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OPERATORS IN COWEN-DOUGLAS CLASSES

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ABSTRACT. The paper introduces a new approach to the Cowen-Douglas theory based on the notion of a spanning holomorphic cross-section. This approach is less geometric and enables one to obtain several additional results, including one about the similarity of operators in the Cowen-Douglass classes and another about the representation of these operators as the adjoint of multiplication by z on certain Hilbert spaces of holomorphic functions.

1. Introduction

Let H be a (separable) Hilbert space and let Ω be a domain in the complex plane $\mathbb{C}$. For a positive integer n the Cowen-Douglas class $B_n(\Omega)$ consists of bounded linear operators T on H with the following properties:

(1) $\mathrm{Ran}(\lambda I - T) = H$ for every $\lambda \in \Omega$.
(2) $\dim[\ker(\lambda I - T)] = n$ for every $\lambda \in \Omega$.
(3) $\mathrm{Span}\{\ker(\lambda I - T)\colon \lambda \in \Omega\} = H$.

Here I is the identity operator on H and Span{ } denotes the closed linear span of a collection of sets in H.

The first systematic study of the classes $B_n(\Omega)$ was made by Cowen and Douglas in [1]. Among the subsequent contributions to the subject we mention [2]. Several ideas in the present paper can easily be traced to [2].

For an operator $T \in B_n(\Omega)$ the mapping

$$z \mapsto \ker(zI - T), \qquad z \in \Omega,$$

gives rise to a Hermitian holomorphic vector bundle, denoted E_T, over Ω. It was shown in [1] that two operators S and T in $B_n(\Omega)$ are unitarily equivalent if and only if the corresponding Hermitian bundles E_S and E_T are equivalent. As a consequence of this, it was shown in [1] that the curvature function of E_T is a complete set of unitary invariants for operators T in $B_1(\Omega)$.

In this paper we introduce an approach to the Cowen-Douglas theory based on the notion of a *spanning holomorphic cross-section*. This new approach is less geometric and enables us to obtain several additional results, including one about the similarity

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of operators in $B_n(\Omega)$ and another about the representation of operators in $B_n(\Omega)$ as the adjoint of multiplication by z on certain Hilbert spaces of holomorphic functions.

Recall that a holomorphic cross-section of the Hermitian bundle E_T is a holomorphic function $\gamma: \Omega \to H$ such that for every $z \in \Omega$, the vector $\gamma(z)$ belongs to the fibre of E_T over z. We call γ a *spanning holomorphic cross-section* if

$$\mathrm{Span}\{\gamma(z): z \in \Omega\} = H.$$

We can now state the main result of the paper.

THEOREM A. *Suppose $n \geq 1$ and $T \in B_n(\Omega)$. Then the Hermitian bundle E_T possesses a spanning holomorphic cross-section.*

As a corollary of this result we obtain the following representation for operators in $B_n(\Omega)$.

THEOREM B. *Every operator $T \in B_n(\Omega)$ is unitarily equivalent to the adjoint of multiplication by z on a Hilbert space of holomorphic functions on the domain $\overline{\Omega} = \{\bar{z}: z \in \Omega\}$.*

This representation was mentioned in [1] in the case $n = 1$. Partial results in this respect were also obtained in [2].

As another application of our main result we will determine when two operators in $B_n(\Omega)$ are similar or quasi-similar. We will also determine the commutant of an operator in $B_n(\Omega)$. To state our results in this direction we need to define some relations between reproducing kernels.

Let K_1 and K_2 be two reproducing kernels on Ω. If there exists a constant $C > 0$ such that $CK_2 - K_1$ is still a reproducing kernel on Ω, then we say that K_1 is dominated by K_2 and denote the relation by $K_1 \prec K_2$. We say that K_1 and K_2 are similar, and we denote the relation by $K_1 \sim K_2$, if both $K_1 \prec K_2$ and $K_2 \prec K_1$.

For a holomorphic cross-section γ of E_T it is easy to see that the function

$$K_\gamma(z, w) = \langle \gamma(z), \gamma(w) \rangle, \qquad z, w \in \Omega,$$

is a reproducing kernel on Ω. If γ_S and γ_T are holomophic cross-sections of E_S and E_T, respectively, then we write $\gamma_S \prec \gamma_T$ when $K_{\gamma_S} \prec K_{\gamma_T}$, and we write $\gamma_S \sim \gamma_T$ when $K_{\gamma_S} \sim K_{\gamma_T}$.

THEOREM C. *Suppose S and T are operators in $B_n(\Omega)$. Then S and T are similar if and only if there exist spanning holomorphic cross-sections γ_S and γ_T for E_S and E_T, respectively, such that $\gamma_S \sim \gamma_T$.*

Analogous results for unitary equivalence and quasi-similarity of two operators in $B_n(\Omega)$ will also be proved.

THEOREM D. *Let $T \in B_n(\Omega)$ and fix a spanning holomorphic cross-section γ_0 of E_T. Then the commutant of T, denoted $(T)'$, is in a one-to-one correspondence with the set of holomorphic cross-sections γ of E_T with $\gamma \prec \gamma_0$.*

We will also show how our method can be applied to study pull-back bundles of holomorphic maps into Grassmannians. Such bundles are studied in [1] and form the basis for the analysis there. In particular, we will give another proof of the rigidity theorem in [1] in the special case $\Omega \subset \mathbb{C}$, and we will obtain a companion result about the similarity of pull-back maps.

