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Enumeration 1-Genericity in the Local Enumeration Degrees

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Abstract We discuss a notion of forcing that characterizes enumeration 1-genericity, and we investigate the immunity, lowness, and quasiminimality properties of enumeration 1-generic sets and their degrees. We construct an enumeration operator Δ such that, for any A, the set Δ^A is enumeration 1-generic and has the same jump complexity as A. We deduce from this and other recent results from the literature that not only does every degree a bound an enumeration 1-generic degree b such that $a' = b'$, but also that, if a is nonzero, then we can find such b satisfying $0_e < b < a$. We conclude by proving the existence of both a nonzero low and a properly Σ^0_2 nonsplittable enumeration 1-generic degree, hence proving that the class of 1-generic degrees is properly subsumed by the class of enumeration 1-generic degrees.

1 Introduction

Enumeration 1-genericity, a form of 1-genericity appropriate for positive reducibilities, was introduced by Badillo and Harris [4] and used as a tool to show that there exists a properly Π^0_2 degree b such that any $x \leq b$ contains only Π^0_2 sets. In [4] various questions about the basic properties of enumeration 1-genericity in the enumeration and singleton degrees, and also its relationship with 1-genericity, were investigated. We continue this investigation in Section 3, where in particular we look at immunity and lowness properties of enumeration 1-generic sets. We also address the question of the distribution of the class of enumeration 1-generic degrees and show that it resembles to some extent the distribution of the class of 1-generic degrees, not only over the Π^0_2 degrees, but also globally with respect to the class of total degrees. In Section 4 we study jump inversion of the enumeration 1-generic degrees. In the context of the Σ^0_2 degrees, this work can be seen as an extension of results

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by McEvoy [15] to the effect that every Π^0_2 degree $b \geq 0'_e$ is the jump of a Π^0_1 degree. Indeed, it follows from the results in Section 4 in conjunction with a result from Badillo, Bianchini, Ganchev, Kent, and Sorbi [3] that every Π^0_2 degree $b \geq 0'_e$ is also the jump of a nonzero Σ^0_2 enumeration 1-generic (and hence quasiminimal) degree. Furthermore, we will see that our results also throw light on the phenomenon of Σ^0_2 highness introduced in [15]. Section 5, which concludes the present article, is motivated by the question of how enumeration 1-genericity and 1-genericity may be separated within the enumeration degrees. We approach this question locally, bearing in mind that every 1-generic degree is splittable, by showing the existence of both low and properly Σ^0_2 nonsplittable enumeration 1-generic degrees.

2 Preliminaries

We assume $\{W_e\}_{e\in\omega}$ to be an acceptable listing of computably enumerable (c.e.) sets with associated c.e. approximations $\{W_{e,s}\}_{s\in\omega}$, and $\{D_n\}_{n\in\omega}$ to be the computable listing of finite sets, where D_n denotes the finite set with canonical index n. We also assume $\langle x, y\rangle$ to be a standard computable pairing function over the integers. We use $X^{[e]}$ to denote the set $\{\langle e, x\rangle \mid \langle e, x\rangle \in X\}$ and χ_Y to denote the characteristic function of Y. We say that the set Y is *characteristic* if $Y = X \oplus \overline{X}$ for some set X, and we note that $X \oplus \overline{X} \equiv_e \chi_X$. We use α, β, σ, and so on to denote finite binary strings (i.e., members of $2^{<\omega}$). We denote by $|\alpha|$ the length of α, so that $|\alpha| = \mu x[x \notin \operatorname{dom}\alpha]$. Also, $\alpha \subseteq \beta$ denotes that α is an initial segment of β (similarly, we use $\alpha \subseteq f$ if $f \in 2^\omega$).

A set A is defined to be *enumeration reducible* to a set B ($A \leq_e B$) if there exists an effective procedure that, given *any* enumeration of B, enumerates A. More formally (see Friedberg and Rogers [8]), $A \leq_e B$ if and only if there exists a c.e. set W such that for all $x \in \omega$,

$$x \in A \qquad \text{iff} \qquad \exists n\big[\langle x, n\rangle \in W \;\&\; D_n \subseteq B\big]. \tag{2.1}$$

We define $\{\Phi_e\}_{e\in\omega}$ to be the effective listing of enumeration operators such that for any set X,

$$\Phi^X_e = \big\{x \;\big|\; \exists n\big[\langle x, n\rangle \in W_e \;\&\; D_n \subseteq X\big]\big\}.$$

Also, for any e, we use the notation $\Phi^X_{e,s}$ to define the finite approximation to Φ^X_e, derived from $W_{e,s}$. For simplicity, we allow a certain amount of ambiguity in our notation, by sometimes equating W_e with the operator Φ_e, and in the case of finite sets, using the letter D or similar to denote both a finite set and its index in the listing of finite sets specified above.

We use the notation x for the equivalence classes of $\leq_e$ or, in other words, the enumeration degrees, whereas $\deg_e(X)$ is the notation for the $\leq_e$-degree of X. Also, 0_e is the degree of the c.e. sets, $\mathcal{D}_e$ denotes the structure of enumeration degrees, and $\mathcal{D}_e(\leq x)$ denotes the substructure of $\mathcal{D}_e$ over the class of degrees $\{y \mid y \leq x\}$ (we say that such a class is a *prime ideal* of $\mathcal{D}_e$). We remind the reader that $\mathcal{D}_e$ and the substructures of the form $\mathcal{D}_e(\leq x)$ are upper semilattices.

We assume the reader to be conversant with Turing ($\leq_T$) and other basic reducibilities for which we use similar notation to the above. $\mathcal{K}$ denotes the standard halting set for Turing machines, whereas the *enumeration semihalting set* relative to X is defined to be the set $K_X = \{x \mid x \in \Phi^X_x\}$ and the *enumeration jump* of X is defined to be the set $J_X = K_X \oplus \overline{K_X}$. The jump of enumeration degree x is written x',

$0'_e$ denotes $\deg_e(J_\emptyset)$, and $0''_e$ denotes $\deg_e(J_\emptyset^{(2)})$. We say that x is *low* if $x' = 0'_e$, and *high* if $x' = 0''_e$. Using the notation specified above, $\mathcal{D}_e(\leq 0'_e)$ denotes the upper semilattice of enumeration degrees comprising precisely the class of Σ^0_2 enumeration degrees. Note that we refer to the latter as the *local* structure of the enumeration degrees, as opposed to the *global* structure $\mathcal{D}_e$.

We denote by ι the canonical embedding of the Turing degrees into the enumeration degrees induced by the map $X \mapsto X \oplus \overline{X}$. We note that ι preserves join and jump.

