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A TRANSFORMATION FORMULA FOR DOUBLE HYPERGEOMETRIC SERIES

R. P. SINGAL

Carlitz [4], Pandey and Saran [6] and the author [8], [9] gave a number of transformation formulae of Kampé de Fériet double hypergeometric series. Since transformation formulae play an important role, the object of this paper is to obtain a transformation formula for the double series and to derive two Cayley Orr type results and a Watson sum for the double series.

1. Kampé de Fériet double hypergeometric series [1] in the notation of [7] is defined as

$$F^{p,q}_{r,s}\left[\begin{matrix} a_p : b_q; b_q'; \\ c_r : d_s; d_s'; \end{matrix}\right] = \sum_{m,n=0}^{\infty} \frac{(a_p)_{m+n}(b_q)_m(b_q')_n}{(c_r)_{m+n}(d_s)_m(d_s')_n m!n!}$$

where $p + q \leqq r + s + 1$ and where a_p stands for the sequence $a_1, a_2, \cdots, a_p$; $(a)_m = \Gamma(a+m)/\Gamma(a)$.

The main result to be proved is

$$F^{1,3}_{1,2}\left[\begin{matrix} a : b, c, -n; d-b, c', -m; \\ d : e, 1+a+b+c-d-e-n; \\ e', 1+a+c'-b-e'-m; \end{matrix}\right]$$

$$\text{(1.1)} = \frac{(e-c)_n(e'-c')_m(e+d-b-a)_n(e'+b-a)_m}{(e)_n(e')_m(e+d-a-b-c)_n(e'+b-c'-a)_m}$$

$$\times F^{1,3}_{1,2}\left[\begin{matrix} d-a : d-b, c, -n; b, c', -m; \\ d : d+e-a-b, 1+c-e-n; \\ e'+b-a, 1+c'-e'-m; \end{matrix}\right].$$

2. **Proof of (1.1).**

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$$F = F^{1,3}_{1,2}\left[\begin{matrix} a : b, c, -n; d - b, c', -m; \\ d : e, 1 + a + b + c - d - e - n; \end{matrix}\right.$$

$$\left.\begin{matrix} \\ e', 1 + a + c' - b - e' - m; \end{matrix}\right]$$

$$= \sum_{p=0}^{n} \frac{(a)_p(b)_p(c)_p(-n)_p}{(d)_p(e)_p(1 + a + b + c - d - e - n)_p p!}$$

$$\times {}_4F_3\left[\begin{matrix} a + p, d - b, c', -m; \\ d + p, e', 1 + a + c' - b - e' - m; \end{matrix}\right]$$

Now using relation [2, 7.2(1)]

$${}_4F_3(x, y, z, -n; u, v, 1 + x + y + z - u - v - n;)$$

$$= \frac{(v - z)_n (u + v - x - y)_n}{(v)_n(u + v - x - y - z)_n}$$

$$\times {}_4F_3(u - x, u - y, z, -n; u, 1 - v + z - n, u + v - x - y)$$

we have

$$F = \frac{(e' - c')_m(e' + b - a)_m}{(e')_m(e' + b - a - c')_m}$$

$$\times \sum_{p=0}^{n} \frac{(a)_p(b)_p(c)_p(-n)_p}{(d)_p(e)_p(1 + a + b + c - d - e - n)_p p!}$$

$$\times {}_4F_3(d - a, b + p, c', -m;$$

$$d + p, e' + b - a, 1 + c' - e' - m;)$$

$$= \frac{(e' - c')_m(e' + b - a)_m}{(e')_m(e' + b - a - c')_m} \tag{2.1}$$

$$\times F^{1,3}_{1,2}\left[\begin{matrix} b : a, c, -n; d - a, c', -m; \\ d : e, 1 + a + b + c - d - e - n; \end{matrix}\right.$$

$$\left.\begin{matrix} \\ e' - a + b, 1 + c' - e' - m; \end{matrix}\right].$$

Repeating the process again on the right side of (2.1) for the second series we have the required result (1.1).

Allowing $a \to \infty$ in (2.1) and (1.1) we have

$$F_{1,1}^{0,3}\left[\begin{matrix}-: b, c, -n; d-b, c', -m; \\ d: e, e';\end{matrix}\right] = \frac{(e'-c')_m}{(e')_m} F_{1,1}^{1,2}\left[\begin{matrix}b: c, -n; c', -m; \\ d: e, 1+c'-e'-m;\end{matrix}\right] \tag{2.2}$$

and

$$\begin{aligned} F_{1,1}^{0,3}\left[\begin{matrix}-: b, c, -n; d-b, c', -m; \\ d: e, e';\end{matrix}\right] &= \frac{(e'-c')_m(e-c)_n}{(e')_m(e)_n} \\ &\times F_{1,1}^{0,3}\left[\begin{matrix}-: d-b, c, -n; b, c', -m; \\ d: 1+c-e-n, 1+c'-e'-m;\end{matrix}\right], \end{aligned} \tag{2.3}$$

a result due to the writer [8].

Allowing $n \to \infty$, $m \to \infty$ in (2.1) and (1.1) we have two transformation formulae for $F_{1,1}^{1,2}$.