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2. Spanning holomorphic cross-sections

In this section we show that the Hermitian bundle E_T, $T \in B_n(\Omega)$, always has a spanning holomorphic cross-section. This will enable us to represent every operator T in $B_n(\Omega)$ as the adjoint of multiplication by z on a Hilbert space of holomorphic functions in $\overline{\Omega}$.

LEMMA 1. *Every Hermitian holomorphic bundle has a global holomorphic frame.*

This result is well-known in complex geometry and is usually referred to as Grauert's theorem. See [4] or [1].

More specifically, the lemma above says that if E is a Hermitian holomophic vector bundle of rank n over Ω, then there exist holomophic cross-sections $\gamma_1, \dots, \gamma_n$ such that for every $z \in \Omega$ the vectors $\gamma_1(z), \dots, \gamma_n(z)$ form a basis for the fibre space at z. Thus a Hermitian holomorphic bundle is trivial as a holomorphic vector bundle. In particular, to study Hermitian bundles such as E_T, $T \in B_n(\Omega)$, we must look for structures which are not only holomorphic but also Hermitian. The notion of a spanning holomorphic cross-section in the context of E_T is such a structure.

To prove the existence of spanning holomorphic cross-sections in E_T we first recall a classical notion from complex analysis. Let X be a vector space of holomorphic functions in Ω. A set $Z \subset \Omega$ is called a uniqueness set for X if the only function in X that vanishes on Z is the zero function. By the identity theorem, Z is a uniqueness set for X whenever Z has an accumulation point in Ω.

LEMMA 2. *Suppose X is a Banach space consisting of holomorphic functions in Ω such that point evaluations are uniformly bounded linear functionals on compact subsets of Ω. Then there exists a sequence $\{a_n\}$ in Ω such that:*

(1) *$\{a_n\}$ has no accumulation point in Ω.*
(2) *$\{a_n\}$ is a uniqueness set for X.*

Proof. First assume $\Omega = \mathbb{D}$, the open unit disk in $\mathbb{C}$. For $0 < r < 1$ let

$$M(r) = \sup\{|f(z)|\colon \|f\| \le 1, |z| = r\}.$$

If f is a unit vector in X with zeros $\{a_n\}$, repeated according to multiplicity, such that $f(0) \neq 0$, then by Jensen's formula,

$$\log|f(0)| + \sum_{k=1}^{n_r} \log \frac{r}{|a_k|} \leq \log(M(r)),$$

or

$$\log|f(0)| + \sum_{k=1}^{n_r} \log \frac{1}{|a_k|} \leq \log(M(r)) + n \log \frac{1}{r},$$

where $r \in (0, 1)$ is any number such that f has no zeros on $|z| = r$, and $a_1, \dots, a_{n_r}$ are the zeros of f in $|z| < r$. It follows that the zeros of any function in X must approach the unit circle at a certain rate. If we now choose an increasing sequence $\{r_n\}$ in $(0, 1)$ such that $r_n \to 1$ at a lower rate, then any function f in X vanishing on $\{r_n\}$ must be identically zero, so that the sequence $\{r_n\}$ is a uniqueness set for X without an accumulation point in $\mathbb{D}$.

By using a conformal mapping, we see that the desired result is also true for any open disk in $\mathbb{C}$. Furthermore, we can choose the uniqueness sequence so that it lies on any given radius of the disk.

In the case of a general domain $\Omega \subset \mathbb{C}$, we can choose an open disk D in Ω such that some boundary point z_0 of D lies in the boundary of Ω. Let X_D be the space consisting of the restrictions to D of functions in X. By the identity theorem, the restriction of $f \in X$ to the disk D is a one-to-one linear mapping. Therefore, X_D is a Banach space of holomorphic functions in D with the norm inherited from X. Now choose a sequence $\{z_n\}$ in D such that $\{z_n\}$ is a uniqueness set for X_D and such that $z_n \to z_0$ as $n \to +\infty$. If $f \in X$ and $f(a_n) = 0$ for every n, then the restriction of f to D must be identically zero, and hence $f(z) = 0$ for all $z \in \Omega$. Thus $\{a_n\}$ is a uniquenss set for X without an accumulation point in Ω. $\square$

LEMMA 3. *Suppose γ: $\Omega \to H$ is holomorphic. For every $x \in H$ define a holomorphic function $\widehat{x}$: $\Omega \to \mathbb{C}$ by*

$$\widehat{x}(z) = \langle \gamma(z), x \rangle, \qquad z \in \Omega.$$

Then the set

$$H_\gamma = \{\widehat{x}\colon\ x \in H\}$$

can be made into a Banach space such that point evaluations are uniformly bounded linear functionals on compact subsets of Ω.

Proof. Let

$$H_0 = \mathrm{Span}\{\gamma(z)\colon\ z \in \Omega\}.$$

Then

$$H_\gamma = \{\widehat{x}\colon\ x \in H_0\},$$

and the mapping $x \mapsto \widehat{x}$ from H_0 onto H_γ is one-to-one. Define a norm on H_γ by $\|\widehat{x}\| = \|x\|$. Then H_γ becomes a Banach space with point evaluations being uniformly bounded linear functionals on compact subsets of Ω. $\square$

Suppose H is a Hilbert space and $\gamma_1, \dots, \gamma_n$ are holomorphic functions from Ω into H. We say that $\gamma_1, \dots, \gamma_n$ span H if

$$H = \operatorname{Span}\{\gamma_k(z)\colon\ 1 \le k \le n, z \in \Omega\}.$$

LEMMA 4. *Suppose H is a Hilbert space. If γ_1 and γ_2 are two holomorphic functions from Ω into H, and if they span H, then there exists a holomorphic function φ from Ω into $\mathbb{C}$ such that $\gamma = \varphi\gamma_1 + \gamma_2$ also spans H.*