Definition 2.1 (Harris [10], Lachlan and Shore [14]) A uniformly computable sequence of finite sets $\{X_s\}_{s\in\omega}$ is said to be a *good approximation* to the set X if

(1) $\forall s(\exists t \geq s)[X_t \subseteq X]$,
(2) $\forall x[x \in X \text{ iff } \exists t(\forall s \geq t)[X_s \subseteq X \Rightarrow x \in X_s]]$.

In this case, we say that X is *good approximable*. Moreover, if (2) is replaced by the condition $\forall x[x \in X \text{ iff } \exists t(\forall s \geq t)[x \in X_s]]$, then $\{X_s\}_{x\in\omega}$ is said to be a *good* Σ^0_2 *approximation*.

Lemma 2.2 (Jockusch [11]) *X is Σ^0_2 if and only if X has a good Σ^0_2 approximation.*

In other words, the sets underlying $\mathcal{D}_e(\leq 0')$ all have good Σ^0_2 approximations.

Lemma 2.3 (see [4]) *If a is a good enumeration degree, then, for every $A \in a$, $K_A \leq_e \overline{K_A}$. In other words, $J_A \equiv_e \overline{K_A}$.*

Lemma 2.4 (McEvoy and Cooper [16]) *An enumeration degree x is low if and only if x contains only Δ^0_2 sets.*

Definition 2.5 An enumeration degree x containing only Σ^0_2 (Π^0_2) sets is *properly* Σ^0_2 (Π^0_2) if it contains no Δ^0_2 sets, and is *downwards properly* Σ^0_2 if every $y \in \{z \mid 0_e < z \leq x\}$ is properly Σ^0_2. $x < 0'_e$ is *cuppable* if there exists $y < 0'_e$ such that $0'_e = x \cup y$, and *noncuppable* otherwise.

Given an arithmetical predicate Γ (e.g., $\Gamma \in \{\Delta^0_2, \Pi^0_2\}$), we sometimes use the shorthand $A \in \Gamma$ if A is a Γ set. Moreover, we say that an enumeration degree a is Γ if a contains a set $A \in \Gamma$.

Lemma 2.6 (Cooper, Sorbi, and Yi [6]) *If $0_e < x < 0'_e$ is Δ^0_2, then x is cuppable.*

Corollary 2.7 (see [6]) *Every noncuppable $0_e < x < 0'_e$ is downwards properly Σ^0_2.*

Notation Suppose that $\{X_s\}_{s\in\omega}$ and $\{\Phi_s\}_{s\in\omega}$ are approximations to some set X and enumeration operator Φ. We use the shorthand $\Phi^X[s] = \Phi_s^{X_s}$. For clarity, we also sometimes use the shorthand $X[s]$ instead of X_s.

3 Enumeration 1-Genericity

We now define the notion of enumeration 1-genericity. We discuss the basic properties of this notion and investigate its relationship with 1-genericity. We also delineate restrictions to the class of enumeration 1-generic degrees by exhibiting two properties inherent to it. We begin with a reminder of the definition of 1-genericity.

Definition 3.1 A set A is said to be 1-*generic* if for any c.e. set $W \subseteq 2^{<\omega}$ there exists $\alpha \subseteq \chi_A$ such that either $\alpha \in W$ or for all β such that $\alpha \subseteq \beta$, $\beta \notin W$.

Notation We use $\mathcal{F}$ to denote the class of finite subsets of ω. We will follow the convention that the letters D, E, F always denote members of $\mathcal{F}$ (although we often also specify that the set denoted is finite). In particular, $\exists E$ is shorthand for $\exists E \in \mathcal{F}$.

Definition 3.2 A set A is defined to be *enumeration* 1-*generic* if, for all c.e. sets $W \subseteq \mathcal{F}$, either there exists a finite set $D \subseteq A$ such that $D \in W$, or a finite set $E \subseteq \overline{A}$ such that, for every $D \in W$, $D \cap E \neq \emptyset$.

Forcing and enumeration 1-genericity We start by inspecting a notion of forcing which gives rise to the enumeration 1-generic sets. (We assume that the reader is familiar with forcing in arithmetic and refer to Shore [17] for an introduction on this topic.) Let $\mathcal{P}$ be a partial ordering. $V \subseteq \mathcal{P}$ is *open* if, for every $p, q \in \mathcal{P}$, if $p \leq q$ and $q \in V$, then $p \in V$. V is *dense* along $\mathcal{A} \subseteq \mathcal{P}$ if, for every $p \in \mathcal{A}$, there is a $q \leq p$, $q \in V$. $\mathcal{A}$ *meets* V if $\mathcal{A} \cap V \neq \emptyset$ and that $\mathcal{A} \subseteq \mathcal{P}$ is a *filter* if $\mathcal{A}$ is closed upwards with respect to the partial ordering and every two conditions in $\mathcal{A}$ have a common lower bound in $\mathcal{A}$. A filter $\mathcal{A}$ is *generic* if $\mathcal{A}$ meets every open set V, which is dense along $\mathcal{A}$.

The standard definition of a 1-generic set is derived from Cohen's notion of forcing on the partial ordering of finite binary strings $2^{<\omega}$ ordered by inclusion, by limiting the amount of genericity required. G is 1-generic if it is derived from a filter $\mathcal{G}$ on $2^{<\omega}$, which meets every Σ^0_1 open subset of $2^{<\omega}$ which is dense along $\mathcal{G}$. One of the key features of 1-generic sets of natural numbers is that every Σ^0_1 statement in the language of arithmetic with an additional predicate for $\mathcal{G}$ is decided by some initial segment of G; that is, either it or its negation is forced by some finite binary string $\sigma \in \mathcal{G}$.

An equivalent way to define a 1-generic set G is as follows. Let $\mathcal{P}_{\mathcal{F}}$ be the partial ordering with elements pairs of disjoint finite sets $\langle D, E\rangle$ ordered by $\langle D_1, E_1\rangle \leq \langle D_2, E_2\rangle$ if and only if $E_1 \supseteq E_2$ and $D_1 \supseteq D_2$. A filter $\mathcal{G} \subseteq \mathcal{P}_{\mathcal{F}}$, which for every n meets the set $V_n = \{\langle D, \emptyset\rangle \mid n \in D\}$, whenever it is dense along $\mathcal{G}$, defines a set G and its complement $\overline{G}$; namely, $G = \bigcup_{\langle D,E\rangle \in \mathcal{G}} D$ and $\overline{G} = \bigcup_{\langle D,E\rangle \in \mathcal{G}} E$. It is fairly easy to check that a set G is 1-generic if and only if it is obtained from a filter $\mathcal{G} \subseteq \mathcal{P}_{\mathcal{F}}$ which meets every Σ^0_1 open subset of $\mathcal{P}_{\mathcal{F}}$, which is dense along $\mathcal{G}$.

