{cd}\,(G \mid F') \cup \mathrm{cd}\,(G/F' \mid E'/F') = \mathrm{cd}\,(G \mid F') \cup \{|E : F|\}$. We determine that $\mathrm{cd}\,(G \mid N') \subseteq \mathrm{cd}\,(G \mid F') \cup \{|E : F|\}$. We claim that the graph $\Gamma(G \mid N')$ is connected. To do this, let $a, b \in \mathrm{cd}\,(G \mid N')$ be arbitrary distinct elements of $\mathrm{cd}\,(G \mid N')$. Recall that $\mathrm{cd}\,(G \mid F')$ consists of degrees that divide $|P \parallel E : F|$ and are divisible by $p|B|$, where P is a normal Sylow p-subgroup of G for some prime p and $|B|$ is some divisor of $|E : F|$. As $\mathrm{cd}\,(G \mid N') \subseteq \mathrm{cd}\,(G \mid F') \cup \{|E : F|\}$, we obtain that either $a, b \in \mathrm{cd}\,(G \mid F')$ or $a \in \mathrm{cd}\,(G \mid F')$ and $b = |E : F|$. If $a, b \in \mathrm{cd}\,(G \mid F')$, then p divides both a and b, and so $(a, b) > 1$. If $a \in \mathrm{cd}\,(G \mid F')$ and $b = |E : F|$, the $|B|$ divides both a and b, and hence $(a, b) > 1$. Thus, in both cases, a and b are joined. We deduce that the graph $\Gamma(G \mid N')$ is connected as claimed.

Since the graph $\Gamma(G \mid N)$ is disconnected, we determine that $\mathrm{cd}\,(G \mid N')$ is a proper subset of $\mathrm{cd}\,(G \mid N)$. Also, as $N' \subseteq F$ is nilpotent, we deduce by Corollary 3.3 of [**3**] that $\mathrm{dl}\,(N') \leq |\mathrm{cd}\,(G \mid N')|$. We conclude that

$$\mathrm{dl}\,(N) \leq \mathrm{dl}\,(N') + 1 \leq |\mathrm{cd}\,(G \mid N')| + 1 \leq |\mathrm{cd}\,(G \mid N)|,$$

where the last inequality holds because $\mathrm{cd}\,(G \mid N')$ is a proper set of $\mathrm{cd}\,(G \mid N)$. This yields the desired conclusion. $\square$

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