Proof. Clearly, we can assume $\gamma_2 \neq 0$. Let

$$X = \{\widehat{x}\colon\ x \in H\},$$

where

$$\widehat{x}(z) = \langle \gamma_2(z), x\rangle, \qquad z \in \Omega.$$

By Lemma 3, the space X can be made into a Banach space with point evaluations on compact subsets of Ω being uniformly bounded linear functionals on X. Applying Lemma 2, we obtain a uniqueness sequence $\{a_n\}$ for X without an accumulation point in Ω. By a classical theorem of Weierstrass (see Theorem 15.11 in [7], for example), there exists a holomorphic function $\varphi\colon\ \Omega \to \mathbb{C}$ such that φ vanishes exactly on $\{a_n\}$.

Now consider

$$\gamma(z) = \varphi(z)\gamma_1(z) + \gamma_2(z), \qquad z \in \Omega.$$

If x in H is orthogonal to $\gamma(z)$ for every $z \in \Omega$, then

$$\varphi(z)\langle\gamma_1(z), x\rangle + \langle\gamma_2(z), x\rangle = 0$$

for every $z \in \Omega$. It follows that $\langle\gamma_2(z), x\rangle = 0$ whenever z is a zero of φ. Since the zero set of φ is a uniqueness set for X, we conclude that $\langle\gamma_2(z), x\rangle = 0$ for every $z \in \Omega$, and therefore $\langle\gamma_1(z), x\rangle = 0$ for every $z \in \Omega$. Since γ_1 and γ_2 span H, we must have $x = 0$, and hence γ also spans H. $\square$

THEOREM 5. *Let H be a Hilbert space and $\gamma_1, \dots, \gamma_n$ be holomorphic functions from Ω into H which span H. Then there exist holomorphic functions $\varphi_1, \dots, \varphi_n$ from Ω into $\mathbb{C}$ such that the function*

$$\gamma(z) = \varphi_1(z)\gamma_1(z) + \cdots + \varphi_n(z)\gamma_n(z), \qquad z \in \Omega,$$

also spans H.

Proof. We prove this by induction on n. The case $n = 1$ is trivial and the case $n = 2$ is just Lemma 4. Now assume the result is true for some positive integer n and assume that $\gamma_1, \dots, \gamma_n, \gamma_{n+1}$ are holomorphic functions from Ω into H which span H. Let H_1 be the closed linear span in H of the set

$$\{\gamma_n(z)\colon z \in \Omega\} \cup \{\gamma_{n+1}(z)\colon z \in \Omega\}.$$

By Lemma 4, there exists a holomorphic function h from Ω into H_1 such that the function $h\gamma_n + \gamma_{n+1}$ spans H_1. It follows that H is spanned by the n functions

$$\gamma_1, \dots, \gamma_{n-1}, h\gamma_n + \gamma_{n+1}.$$

By the induction hypothesis, there exist holomorphic functions $\varphi_1, \dots, \varphi_n$ from Ω into H such that the function

$$\gamma = \varphi_1\gamma_1 + \cdots + \varphi_{n-1}\gamma_{n-1} + \varphi_n(h\gamma_n + \gamma_{n+1})$$

spans H. □

COROLLARY 6. *Suppose n is a positive integer and T is an operator in $B_n(\Omega)$. Then the Hermitian bundle E_T admits a spanning holomorphic cross-section.*

Proof. By Lemma 1, the Hermitian bundle E_T admits a global holomorphic frame $\gamma_1, \dots, \gamma_n$. By the condition (3) in the definition of $B_n(\Omega)$, the functions $\gamma_1, \dots, \gamma_n$ span H. The desired result now clearly follows from the theorem above. □

Observe that if $\{a_n\}$ is a uniqueness set for a space X and $\{b_n\}$ is a uniqueness set for another space Y, then the union $\{a_n\} \cup \{b_n\}$ (counting multiplicity) is a uniqueness set for both X and Y. Therefore, by modifying the proofs of Lemma 4 and Theorem 5 only slightly, we can generalize Theorem 5 to the case of finitely many Hilbert spaces as follows.

THEOREM 7. *Suppose $H_1, \dots, H_m$ are Hilbert spaces and n is a positive integer. If for every $1 \le k \le m$ the functions $\gamma_{k1}, \dots, \gamma_{kn}$ are holomorphic from Ω into H_k and span H_k, then there exist holomorphic functions $\varphi_1, \dots, \varphi_n$ from Ω into $\mathbb{C}$ such that the function*

$$\gamma_k = \varphi_1\gamma_{k1} + \cdots + \varphi_n\gamma_{kn}$$

also spans H_k for every $1 \le k \le m$.

The main point here is that we can choose one set of coefficient functions $\{\varphi_1, \dots, \varphi_n\}$ which works for all the spaces H_k simultaniously. Later on we will need this slightly stronger process of generating spanning holomorphic cross-sections in certain Hermitian bundles.