Enumeration 1-generic sets are also obtained from filters $\mathcal{G} \subseteq \mathcal{P}_{\mathcal{F}}$. The genericity requirements for these filters are limited further to only positive requirements. Let us call a set of conditions $V \subseteq \mathcal{P}_{\mathcal{F}}$ *positive* if and only if whenever $\langle D, E\rangle \in V$, we also have that $\langle D, \emptyset\rangle \in V$. Then G is enumeration 1-generic if and only if it is obtained from a filter $\mathcal{G} \subseteq \mathcal{P}_{\mathcal{F}}$ which meets every Σ^0_1 positive open subset of $\mathcal{P}_{\mathcal{F}}$, which is dense along $\mathcal{G}$. Similarly, we can characterize enumeration 1-genericity syntactically: a filter $\mathcal{G}$ gives rise to an enumeration 1-generic set if and only if every positive Σ^0_1 statement in the language of arithmetic with an additional predicate for $\mathcal{G}$ is decided by some condition in $\mathcal{G}$, where a positive Σ^0_1 statement is one obtained from Σ^0_1 statements in arithmetic (that do not mention $\mathcal{G}$) and statements of the form "$\langle D, \emptyset\rangle \in \mathcal{G}$," closed under conjunctions and existential quantification.

In the same way that 1-genericity has a natural characterization in terms of Turing functionals, we find that enumeration 1-genericity can be characterized in terms of enumeration operators as follows.

Proposition 3.3 (see [4]) *A is enumeration 1-generic if and only if, for every $e \in \omega$, either $e \in \Phi_e^A$ or, for some finite set $E \subseteq \overline{A}$, $e \notin \Phi_e^{\omega - E}$.*

Remark Note that if A is enumeration 1-generic, then A is infinite. Indeed, suppose that $A \subseteq \omega \restriction n$ for some $n \geq 0$. Then by enumeration 1-genericity of A, there exists a finite set E such that $D \cap E \neq \emptyset$ for every $D \in \{\{m\} \mid m \geq n\}$, which is an obvious contradiction. However, the notion of enumeration 1-genericity is weak in the sense that there are clearly enumeration 1-generic sets which are c.e.—the obvious example being ω itself.

In view of the above observations, we now consider how the definition of enumeration 1-genericity might be strengthened. The next lemma shows that coinfiniteness is an obvious candidate for this.

Lemma 3.4 *If A is enumeration 1-generic and coinfinite, then $\overline{A}$ is hyperimmune. Thus A is not Π_1^0.*

Proof Suppose that there exists a sequence of mutually disjoint finite sets $\{D_{f(i)}\}_{i \in \omega}$ with f computable such that, for all i, $D_{f(i)} \cap \overline{A} \neq \emptyset$. Let $W = \{D_{f(i)} \mid i \in \omega\}$. By enumeration 1-genericity there exists a finite set $E \subseteq \overline{A}$ such that for all $D \in W$, $D \cap E \neq \emptyset$. This is an obvious contradiction since W contains mutually disjoint finite sets. Hence $\overline{A}$ is hyperimmune. $\square$

However, in the context of the enumeration degrees, coinfiniteness does not confer nontriviality to the notion of enumeration 1-genericity, as we now see.

Proposition 3.5 *There exists a coinfinite c.e. enumeration 1-generic set A.*

Proof The proof involves enumerating a set A in stages so as to satisfy, for all $e \in \omega$, the following requirements:

$$R: \quad |A| \text{ is infinite},$$

$$P_e: \quad (\exists D \in W_e)[D \subseteq B] \vee (\exists E \subseteq \overline{B})(\forall D \in W_e)[D \cap E \neq \emptyset].$$

To do this, we use a standard finite injury construction in which at every stage s a finite approximation A_s is defined such that $A = \bigcup_{s \in \omega} A_s$. Each requirement P_e works with its own *restraint* witness $x(e,s) \in \omega$ defined at the end of stage s and its own *avoidance* parameter $\Omega(e, s+1) = \{x(i,s) \mid i \leq e\}$ defined at the beginning of stage $s+1$. The requirement P_e is said to be *satisfied* at stage $s+1$ if there exists $D \in W_e[s]$ such that $D \subseteq A_s$. Likewise, P_e is said to *require attention* at stage $s+1$ if it is not satisfied and there exists $D \in W_e[s+1]$ such that $D \cap \Omega(e, s+1) = \emptyset$. The construction is defined as follows.

Stage 0. Define $x(e, 0) = e$ for all $e \geq 0$.

Stage $s+1$. If there is no $e < s$ such that P_e requires attention, then reset $x(e, s+1) = x(e,s)$ for all $e \geq 0$ and go to stage $s+2$. Otherwise, let e be the least such index. Enumerate into A the least set $D \in W_e[s+1]$ such that $D \cap \Omega(e, s+1) = \emptyset$; that is, set $A_{s+1} = A_s \cup D$. Reset $x(i, s+1) = x(i, s)$ for all $i \leq e$ and, letting $\hat{s} = \max\{x(e,s), s\}$, set $x(j, s+1) = \hat{s} + j$ for all $j > e$, and go to stage $s+2$.

The verification of the construction is a straightforward induction argument over index e. Note first that

$$x(i,s) \notin A_s \quad \text{for all } i, s \geq 0. \tag{3.1}$$

Proceed by assuming that index e and stage s_e are such that $x(i,s) = x(i,s_e)$ for all $i \leq e$ and $s \geq s_e$—and accordingly let $x(i) = x(i,s_e)$—and are also such that, for all $j < e$, P_j does not require attention at any such stage s. Let $\Omega(e) = \{x(i) \mid i \leq e\}$, and note that it follows from (3.1) that $\Omega(e) \subseteq \overline{A}$. Clearly, if there exists $D \in W_e$ such that $D \cap \Omega(e) = \emptyset$ and P_e has not been satisfied before stage s_e, then P_e will receive attention at some stage $s \geq s_e$. Thus clearly P_e will be satisfied in the limit (since $D \cap \Omega(e) \neq \emptyset$ for all $D \in W_e$ otherwise). Moreover, as P_e only requires attention at most once after stage s_e there exists a corresponding stage $s_{e+1} \geq s_e$ such that $x(i,s) = x(i,s_{e+1})$ for all $i \leq e+1$ and $s \geq s_{e+1}$ and such that, at any such stage s and any index $j < e+1$, no requirement P_j requires attention at stage s. We can therefore conclude that $x(e)$ is defined for all e, that $\Omega = \{x(e) \mid e \in \omega\} \subseteq \overline{A}$, so that R is satisfied (since clearly $x(i) \neq x(j)$ for all $i \neq j$ by construction), and that P_e is satisfied for all $e \geq 0$. □

Remark It follows from Lemma 3.4 that every coinfinite c.e. enumeration 1-generic set is hypersimple.

Another obvious way of strengthening enumeration 1-genericity is to impose symmetricity of this notion over a set A and its complement.