Similarly letting $b \to \infty$ in (1.1) we get

$$\begin{aligned} F_{1,1}^{1,2}\left[\begin{matrix}a: c, -n; c', -m; \\ d: e, e';\end{matrix}\right] &= \frac{(e-c)_n(e'-c')_m}{(e)_n(e')_m} \\ &\times F_{1,1}^{1,2}\left[\begin{matrix}d-a: c, -n; c', -m; \\ d: 1+c-e-n, 1+c'-e'-m;\end{matrix}\right], \end{aligned} \tag{2.4}$$

a result due to Pandey and Saran [6].

Taking $d = 1-a-n-m$, $e = 1-c-n$, $e' = 1-c'-m$ in (2.4) we get

$$\begin{aligned} F_{1,1}^{1,2}\left[\begin{matrix}1-2a-m-n: -n, c; -m, c'; \\ 1-a-m-n: 2c, 2c';\end{matrix}\right] &= \frac{(c)_n(c')_m}{(2c)_n(2c')_m} \\ &\times F_{1,1}^{1,2}\left[\begin{matrix}a: -n, c; -m, c'; \\ 1-a-m-n: 1-c-n, 1-c'-m;\end{matrix}\right] \end{aligned} \tag{2.5}$$

The right side of (2.5) can be summed by the result [9, (4.3)]

$$\begin{aligned} &F_{1,1}^{1,2}\left[\begin{matrix}d: a, b; a', b'; \\ 1+a+a'-d: 1+a-b, 1+a'-b';\end{matrix}\right] \\ &= \frac{\Gamma(1+a+a'-d)\Gamma(1-2d)}{\Gamma(1+a+a'-2d)\Gamma(1-d)} \\ &\times \frac{\Gamma(\frac{1}{2}+d)\Gamma(b'-a'/2)\Gamma(b-a/2)\Gamma(d-a'+b')}{\Gamma(\frac{1}{2})\Gamma(d+b'-a'/2)\Gamma(d+b-a/2)\Gamma(b'-a')} \\ &\times \frac{\Gamma(d-a+b)\Gamma(1-d-b-b')}{\Gamma(b-a)\Gamma(1-b-b')} \end{aligned} \tag{2.6}$$

provided either d or $a/2$, $a'/2$ are negative integers and $2d = 1 + a + a' - 2b - 2b'$.

Hence we have the sum

$$F^{1,2}_{1,1}\left[\begin{matrix} 1 - 2a - 2m - 2n : -2n, c; -2m, c'; \\ 1 - a - 2m - 2n : 2c, 2c'; \end{matrix}\right] \tag{2.7}$$
$$= \frac{(\frac{1}{2})_{m+n}(a)_{m+n}(a + c)_{2n}(a + c')_{2m}}{(c + \frac{1}{2})_n(c' + \frac{1}{2})_m(a)_{2m+2n}(a + c)_n(a + c)_m}$$

provided $a + c + c' + m + n = \frac{1}{2}$ which can be called a Watson sum for the double series.

Taking $e = e' = a$ in (1.1) and making $m, n \to \infty$ we get a result due to Carlitz [5].

Also taking $e = e' = a$ and $c' = d - c$ in (1.1) we can get a sum for $F^{1,3}_{1,2}$.

3. Finally we give two Cayley Orr type results. It can be easily proved that by proper choice of parameters, (1.1) gives us the following results: If

$$(1 - X)^{c+a-e-b'}(1 - Y)^{c'+a-e'-b} \times F^{1,2}_{1,1}\left[\begin{matrix} a : b, c; b', c'; \\ b + b' : e, e'; \end{matrix} X, Y\right] = \sum_{m,n=0}^{\infty} A_{m,n}X^nY^m \tag{3.1}$$

then

$$(1 - X)^{c-e}(1 - Y)^{c'-e'} \times F^{1,2}_{1,1}\left[\begin{matrix} b + b' - a : b', c; b, c'; \\ b + b' : e + b' - a, e' + b - a; \end{matrix} X, Y\right]$$
$$= \sum_{m,n=0}^{\infty} \frac{(e)_n(e')_m}{(e + b' - a)_n(e' + b - a)_m} A_{m,n}X^nY^m$$

and, if

$$(1 - X)^{-a}(1 - Y)^{-a}(a : c, c'; b + b'; X/(X - 1), Y/(Y - 1)) = \sum_{m,n=0}^{\infty} A_{m,n}X^nY^m \tag{3.2}$$

then

$$(1-X)^{c-b'}(1-Y)^{c'-b}F_{1,1}^{1,2}\left[\begin{matrix} a : b, c; b', c'; \\ b+b' : a, a; \end{matrix} X, Y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(b')_n(b)_m}{(a)_n(a)_m} A_{m,n}X^nY^m.$$

It may be remarked that if we put $Y = 0$ in (3.1) we get a result due to Bhatt [3] in the correct form and (3.2) gives us the following result: if

$$(3.3) \qquad (1-X)^{-a}{}_2F_1(a, c; b+b'; X/(X-1)) = \sum_{m=0}^{\infty} A_mX^m$$

then

$$(1-X)^{c-b'}{}_2F_1(b, c; b+b'; X) = \sum_{m=0}^{\infty} \frac{(b')_m}{(a)_m} A_mX^m.$$

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Thapar College of Engineering, Patiala, India