THEOREM 8. *Suppose n is any positive integer and T is an operator in $B_n(\Omega)$. Then T is unitarily equivalent to the adjoint of multiplication by z on some reproducing Hilbert space of holomorphic functions in $\overline{\Omega}$.*

Proof. Fix a spanning holomorphic cross-section γ for the Hermitian bundle E_T. Let

$$\widehat{H} = \{\widehat{x}\colon x \in H\},$$

where H is the Hilbert space on which the operator T acts and

$$\widehat{x}(z) = \langle x, \gamma(\bar{z})\rangle, \qquad z \in \overline{\Omega}.$$

Note that $\widehat{x}$ here is different from the $\widehat{x}$ used earlier in Lemmas 3 and 4.

It is clear that $\widehat{H}$ is a complex vector space consisting of holomorphic functions in $\overline{\Omega}$. Since γ is spanning, the mapping $U\colon H \to \widehat{H}$ defined by $U(x) = \widehat{x}$ is linear and one-to-one. Now define an inner product $\langle\ ,\ \rangle_*$ on $\widehat{H}$ by

$$\langle \widehat{x}, \widehat{y}\rangle_* = \langle x, y\rangle, \qquad x, y \in H,$$

where $\langle\ ,\ \rangle$ is the inner product on the Hilbert space H. Then $\widehat{H}$ becomes a Hilbert space with point evaluations on compact subsets of $\overline{\Omega}$ being unformly bounded linear functionals, and the mapping $U\colon H \to \widehat{H}$ is a unitary transformation.

Let $S = UTU^*$. We show that S^*, the adjoint of S, is the operator of multiplication by z on $\widehat{H}$. Fix $x \in H$. Then for any $z \in \overline{\Omega}$ we have

$$\begin{aligned} S^*(\widehat{x})(z) &= UT^*U^*(\widehat{x})(z) = UT^*(x)(z) \\ &= \widehat{(T^*x)}(z) = \langle T^*x, \gamma(\bar{z})\rangle \\ &= \langle x, T\gamma(\bar{z})\rangle = \langle x, \bar{z}\gamma(\bar{z})\rangle \\ &= z\langle x, \gamma(\bar{z})\rangle = z\widehat{x}(z). \end{aligned}$$

This shows that T is unitarily equivalent to the adjoint of multiplication by z on the space $\widehat{H}$. $\square$

From the above representation it is clear when an operator in $B_n(\Omega)$ is reducible. More specifically, an operator $T \in B_n(\Omega)$ is reducible if and only if $T = T_1 \oplus T_2$, where $T_1 \in B_{n_1}(\Omega)$ and $T_2 \in B_{n_2}(\Omega)$ with $n_1 + n_2 = n$. But this follows easily from the definition of $B_n(\Omega)$ anyway. In particular, every operator in $B_1(\Omega)$ is irreducible.

If we represent $T \in B_n(\Omega)$ as the adjoint of multiplication by z on a reproducing Hilbert space H, then T is reducible if and only if there exist mutually orthogonal (non-trivial) z-invariant subspaces in H.

3. Reproducing kernels

Our analysis of operators in $B_n(\Omega)$ depends very much on the notion of reproducing kernels. Thus in this section we gather the necessary concepts and results from the general theory of reproducing kernels that we shall use later on.

We are primarily interested in holomorphic reproducing kernels. Thus we consider Hilbert spaces H consisting of holomorphic functions in a domain Ω. If point evaluation at every point in Ω is a bounded linear functional on H, then H has a reproducing kernel $K(z, w)$. The following result of Moore (see [6] or [8]) is the basis for the general theory of reproducing kernels.

THEOREM 9. *A function $K\colon \Omega \times \Omega \to \mathbb{C}$ is the reproducing kernel of a Hilbert space if and only if*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j}\, K(z_j, z_i) \geq 0$$

for every positive integer n, every collection $\{z_1, \ldots, z_n\}$ in Ω, and every sequence $\{c_1, \ldots, c_n\}$ in $\mathbb{C}$.

Note that if K is the reproducing kernel of a Hilbert space H, then the double sum above is equal to the norm of the vector

$$x = \sum_{k=1}^{n} c_k K(\cdot, z_k)$$

in H and therefore is non negative. Conversely, if a function K satisfies the positivity condition in the theorem above, then one considers the vector space H_0 consisting of all functions of the form

$$f(z) = \sum_{k=1}^{n} c_k K(z, z_k), \qquad z \in \Omega,$$

where $n \geq 1$ and $\{z_1, \ldots, z_n\} \subset \Omega$. The positivity condition on K enables us to define an inner product on H_0 as follows:

$$\left\langle \sum_{k=1}^{n} a_k K(\cdot, z_k), \sum_{k=1}^{m} b_k K(\cdot, w_k) \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \overline{b_j}\, K(w_j, z_i).$$

It is then easy to check that K is the reproducing kernel of the Hilbert space completion of H_0.

As an application of the above theorem we give the following example of generating reproducing kernels from a given Hilbert space.

PROPOSITION 10. *Suppose H is a Hilbert space and $\gamma\colon \Omega \to H$ is a holomorphic function. Then the function*

$$K(z, w) = \langle \gamma(z), \gamma(w) \rangle$$

is the reproducing kernel of a Hilbert space of holomorphic functions in Ω.

Proof. The positivity condition is straightforward in this case. We omit the simple details. □

Note that the Hilbert space $\widehat{H}$ in the proof of Theorem 8 possesses a reproducing kernel. In fact, the reproducing kernel is given by

$$K(z, w) = \langle \gamma(\bar{w}), \gamma(\bar{z}) \rangle, \qquad z, w \in \overline{\Omega}.$$

To check this, fix $x \in H$ and $w \in \overline{\Omega}$, and write

$$K_w(z) = K(z, w) = \widehat{\gamma(\bar{w})}(z), \qquad z \in \overline{\Omega}.$$

Then

$$\langle \widehat{x}, K_w \rangle_* = \langle x, \gamma(\bar{w}) \rangle = \widehat{x}(w).$$

Thus the kernel function K above does have the desired reproducing property.