Definition 3.6 (see [4]) A set A is defined to be *symmetric enumeration (s.e.) 1-generic* if both A and $\overline{A}$ are enumeration 1-generic.

Now, unlike enumeration 1-genericity alone, s.e. 1-genericity does confer nontriviality in the context of the enumeration degrees.

Proposition 3.7 (see [4]) *If A is s.e. 1-generic, then $A \notin \Sigma^0_1 \cup \Pi^0_1$.*

Remark Note that the proposition above follows directly from Lemma 3.4.

Note that, by definition, the class of enumeration 1-generic degrees subsumes the class of s.e. degrees. Similarly, we find that the class of s.e. degrees subsumes the class of 1-generic degrees.

Proposition 3.8 (see [4]) *If A is 1-generic, then A is s.e. 1-generic.*

On the other hand, we find that enumeration 1-generic sets display a certain form of *lowness*.

Lemma 3.9 *For every enumeration 1-generic set A, $J_A \equiv_e A \oplus \overline{A} \oplus J_\emptyset$.*

Proof We know that $K_A \equiv_e A$. Moreover, for every finite set E, the set $\overline{\Phi_e^{\omega - E}}$ is enumeration reducible to $J_\emptyset$ uniformly in e and E, via (say) the operator $\Phi_{g(e,E)}$. By enumeration 1-genericity of A, $\overline{K_A} = \{e \mid \exists E[e \in \Phi_{g(e,E)}^{J_\emptyset} \;\&\; E \subseteq \overline{A}]\}$ and hence $\overline{K_A} \leq_e \overline{A} \oplus J_\emptyset$.

We conclude that $J_A = K_A \oplus \overline{K_A} \equiv_e A \oplus \overline{A} \oplus J_\emptyset$. □

Corollary 3.10 *If G is s.e. 1-generic, then $J_G \equiv_e J_{\overline{G}}$. If G is 1-generic, then $J_G \equiv_e J_{\overline{G}} \equiv_e J_{G \oplus \overline{G}}$.*

Proof If G is s.e. 1-generic, then both G and $\overline{G}$ are enumeration 1-generic. By Lemma 3.9, $J_G \equiv_{\mathrm{e}} G \oplus \overline{G} \oplus J_\emptyset \equiv_{\mathrm{e}} J_{\overline{G}}$.

If G is 1-generic, then we know that its jump in the Turing degrees also behaves this way: $G' \equiv_{\mathrm{T}} G \oplus \emptyset'$. Thus, as ι preserves join and jump, $J_{G\oplus\overline{G}} \equiv_{\mathrm{e}} G \oplus \overline{G} \oplus J_\emptyset$. □

Corollary 3.11 *If $A \in \Pi^0_2$ is enumeration 1-generic, then $J_A \equiv_{\mathrm{e}} A \oplus J_\emptyset$. In particular, if A is Δ^0_2, then $\deg_{\mathrm{e}}(A)$ is low.*

Remark A straightforward argument shows that any Δ^0_2 approximation to A is in fact both *low*—in the sense of [16]—and *good* as defined in Definition 2.1.

Corollary 3.11 suggests a way of delineating the distribution of enumeration 1-generic degrees within the Δ^0_2 degrees. However, we will see below in Proposition 4.12 that there exists a Σ^0_2 enumeration 1-generic degree in which not every set is enumeration 1-generic. Accordingly, Corollary 3.11 could be applied directly to the Δ^0_2 degrees themselves (and not just to individual sets) only if the latter phenomenon can be ruled out in the case of the Δ^0_2 degrees. Indeed, its presence in this context would suggest the existence of Δ^0_2 enumeration 1-generic degrees that are nonlow.

The above discussion leads us to the question of what overall restrictions there are to the distribution of the enumeration 1-generic degrees.

Lemma 3.12 (see [4]) *If $B \in \Pi^0_2$ is enumeration 1-generic, then the class $\mathcal{B} = \{X \mid X \leq_{\mathrm{e}} B\}$ is Π^0_2 in a uniform manner.*

Note that this lemma is a simple corollary of Corollary 3.11 if every such B is in fact Δ^0_2. We now see that this is not the case.

Proposition 3.13 (see [4]) *There exists a properly Π^0_2 enumeration degree b such that the principal ideal $\mathcal{D}_{\mathrm{e}}(\leq b)$ only contains Π^0_2 sets.*

Proof Let A be a set such that $\deg_{\mathrm{e}}(A)$ is properly Σ^0_2 and $\overline{A}$ is enumeration 1-generic. For example, take A to be the 1-generic set with noncuppable enumeration degree constructed in the proof of [4, Theorem 3.2]. Let $B = \overline{A}$ and $b = \deg_{\mathrm{e}}(B)$. Then B is enumeration 1-generic and hence $\mathcal{D}_{\mathrm{e}}(\leq b)$ only contains Π^0_2 sets by Lemma 3.12. Moreover, no set X in b is Δ^0_2 since this would imply that B is Δ^0_2 in contradiction with the definition of $A = \overline{B}$. □

The above is a first illustration of a natural restriction of the class of enumeration 1-generic degrees within $\mathcal{D}_{\mathrm{e}}$. However, these results tell us nothing further (to Corollary 3.11) about the local structure of Σ^0_2 degrees. For example, is $0'_{\mathrm{e}}$ enumeration 1-generic? The final result of this section not only settles this question but also shows that the distribution of the enumeration 1-generic degrees bears a certain resemblance to the distribution of the 1-generic degrees, both globally and locally within $\mathcal{D}_{\mathrm{e}}$.

Proposition 3.14 *Every enumeration 1-generic degree $0_{\mathrm{e}} < a$ is quasiminimal.*

Proof Suppose that A is an enumeration 1-generic set and that C is a characteristic set such that $C \leq_{\mathrm{e}} A$. Accordingly, let Φ witness this reduction (i.e., $C = \Phi^A$), and consider the c.e. set

$$S = \bigl\{D \bigm| \exists F\, \exists F'\bigl[\langle 2x, F\rangle \in \Phi \;\&\; \langle 2x+1, F'\rangle \in \Phi \;\&\; D = F \cup F'\bigr]\bigr\}.$$

Since C is characteristic, it follows that $D \nsubseteq A$ for all $D \in S$. Hence, by enumeration 1-genericity of A, there exists a finite set $E \subseteq \overline{A}$ such that for all $D \in S$, $D \cap E \neq \emptyset$. However, this implies that $C = \Phi^{\omega - E}$.

Indeed, clearly $C \subseteq \Phi^{\omega-E}$ (as $A \subseteq \omega - E$). Suppose that there exists $y \in \Phi^{\omega-E} \setminus C$. Then $y = 2x + i$ for some $i \in \{0, 1\}$. Without loss of generality, suppose that $i = 0$. Accordingly, there is a finite set $F \subseteq \omega - E$ such that $\langle 2x, F\rangle \in \Phi$. Since C is characteristic and $2x \notin C$, it follows that $2x + 1 \in C = \Phi^A$. Hence there exists a finite set F' such that $\langle 2x + 1, F'\rangle \in \Phi$ and $F' \subseteq A \subseteq \omega - E$. Set $D = F \cup F'$. Clearly $D \in S$ whereas, by the above, $D \cap E = \emptyset$. This contradicts the definition of E. Thus $\Phi^{\omega-E} \subseteq C$ and so $C = \Phi^{\omega-E}$; that is, C is c.e. □

4 Enumeration 1-Genericity and Jump Inversion

In this section we show that there is a uniform method for constructing, below any enumeration degree a, an enumeration 1-generic degree b having the same jump complexity as a. We also show that below any nonzero Δ^0_2 degree there exists a nonzero Δ^0_2 enumeration 1-generic degree of lowest possible jump complexity. These results allow us to conclude that every nonzero good enumeration degree strictly bounds a nonzero enumeration 1-generic degree of the same jump complexity. We also consider the relationship between enumeration 1-genericity and Σ^0_2 highness (defined below) which is brought to light by these results. We begin with some further background material.

Notation Given a Σ^0_2 approximation $\{A_s\}_{s\in\omega}$ to a set A, we use the shorthand[1] c_A to denote the computation function relative to $\{A_s\}_{s\in\omega}$ defined by setting, for all $x \in \omega$, $c_A(x) = (\mu s > x)[A_s \restriction x \subseteq A]$.

Definition 4.1 A Σ^0_2 approximation $\{A_s\}_{s\in\omega}$ is said to be *high* if its associated computation function c_A is total and dominates every computable function f (i.e., $c_A(x) > f(x)$ for almost every x). A set A is said to be Σ^0_2 *high* if it has a Σ^0_2 approximation (and so, using standard terminology, an enumeration degree is Σ^0_2 *high* if it contains such a set).

Lemma 4.2 (Shore and Sorbi [18]) *A degree $a \le 0'_{\mathrm{e}}$ is high if and only if it is Σ^0_2 high.*

We now proceed with the main result of this section.

Proposition 4.3 *There exists an enumeration operator Δ, such that, for every A, Δ^A is enumeration 1-generic and $J_{\Delta^A} \equiv_{\mathrm{e}} J_A$.*

Proof We construct a c.e. operator Δ so that for every e the following requirement is met:

$$P_e: \quad \forall A\big[(\exists D \in W_e)[D \subseteq \Delta^A] \vee (\exists E \subseteq \overline{\Delta^A})(\forall D \in W_e)[D \cap E \neq \emptyset]\big].$$

The construction is a finite injury construction in stages. At every stage s we construct a c.e. set Δ_s. The intent is that $\Delta = \bigcup_{s\in\omega} \Delta_s$ is the required operator. For every e we will have a coding location d_e. The coding locations are our tool to code, for every A, the bits of K_A. $F_e = \{d_j \mid j \le e\}$ is the set of all coding locations for higher priority requirements. Note that F_e is restrained in the sense that, if we are acting to satisfy the $(e + 1)$th genericity requirement P_{e+1}, then we are not allowed to enumerate into Δ axioms for the elements in F_e. Depending on the oracle A, each nonempty set $E \subseteq F_e$ may or may not also be a subset of $\overline{\Delta^A}$. In fact, it may be the case that $E = \emptyset$ is the only subset of both F_e and $\overline{\Delta^A}$. So we will make sure that all possibilities are covered in the way that we meet the requirement P_e. In particular, at any moment in the construction we will know the status of a requirement—satisfied or not—and, if the requirement is not yet satisfied, then we will know how far we have gone towards satisfying it. Every finite subset of $S \subseteq F_e$ will be announced as either *covered* or *not yet covered*. The intuition behind this notion is that if the finite subset S turns out to be a subset of the oracle A and there is a finite set D in W_e such that $D \cap (F_e \setminus S) = \emptyset$, then we will ensure that $D \subseteq \Delta^A$ by enumerating axioms for all the elements in $D \setminus S$. In other words, we have satisfied the requirement P_e, provided that S is a subset of the oracle. As there are finitely many subsets of F_e, these actions will be performed finitely many times. Once every such finite subset is covered, we will announce that P_e is *satisfied* (for now). Later in the construction, however, we might announce that P_e is *not yet satisfied* if a higher priority requirement acts.

The Construction. At stage 0, $\Delta_0 = \emptyset$ and, for all e, d_e is undefined, P_e is announced as not yet satisfied, and every subset of F_e is announced as not yet covered.

Remark During the construction, we only take care to define F_e (as specified above) when d_e is defined. If d_e is not defined, then the value of F_e is unimportant.

We will say that P_e *requires attention* at stage $s + 1$ if P_e is (announced as) not yet satisfied and one of the following is true.

(1) The coding location d_e is undefined.
(2) There is a finite set $D \in W_e[s]$ such that $D \cap F_e = S$ is not yet covered.

At stage $s + 1$, we let e be the least number for which the requirement P_e requires attention at stage $s + 1$.

(1) If the required attention is because the coding location is undefined, then we define the value of d_e to be the least number for which there is no axiom in Δ_s. Then we set $\Delta_{s+1} = \Delta_s \cup \{\langle d_e, D\rangle \mid \langle e, D\rangle \in \Phi_e\}$. Next we injure all lower priority requirements by announcing all of them as not yet satisfied, making d_j for $j > e$ undefined and announcing all finite subsets of F_j not yet covered for $j > e$. (Note that even though Δ_{s+1} is not necessarily finite, it is c.e. and contains axioms for finitely many elements. Thus this step is computable.)
(2) Otherwise, pick the least finite set $D \in W_e[s]$ such that $D \cap F_e = S$ is not yet covered. Announce that all sets X, such that $S \subseteq X \subseteq F_e$, are covered. If all subsets of F_e are covered, then announce the requirement P_e satisfied. Then set $\Delta_{s+1} = \Delta_s \cup \{\langle n, \emptyset\rangle \mid n \in D \setminus S\}$. Again we injure all lower priority requirements by announcing all of them as not yet satisfied, making d_j for $j > e$ undefined and announcing all finite subsets of F_j not yet covered for $j > e$.

If no requirement requires attention at stage $s + 1$, then set $\Delta_{s+1} = \Delta_s$.

End of construction.