In general, if $T \in B_n(\Omega)$ and if γ is a holomorphic cross-section of E_T, then it is often more desirable for us to consider the kernel

$$K(z, w) = \langle \gamma(\bar{w}), \gamma(\bar{z}) \rangle$$

on $\overline{\Omega}$, instead of the kernel

$$K(z, w) = \langle \gamma(z), \gamma(w) \rangle$$

on Ω.

Definition 11. Suppose K_1 and K_2 are two reproducing kernels on Ω. If there exists a constant $C > 0$ such that $CK_2 - K_1$ is still a reproducing kernel, then we write $K_1 \prec K_2$.

By Theorem 9, two reproducing kernels K_1 and K_2 satisfy $K_1 \prec K_2$ if and only if there exists a constant $C > 0$ such that

$$\left(K_1(z_i, z_j)\right)_{n \times n} \leq C \left(K_2(z_i, z_j)\right)_{n \times n}$$

as matrices for all $n \geq 1$ and all $\{z_1, \ldots, z_n\}$ in Ω.

Definition 12. Suppose K_1 and K_2 are two reproducing kernels on Ω. If $K_1 \prec K_2$ and $K_2 \prec K_1$, then we write $K_1 \sim K_2$.

The following result provides us with a class of reproducing kernels dominated by a given one.

PROPOSITION 13. *Suppose K is the reproducing kernel of a Hilbert space H of holomorphic functions in Ω. Let φ be any holomophic function in Ω. Then*

$$\varphi(z)K(z,w)\overline{\varphi(w)} \prec K(z,w)$$

if and only if φ is a bounded (pointwise) multiplier on H.

Proof. First assume that the operator of multiplication by φ, denoted M_φ, is bounded on H. Consider the operator $T = M_\varphi^*$ on H. It is easy to see that

$$TK_z = \overline{\varphi(z)}\,K_z, \qquad z \in \Omega,$$

and hence

$$T\left(\sum_{k=1}^n c_k K_{z_k}\right) = \sum_{k=1}^n c_k\,\overline{\varphi(z_k)}\,K_{z_k},$$

for all $n \geq 1$, $\{z_1, \dots, z_n\} \subset \Omega$, and $\{c_1, \dots, c_n\} \subset \mathbb{C}$. Here we use K_λ to denote the function $K_\lambda(w) = K(w, \lambda)$, $w \in \Omega$. Applying $\| \, \|^2$ to both sides of the above identity, we obtain

$$\sum_{i=1}^n \sum_{j=1}^n c_i\,\overline{c_j}\,\overline{\varphi(z_i)}\,\varphi(z_j)K(z_j,z_i) \leq \|T\|^2 \sum_{i=1}^n \sum_{j=1}^n c_i\,\overline{c_j}\,K(z_j,z_i).$$

By Theorem 9 the function

$$\|T\|^2 K(z,w) - \varphi(z)K(z,w)\,\overline{\varphi(w)}$$

is a reproducing kernel, so that

$$\varphi(z)K(z,w)\,\overline{\varphi(w)} \prec K(z,w).$$

Conversely, if

$$\varphi(z)K(z,w)\,\overline{\varphi(w)} \prec K(z,w),$$

then there exists a constant $C > 0$ such that

$$CK(z,w) - \varphi(z)K(z,w)\,\overline{\varphi(w)}$$

is a reproducing kernel. By Theorem 9, we have

$$\sum_{i=1}^n \sum_{j=1}^n c_i\,\overline{c_j}\,\overline{\varphi(z_i)}\varphi(z_j)K(z_j,z_i) \leq C \sum_{i=1}^n \sum_{j=1}^n c_i\,\overline{c_j}\,K(z_j,z_i)$$

for all $n \geq 1$, $\{z_1, \dots, z_n\} \subset \Omega$, and $\{c_1, \dots, c_n\} \subset \mathbb{C}$. It follows that

$$T\left(\sum_{k=1}^n c_k K_{z_k}\right) = \sum_{k=1}^n c_k\,\overline{\varphi(z_k)}\,K_{z_k}$$

extends to bounded operator on H. A simple calculation then shows that T^* must the operator of multiplication by φ. $\square$

In the special case $\Omega = \mathbb{D}$ (the open unit disk in $\mathbb{C}$) and $\|\varphi\|_\infty \leq 1$, the above result can be found in [3].

Recall that to every holomorphic cross-section γ in E_T, $T \in B_n(\Omega)$, there corresponds a reproducing kernel

$$K_\gamma(z, w) = \langle \gamma(z), \gamma(w) \rangle, \qquad z, w \in \Omega.$$

For two holomorphic cross-sections γ_1 and γ_2 in E_{T_1} and E_{T_2}, respectively, we write $\gamma_1 \prec \gamma_2$ if $K_{\gamma_1} \prec K_{\gamma_2}$. Similarly, we write $\gamma_1 \sim \gamma_2$ if $K_{\gamma_1} \sim K_{\gamma_2}$.