Lemma 4.4 *For every e, there is a least stage s_e such that P_e does not get injured by higher priority requirements at stages $t > s_e$. Furthermore, the function $e \mapsto s_e$ is computable by $\mathcal{K}$ (the Turing halting set).*

Proof The proof is by induction. P_0 does not get injured at all, so $s_0 = 0$. Suppose that P_e does not get injured after stage s_e and s_e is least with this property. This means that P_e is injured for the last time at stage s_e. Then at stage $s_e + 1$, P_e requires attention with undefined coding location d_e. By definition of s_e, G_e is the least requirement that requires attention at this stage and hence receives attention. At stage $s_e + 1$, the final value of d_e and the final value of the set F_e are defined. $\mathcal{K}$ can answer recursively every question of the following: "Does there exist a finite set D in the c.e. set W_e which covers the finite set S?" Here D covers S if $D \cap F_e \subseteq S$. So $\mathcal{K}$ can compute which of the finite subsets of F_e get covered by asking $2^{|F_e|}$ such questions. Note that if S can be covered, then it will be covered, because after stage s_e, whenever P_e requires attention, it receives attention. Now $\mathcal{K}$ can run the construction for the number of stages necessary until it reaches a stage at which the last finite set which can be covered, gets covered. This stage is s_{e+1}, the last stage at which P_{e+1} is injured. □

Lemma 4.5 *For every A, the set Δ^A is enumeration 1-generic.*

Proof Fix A and W_e. Let F_e be the final value of this parameter obtained at stage $s_e + 1$. Suppose that there is a finite set $D \in W_e$ such that $D \cap F_e \subseteq \Delta^A$. Then $S = D \cap F_e$ can be covered. By the properties of the construction, it will be covered at some stage t $(> s_e + 1)$. At stage t, we have found a (possibly different) finite set D^* such that $D^* \cap F_e \subseteq S \subseteq \Delta^A$, and for every element $n \in D^* \setminus F_e$, we have enumerated the axiom $\langle n, \emptyset\rangle \in \Delta_t$; hence $D^* \subseteq \Delta^A$.

Otherwise, for every $D \in W_e$, we have that $D \cap F_e \cap \overline{\Delta^A} \neq \emptyset$; that is, the finite set $E = F_e \cap \overline{\Delta^A}$ intersects every member of W_e. □

Lemma 4.6 *For every A, $J_A \equiv_{\mathrm{e}} J_{\Delta^A}$.*

Proof $J_{\Delta^A} \leq_{\mathrm{e}} J_A$ by monotonicity of the enumeration jump. So we only need to show that $J_A \leq_{\mathrm{e}} J_{\Delta^A}$.

We will show that $K_A \leq_{\mathrm{T}} \Delta^A \oplus \mathcal{K}$. Now again, using the fact that ι preserves join and jump and maps $\deg_{\mathrm{T}}(K_A)$ to $\deg_{\mathrm{e}}(K_A \oplus \overline{K_A})$ as also $\deg_{\mathrm{T}}(\Delta^A \oplus \mathcal{K})$ to $\deg_{\mathrm{e}}(\Delta^A \oplus \overline{\Delta^A} \oplus J_\emptyset)$, it will follow that $J_A = K_A \oplus \overline{K_A} \leq_{\mathrm{e}} \Delta^A \oplus \overline{\Delta^A} \oplus J_\emptyset$. As Δ^A is enumeration 1-generic, we know from Lemma 3.9 that $\Delta^A \oplus \overline{\Delta^A} \oplus J_\emptyset \equiv_{\mathrm{e}} J_{\Delta^A}$ and hence that $J_A \leq J_{\Delta^A}$.

To compute $K_A(e)$ we use $\mathcal{K}$ to compute the stage s_e, the last stage at which P_e is injured, and then run stage $s_e + 1$ at which d_e is defined. Now $d_e \in \Delta^A$ if and only if $e \in \Phi_e^A$. This is because at stage $s_e + 1$ we enumerate the only axioms for d_e that ever get enumerated in Δ and they mirror exactly the axioms for e in Φ_e. We use Δ^A to determine this last membership question. □

This concludes the proof of Proposition 4.3. □

Corollary 4.7 *For every enumeration degree $a \in \mathcal{D}_{\mathrm{e}}$, there exists an enumeration 1-generic $b \leq a$ such that $b' = a'$.*

Note that, in the case in which a is low, Corollary 4.7 does not guarantee that $b > 0_{\mathrm{e}}$. However, we can deduce from the following result in [3] that such a degree does indeed exist.

Theorem 4.8 (see [3]) *For every Δ_2^0 enumeration degree $a > 0_{\mathrm{e}}$, there exists $0_{\mathrm{e}} < b \leq a$ such that b is 1-generic.*

Moreover, the next result by Ganchez and Sorbi will allow us to show that the degree b stipulated in Corollary 4.7 can be chosen to be strictly below a.

Theorem 4.9 (Ganchev and Sorbi [9]) *Every enumeration degree $a > 0_e$ bounds a nontrivial initial segment of enumeration degrees whose nonzero elements all have the same jump as a.*

Accordingly, we obtain a stronger version of Corollary 4.7.

Proposition 4.10 *For every enumeration degree $c > 0_e$, there exists enumeration 1-generic $0_e < b < c$ such that $b' = c'$.*

Proof By Theorem 4.9, we can choose $0_e < a < c$ such that $a' = c'$. If c is not low, then we apply Corollary 4.7 to a. If c is low, then we apply Theorem 4.8 in conjunction with Proposition 3.8 to a. In both cases, we obtain an enumeration 1-generic $0_e < b \leq a < c$ such that $b' = c'$. □

Given Lemma 4.2 and Corollary 4.7 applied to the special case when $a = 0_e'$, we might expect there to exist a set A that is both Σ_2^0 high and enumeration 1-generic. We now investigate whether this is the case.

Lemma 4.11 *If A is Σ_2^0 high, then A is not enumeration 1-generic (and hence neither s.e. 1-generic nor 1-generic).*

Proof Let $\{A_s\}_{s\in\omega}$ be a high Σ_2^0 approximation to A with associated computation function c_A. Let $s_A \in \omega$ be such that $c_A(s) > s + 1$ (i.e., the successor function) for all $s > s_A$. Define the c.e. set

$$W = \{A_{s+1}\restriction s \mid s > s_A\},$$

and observe that, by definition of s_A, for all $D \in W$, $D \nsubseteq A$. Suppose that A is enumeration 1-generic. Then there exists a finite set $E \subseteq \overline{A}$ such that $D \cap E \neq \emptyset$ for all $D \in W$. Let

$$m = \max(E \cup \{s_A\}) + 1,$$

and let s_m be such that $s_m + 1 = c_A(m)$ (and so $s_m \geq m$). By definition of c_A, $A_{s_m+1}\restriction m \subseteq A$ (whereas $E \subseteq \overline{A}\restriction m$). Thus, letting $D = A_{s_m+1}\restriction s_m$, we see that $D \in W$ and $D \cap E = \emptyset$, which is a contradiction. Thus A is not enumeration 1-generic. □

Proposition 4.12 *There exists a high enumeration 1-generic degree $b \leq 0_e'$ and sets $B, C \in b$ such that $B \neq C$, and*

(i) *B is not Σ_2^0 high,*
(ii) *C is not enumeration 1-generic.*

Proof Choose $A = J_\emptyset$ in Proposition 4.3, and set $B = \Delta^A$. Then $b = \deg_e(B)$ is high. By Lemma 4.2, b contains a Σ_2^0 high set C. By Lemma 4.11, it follows that $B \neq C$ and that B is a witness for (i), whereas C is a witness for (ii). □

5 Enumeration 1-Genericity and Nonsplitting

We saw in Proposition 3.8 that every 1-generic set A is s.e. 1-generic and hence enumeration 1-generic. We also saw that the class of nonzero enumeration 1-generic degrees shares at least two nontrivial structural properties with the 1-generic degrees, namely, quasiminimality (see Proposition 3.14) and Π_2^0 downward closure (see

Lemma 3.12). So are these two classes identical or is there some other property that separates them? Consider the following property.

Definition 5.1 A degree a is said to be *splittable* if there exist incomparable degrees a_0 and a_1 such that $a = a_0 \cup a_1$. Otherwise a is said to be *nonsplittable*.

In this section we show that splittability is just one such property, and that this separation occurs within the Σ^0_2 degrees.

Proposition 5.2 (Folklore) *Every* 1*-generic enumeration degree a is splittable.*

Note. The proof below is a straightforward adaptation of the proof of this property in the context of *function* 1-genericity given in Copestake [7].

Proof Suppose that $A \in a$ is 1-generic. Define the sets A_0 and A_1 such that, for $i \in \{0, 1\}$, $A_i = \{x \mid 2x + i \in A\}$. Note that by immunity of $\overline{A}$, both A_0 and A_1 are infinite. Clearly $A \equiv_e A_0 \oplus A_1$. Suppose that $A_0 \leq_e A_1$, and let Φ be the enumeration operator witnessing this reduction. Consider the c.e. set $S \subseteq 2^{<\omega}$ defined by setting

$$S = \big\{\tau \;\big|\; \exists x\, \exists D\big[\tau(2x) = 0 \;\&\; x \in \Phi^D \;\&\; D \subseteq \big\{z \;\big|\; \tau(2z + 1) = 1\big\}\big]\big\}.$$

Note that, by definition of Φ, $\tau \notin S$ for any $\tau \subset \chi_A$. Thus (by 1-genericity of A) there exists $\sigma \subset \chi_A$ such that, for all $\tau \supseteq \sigma$, $\tau \notin S$. Now, as A_0 is infinite we can pick x and $D \subseteq A_1$ such that $2x \geq |\sigma|$ and $x \in \Phi^D$. Let $\gamma \supset \sigma$ be any string defined so that $\gamma(2x) = 0$ and $\gamma(2z + 1) = 1$ for all $z \in D$.[2] Then clearly $\gamma \in S$, which is a contradiction. In other words, $A_0 \not\leq_e A_1$. By a similar argument, $A_1 \not\leq_e A_0$. Therefore, letting $a_0 = \deg_e(A_0)$ and $a_1 = \deg_e(A_1)$, we see that the pair a_0, a_1 witnesses the splittability of a. □

In contrast to this, we will show that there exist both a low and a properly Σ^0_2 nonsplittable enumeration 1-generic degree. In order to do this, we transpose the methodology of the low nonsplittability proof of Ahmad and Lachlan [1] onto a tree of strategies construction using techniques formulated by Kent [12] and Kent and Sorbi [13]. We note that Theorem 5.3 can in fact be proved by a straightforward modification of Ahmad and Lachlan's proof (see [1]). However, the reader will notice that the manner in which the tree of strategies construction is applied here not only clarifies the mechanics of the proof (in that the *streams* of *free* numbers used by the splitting strategies are precisely reflected in the structure of the tree of strategies itself), but also that it allows the low nonsplittability version to be easily adapted to show properly Σ^0_2 nonsplittability. We also note that the present construction is an adaptation of Kent's nonsplittability proofs in [12] with the difference that a close interpretation of the elegant e-states method used in [1] is implicitly adhered to in the definition of the tree of strategies.

Theorem 5.3 *There exists a low enumeration* 1*-generic nonsplittable degree $a > 0_e$.*

Proof We will define a set A with Δ^0_2 approximation $\{A_s\}_{s\in\omega}$ satisfying the following requirements.

$$R_{\Psi,\Omega_0,\Omega_1}: \quad A = \Psi^{\Omega_0^A \oplus \Omega_1^A} \Rightarrow A \leq_e \Omega_i^A \text{ for some } i \in \{0, 1\} \text{ or } A \text{ c.e.},$$

$$N_W: \quad A \neq W,$$

$$P_W: \quad (\exists D \in W)[D \subseteq A] \vee (\exists E \subseteq \overline{A})(\forall D \in W)[D \cap E \neq \emptyset].$$

Note our use of shorthand notation in the above (introduced to simplify the presentation), whereby we understand $(\Psi, \Omega_0, \Omega_1) \in \{(\Psi_e, \Omega_{e,0}, \Omega_{e,1})\}_{e\in\omega}$ and where the latter is an acceptable listing of all triples of enumeration operators. Likewise, W ranges over an acceptable listing of c.e. sets $\{W_e\}_{e\in\omega}$. In each case, we assume that the listing is associated with uniform c.e. approximations of the sets/operators involved.

(1) *The tree of strategies*

We define the overall set of *outcomes* to be $\Sigma = \{0, 1, 2\} \cup \{\text{void}\}$ and the set of *tree outcomes* to be $\{0, 1, 2\}$. We fix an arbitrary effective priority ordering $\{L_e\}_{e\in\omega}$ of all R, N, and P requirements. We also define $\mathcal{T} \subseteq \{0, 1, 2\}^{<\omega}$ and we refer to it as the *tree of strategies*. Each node $\alpha \in \mathcal{T}$ will be associated, and so identified, with the strategy for the satisfaction of $L_{|\alpha|}$. We use the notation $\mathcal{R}_{\Psi,\Omega_0,\Omega_1}$ for the set of $R_{\Psi,\Omega_0,\Omega_1}$ strategies and $\mathcal{R}$ for the set of all R strategies. Likewise, for $(\mathcal{Q}, Q) \in \{(\mathcal{N}, N), (\mathcal{P}, P)\}$ we use the notation $\mathcal{Q}_W$ for the set of strategies associated with Q_W, and we let $\mathcal{Q}$ denote the set of all such strategies.

We assign requirements to nodes on $\mathcal{T}$ by induction as follows. Define $\emptyset \in \mathcal{T}$. Given $\alpha \in \mathcal{T}$ we distinguish three cases depending on the requirement L associated with α.