PROPOSITION 14. *Suppose γ is a holomorphic function from Ω into a Hilbert space. Let P be a bounded positive operator on H. Define*

$$K_1(z, w) = \langle \gamma(z), \gamma(w) \rangle, \qquad z, w \in \Omega,$$

and

$$K_2(z, w) = \langle P\gamma(z), \gamma(w) \rangle, \qquad z, w \in \Omega.$$

Then $K_2 \prec K_1$. If P is also invertible, then $K_1 \sim K_2$.

Proof. We already know that K_1 is a reproducing kernel. It is easy to see that K_2 is also a reproducing kernel. In fact, for any sequence $\{c_1, \ldots, c_n\}$ in $\mathbb{C}$ and $\{z_1, \ldots, z_n\}$ in Ω we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} K_2(z_i, z_j) = \left\langle P \sum_{i=1}^{n} c_i \gamma(z_i), \sum_{j=1}^{n} c_j \gamma(z_j) \right\rangle \geq 0.$$

To show that $K_2 \prec K_1$, let $Q_1 = \|P\|I - P$. Then Q_1 is a positive operator and so

$$K(z, w) = \|P\|K_1(z, w) - K_2(z, w) = \langle Q_1\gamma(z), \gamma(w) \rangle$$

is a reproducing kernel.

If P is also invertible, then by the spectral theorem there exists a number $C > 0$ such that $Q_2 = CP - I$ is a positive operator. This implies that

$$K(z, w) = \langle Q_2\gamma(z), \gamma(w) \rangle = CK_2(z, w) - K_1(z, w)$$

is a reproducing kernel. Thus $K_1 \prec K_2$ and hence $K_1 \sim K_2$. $\square$

We will see later that if γ spans the whole space H, then the converse of Proposition 14 is also true.

4. Unitary equivalence

In this section we give another proof of the main theorem in [1], which states that two operators in $B_n(\Omega)$ are unitarily equivalent if and only if their associated Hermitian bundles are equivalent. Our proof here does not use as much complex geometry, and it can be modified to obtain results about the similarity problem for operators in $B_n(\Omega)$.

Let E_1 and E_2 be two Hermitian holomorphic vector bundles over Ω. First recall that E_1 and E_2 are said to be equivalent if there exists a bundle map F from E_1 onto E_2 such that for every $z \in \Omega$ the function F maps the fibre of E_1 at z unitarily onto the fibre of E_2 at z.

Also recall that if T_1 and T_2 are bounded linear operators on Hilbert spaces H_1 and H_2, respectively, then T_1 and T_2 are unitarily equivalent if there exists a unitary transformation $U: H_1 \to H_2$ such that $UT_1 = T_2U$, or $T_1 = U^*T_2U$.

THEOREM 15. *Suppose S and T are operators in $B_n(\Omega)$ for some $n \geq 1$. Then the following conditions are equivalent*:

(1) *The operators S and T are unitarily equivalent.*
(2) *The Hermitian bundles E_S and E_T are equivalent.*
(3) *There exist spanning holomorphic cross-sections γ_S and γ_T in E_S and E_T, respectively, such that $\|\gamma_S(z)\| = \|\gamma_T(z)\|$ for all $z \in \Omega$.*

Proof. First assume S and T are unitarily equivalent. Without loss of generality we may assume S and T both act on the same Hilbert space H. Let $U: H \to H$ be a unitary transformation with $US = TU$. If $Sx = \lambda x$ for some $\lambda \in \Omega$ and $x \in H$, then

$$T(Ux) = U(Sx) = \lambda Ux.$$

Thus U maps the fibre of E_S at λ unitarily onto the fibre of E_T at λ, and so E_S and E_T are equivalent as Hermitian bundles. This proves that (1) implies (2).

Next assume that E_S and E_T are equivalent as Hermitian bundles. Then there exists a bundle map F from E_S onto E_T such that F maps $\ker(\lambda I - S)$ unitarily onto $\ker(\lambda I - T)$ for every $\lambda \in \Omega$. Now fix a global holomorphic frame $\gamma_1, \dots, \gamma_n$ for E_S. It is clear that the functions $F\gamma_1, \dots, F\gamma_n$ form a global holomorphic frame for E_T. By Theorem 7, there exist holomorphic functions $\varphi_1, \dots, \varphi_n$ from Ω into $\mathbb{C}$ such that the functions

$$\gamma_S(z) = \varphi_1(z)\gamma_1(z) + \cdots + \varphi_n(z)\gamma_n(z)$$

and

$$\gamma_T(z) = \varphi_1(z)F\gamma_1(z) + \cdots + \varphi_n(z)F\gamma_n(z)$$

are spanning holomorphic cross-sections for E_S and E_T, respectively. From the linearity of F on every fibre we obtain $\gamma_T(z) = F\gamma_S(z)$, $z \in \Omega$. Since F is also isometric on every fibre, we have $\|\gamma_S(z)\| = \|\gamma_T(z)\|$ for every $z \in \Omega$. This proves that (2) implies (3).

Finally assume that γ_S and γ_T are spanning holomorphic cross-sections in E_S and E_T, respectively, such that $\|\gamma_S(z)\| = \|\gamma_T(z)\|$ for all $z \in \Omega$. Define a function

$$K_S\colon \Omega \times \Omega \to \mathbb{C}$$

by

$$K_S(z, w) = \langle \gamma_S(z), \gamma_S(w) \rangle, \qquad z, w \in \Omega.$$

Define a function K_T similarly. Then $K_S(z, w)$ and $K_T(z, w)$ are both holomorphic in z and anti-holomorphic in w, and $K_S(z, z) = K_T(z, z)$ for all $z \in \Omega$. By a well-known uniqueness theorem in the theory of several complex variables (see Exercise 3 on page 326 of [5], for example), we must have $K_S(z, w) = K_T(z, w)$ for all z and w in Ω. It follows that the mapping U defined by

$$U\left(\sum_{k=1}^{n} c_k \gamma_S(z_k)\right) = \sum_{k=1}^{n} c_k \gamma_T(z_k)$$

extends to a unitary transformation on H. Since γ_S spans H and

$$TU\gamma_S(z) = T\gamma_T(z) = z\gamma_T(z) = US\gamma_S(z)$$

for every $z \in \Omega$, we conclude that $TU = US$, which proves that (3) implies (1). □

5. Similarity in $B_n(\Omega)$

In this section we consider the problem of when two operators in $B_n(\Omega)$ are similar. We will also consider the companion problem of when two operators in $B_n(\Omega)$ are quasi-similar.

Suppose H_1 and H_2 are Hilbert spaces and suppose T_1 and T_2 are bounded linear operators on H_1 and H_2, respectively. Then T_1 is said to be similar to T_2 if there exists a bounded invertible operator $A\colon H_1 \to H_2$ such that $AT_1 = T_2 A$. When T_1 and T_2 are similar, we then write $T_1 \sim T_2$. Also recall that T_1 is quasi-similar to T_2 if there exist bounded linear operators $A\colon H_1 \to H_2$ and $B\colon H_2 \to H_1$, both one-to-one and having dense range, such that $AT_1 = T_2 A$ and $T_1 B = BT_2$.

THEOREM 16. *Suppose S and T are operators in $B_n(\Omega)$ for some $n \geq 1$. Then $S \sim T$ if and only if there exist spanning holomorphic cross-sections γ_S and γ_T in E_S and E_T, respectively, such that $\gamma_S \sim \gamma_T$.*

Proof. Suppose S and T are similar. Without loss of generality we may assume that S and T act on the same Hilbert space H. Thus there exists an invertible operator A on H such that $AS = TA$. This intertwining relation of S and T then implies that A maps $\ker(\lambda I - S)$ into $\ker(\lambda I - T)$; this is also onto by a dimension count. Choose a spanning holomorphic cross-section γ_S in E_S. Then it is easy to see that $\gamma_T = A\gamma_S$ is a spanning holomorphic cross-section of E_T. Since

$$\langle \gamma_T(z), \gamma_T(w)\rangle = \langle A^*A\gamma_S(z), \gamma_S(w)\rangle$$

and A^*A is an invertible positive operator, we have $\gamma_S \sim \gamma_T$ in view of Proposition 14.

On the other hand, if there exist spanning holomorphic cross-sections γ_S and γ_T in E_S and E_T, respectively, such that $\gamma_S \sim \gamma_T$. Then there exists a constant $C > 0$ such that

$$C^{-1}\left\|\sum_{k=1}^{n} c_k\gamma_T(z_k)\right\| \le \left\|\sum_{k=1}^{n} c_k\gamma_S(z_k)\right\| \le C\left\|\sum_{k=1}^{n} c_k\gamma_T(z_k)\right\|$$

for all $n \ge 1$, $\{c_1, \dots, c_n\}$ in $\mathbb{C}$, and $\{z_1, \dots, z_n\}$ in Ω. It follows that the operator A defined by

$$A\left(\sum_{k=1}^{n} c_k\gamma_T(z_k)\right) = \sum_{k=1}^{n} c_k\gamma_S(z_k)$$

extends to a bounded invertible linear operator on H. Since

$$AT\gamma_T(z) = zA\gamma_T(z) = z\gamma_S(z) = S\gamma_S(z) = SA\gamma_T(z)$$

for every $z \in \Omega$, and since γ_T spans H, we conclude that $AT = SA$, and hence S and T are similar. □

THEOREM 17. *Suppose S and T are operators in $B_n(\Omega)$. Then S and T are quasi-similar if and only if there exist spanning holomorphic cross-sections γ_1 and γ_2 in E_S, and σ_1 and σ_2 in E_T, such that $\gamma_1 \prec \sigma_1$ and $\sigma_2 \prec \gamma_2$.*

Proof. The proof is similar to that of Theorem 16. We leave the details to the interested reader. □

6. The commutant of an operator in $B_n(\Omega)$

Let T be a bounded linear operator on a Hilbert space H. Then the commutant of T, denoted $(T)'$, is the algebra of all bounded linear operators S on H such that $ST = TS$.

THEOREM 18. *Let T be an operator in $B_n(\Omega)$ and let γ_0 be a spanning holomorphic cross-section of E_T. Then the commutant of T, $(T)'$, can be identified with the set of all holomorphic cross-sections γ in E_T with the property $\gamma \prec \gamma_0$.*

Proof. First assume A is a bounded linear operator on H which commutes with T. Define $\gamma\colon \Omega \to H$ by

$$\gamma(z) = A\gamma_0(z), \qquad z \in \Omega.$$

Then γ is a holomorphic cross-section in E_T. In fact, for every $z \in \Omega$,

$$T\gamma(z) = TA\gamma_0(z) = AT\gamma_0(z) = zA\gamma_0(z) = z\gamma(z).$$

Thus $\gamma(z)$ lies in the fibre over z in the bundle E_T. Let $C = \|A^*A\|$. Then according to Proposition 14 the function $CK_{\gamma_0} - K_\gamma$ is still a reproducing kernel. Thus $\gamma \prec \gamma_0$.

Next assume that γ is a holomorphic cross-section in E_T with $\gamma \prec \gamma_0$. Then the operator A defined by

$$A\left(\sum_{k=1}^{n} c_k\gamma_0(z_k)\right) = \sum_{k=1}^{n} c_k\gamma(z_k)$$

extends to a bounded linear operator on H. Furthermore, for every $z \in \Omega$,

$$AT\gamma_0(z) = zA\gamma_0(z) = z\gamma(z) = T\gamma(z) = TA\gamma_0(z).$$

Since γ_0 is spanning, we conclude that $AT = TA$, and so A is in the commutant algebra of T. □

Note that in the case $n = 1$, if we represent T as the adjoint of multiplication by z on a certain Hilbert space H of holomorphic functions in $\overline{\Omega}$, then $(T)'$ consists exactly of those multiplication operators M_φ, where φ is a holomorphic mulitiplier of H. By the closed graph theorem, such a φ is necessarily bounded on Ω. Therefore, the commutant of every operator in $B_1(\Omega)$ is isomorphic to a weakly closed subalgebra of $H^\infty(\overline{\Omega})$.

Also, Theorem 18 can be generalized as follows. Suppose S and T are operators in $B_n(\Omega)$. An operator A on H is said to intertwine S and T if $AS = TA$. Fix a spanning holomorphic cross-section γ_S for E_S. Then an operator A on H intertwines S and T if and only if there exists a holomorphic cross-section γ_T in E_T such that $\gamma_T = A\gamma_S \prec \gamma_S$.

7. Pull-back bundles of a Grassmannian

For a separable Hilbert space H and a positive integer n let $\mathrm{Gr}(n, H)$ denote the Grasmann manifold consisting of all n-dimensional subspaces of H. We will be interested in functions from a domain in $\mathbb{C}$ into $\mathrm{Gr}(n, H)$.

Let Ω be a domain in $\mathbb{C}$ and let $f\colon \Omega \to \mathrm{Gr}(n, H)$ be a function. We say that f is holomorphic if for every point $z_0 \in \Omega$ there exists a neighborhood V of z_0 in Ω and

n holomorphic functions $\gamma_1, \ldots, \gamma_n$ from V into H such that

$$f(z) = \mathrm{Span}\{\gamma_1(z), \ldots, \gamma_n(z)\}$$

for all $z \in V$.

Suppose $f\colon \Omega \to \mathrm{Gr}(n, H)$ is a holomorphic function. Then f induces a natural Hermitian holomorphic bundle E_f as follows:

$$E_f = \{(x, z) \in H \times \Omega\colon x \in f(z)\}.$$

The associated projection $\pi\colon E_f \to \Omega$ is of course given by $\pi(x, z) = z$. The bundle E_f will be called the pull-back bundle of the Grassmannian $\mathrm{Gr}(n, H)$ induced by f. It is clear that the fibre of E_f at z is just $f(z)$.

For every operator $T \in B_n(\Omega)$ the associated bundle E_T is a pull-back of the Grassmannian $\mathrm{Gr}(n, H)$. In fact, if we define

$$f\colon \Omega \to \mathrm{Gr}(n, H)$$

by

$$f(z) = \ker(zI - T), \qquad z \in \Omega,$$

then f is holomorphic and $E_T = E_f$.

Let X be a subspace of H and A be an operator on H. In the definition below we will use AX to denote the set $\{Ax\colon x \in X\}$.

Definition 19. Let f and g be two holomorphic functions from Ω into $\mathrm{Gr}(n, H)$. We say that f and g are congruent if there exists a unitary operator U on H such that $f(z) = Ug(z)$ for every $z \in \Omega$. And we say that f and g are similar if there exists a bounded invertible operator A on H such that $f(z) = Ag(z)$ for every $z \in \Omega$.

THEOREM 20. *Suppose Ω is a domain in $\mathbb{C}$ and*

$$f, g\colon \Omega \to \mathrm{Gr}(n, H)$$

are holomorphic functions such that

$$H = \mathrm{Span}\{f(z)\colon z \in \Omega\} = \mathrm{Span}\{g(z)\colon z \in \Omega\}.$$

Then the following conditions are equivalent:

(1) *f and g are congruent.*
(2) *E_f and E_g are equivalent.*
(3) *There exist spanning holomorphic cross-sections γ_f and γ_g in E_f and E_g, respectively, such that $\|\gamma_f(z)\| = \|\gamma_g(z)\|$ for all $z \in \Omega$.*

Proof. The proof is similar to that of Theorem 15. We omit the details. □

The equivalence of (1) and (2) above is the rigidity theorem in [1] in the special case $\Omega \subset \mathbb{C}$. Our method here only works for $\Omega \subset \mathbb{C}$.

One consequence of the (proof of the) theorem above is that for pull-back bundles induced by maps into a Grassmannian, local equivalence of E_f and E_g implies (global) equivalence. As was remarked in [1], this is not true for general Hermitian holomorphic bundles.

THEOREM 21. *Suppose Ω is a plane domain and*

$$f, g\colon \Omega \to \mathrm{Gr}(n, H)$$

are holomorphic with

$$H = \mathrm{Span}\{f(z)\colon z \in \Omega\} = \mathrm{Span}\{g(z)\colon z \in \Omega\}.$$

Then f and g are similar if and only if there exist spanning holomorphic cross-sections γ_f and γ_g in E_f and E_g, respectively, such that $\gamma_S \sim \gamma_g$.

Proof. Again the proof is similar to that of Theorem 16. We omit the details. □

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