Case 1. $\alpha \in \mathcal{R}$: define $\alpha^\frown\langle n\rangle \in \mathcal{T}$ for $n \in \{0, 1, 2\}$.
Case 2. $\alpha \in \mathcal{N}$: define $\alpha^\frown\langle n\rangle \in \mathcal{T}$ for all $n \in \{0, 1\}$.
Case 3. $\alpha \in \mathcal{P}$: define $\alpha^\frown\langle n\rangle \in \mathcal{T}$ for all $n \in \{0, 1\}$.

(2) *Notation and terminology for strings*

We use standard notation and terminology for strings as found, for example, in Soare [19]. Accordingly, we use $\leq$ and $<$ ($\subseteq$ and $\subset$) to denote, respectively, nonstrict and strict lexicographical ordering (inclusion) on $\mathcal{T}$.[3] We have that $\sigma <_L \tau$ denotes $\sigma < \tau$, but $\sigma \nsubseteq \tau$.

(3) *Environment parameters*

Local parameters for $\alpha \in \mathcal{R}_{\Psi,\Omega_0,\Omega_1}$. $R(\alpha, s) \in \{0, 1, 2, \text{void}\}$ is the *outcome* parameter, and $\Gamma_{\alpha,0}[s]$ and $\Gamma_{\alpha,1}[s]$ are finite approximations to enumeration operators constructed so as to (possibly) witness $A \leq_e \Omega_0^A$ or $A \leq_e \Omega_1^A$. (Note that, for $i \in \{0, 1\}$, we use Γ_i as shorthand for $\Gamma_{\alpha,i}$ when there is no danger of ambiguity.) Outcome $R(\alpha, s) = j$ for $j \leq 1$ corresponds to α's belief that if $A = \Psi^{\Omega_0^A \oplus \Omega_1^A}$, then $A \leq_e \Omega_j^A$ (as witnessed by Γ_j in the limit). Likewise, under the same assumption, $R(\alpha, s) = 2$ corresponds to α's belief that A is c.e. (contradicting the definition of a). For ease of description in the construction, α also has a *dummy witness* parameter $x(\alpha, s) = -1$.

Local parameters for $\alpha \in \mathcal{N}_W$. $N(\alpha, s) \in \{0, 1, \text{void}\}$ is the *outcome* parameter and $x(\alpha, s) \in \{-1\} \cup \omega$ is the *witness* parameter associated with α. Outcome $R(\alpha, s) = 0$ corresponds to α's knowledge that $x(\alpha, s) \in W$ and belief that $x(\alpha, s) \notin A$ (which will be vindicated if α is not initialized at any stage $t > s$). $N(\alpha, s) = 1$, on the other hand, means that α believes that $x(\alpha, s) \in A \setminus W$.

Local parameters for $\alpha \in \mathcal{P}_W$. $P(\alpha, s) \in \{0, 1, \text{void}\}$ is the *outcome* parameter and $x(\alpha, s) = -1$ is a *dummy witness* parameter for α. $P(\alpha, s) = 0$ corresponds to α's belief that there is some $D \in W$ such that $D \subseteq A$ (which will be vindicated if α is on the true path and is not initialized at any stage $t > s$). $P(\alpha, s) = 1$, on the other hand, corresponds to α's belief that there is no such D in W.

The stream for any $\alpha \in \mathcal{T}$. $S(\alpha, s) = \{x(\beta, s) \mid x(\beta, s) \geq 0 \ \& \ \alpha \subseteq \beta\}$ is the (finite) *stream* associated with α at stage s and corresponds to the set of numbers already processed by the construction at stage s and which are (roughly speaking) available for processing by α at stage $s + 1$. Note that, by definition, $x(\alpha, s) \notin S(\alpha^\frown\langle n\rangle, s)$ for any $n \in \{0, 1, 2\}$. (This observation is significant for the construction in the case $\alpha \in \mathcal{N}$ and trivial otherwise.)

Global parameters for stage $s + 1$. Each stage $s+1$ has the following parameters.

(i) $z(s + 1, t) \in \omega \cup \{\text{break}\}$ is a *floating witness* which is passed down the $s + 1$ stage approximation to the true path. When $t = 0$, $z(s + 1, t)$ starts life by denoting the number s. For $t \geq 0$, the witness $z(s + 1, t)$ is passed to the strategy α of length t eligible to act at substage $t + 1$ provided that $z(s + 1, t) \neq$ break. The strategy α decides whether (a) to set $z(s + 1, t + 1) =$ break, thus causing stage $s + 1$ to terminate,[4] (b) to reallocate $z(s + 1, t + 1)$ to some number belonging to its s stage *stream*, or (c) to reset $z(s + 1, t + 1) = z(s + 1, t)$. In case (a), the strategy α either sets $x(\alpha, s + 1) = z(s + 1, t)$ or *dumps* $z(s + 1, t)$ into A,[5] whereas in case (b), α always dumps $z(s + 1, t)$ into A. Note that case (a) corresponds to $\alpha \in \mathcal{N} \cup \mathcal{P}$ and case (b) to $\alpha \in \mathcal{R}$, whereas case (c) may apply to any strategy α. Also note that in cases (b) and (c), the new value of the floating witness $z(s + 1, t + 1)$ is passed to the strategy $\alpha^\frown\langle i\rangle$ of length $t + 1$ eligible to act at stage $t + 2$.

(ii) $D(s + 1, t) \in \mathcal{F}$ is a record, established at substage t, that defines a set of numbers that will be *dumped* at the end of stage $s + 1$. When $t = 0$, $D(s + 1, t)$ starts life as $\emptyset$. $D(s + 1, t + 1)$ is defined provided that $z(s + 1, t) \neq$ break (i.e., the stage has not yet terminated), and in this case, $D(s + 1, t) \subseteq D(s + 1, t + 1)$.

(iii) $D(s + 1)$ is the overall set of numbers *dumped* into A at the end of stage $s + 1$. Thus, by definition, $D(s + 1) = D(s + 1, |\beta_s| + 1)$, where β_s is the s stage approximation to the true path.

Initialization. For $(Q, \mathcal{Q}) \in \{(R, \mathcal{R}), (N, \mathcal{N}), (P, \mathcal{P})\}$ and any $\alpha \in \mathcal{Q}$, we say that "void" is the *initial value* of $Q(\alpha, s)$ and that -1 is the initial value of $x(\alpha, s)$. For $\alpha \in \mathcal{R}$, we say that $\emptyset$ is the initial value of $\Gamma_{\alpha,i}$ for $i \in \{0, 1\}$. The *initialization* of a node $\alpha \in \mathcal{T}$ is the process of resetting its associated parameters to their initial values.

The Construction. The construction proceeds in stages $s \in \omega$. At each stage s the construction defines the following finite sets. $D_A[s]$ is the set of numbers already ***D**umped* into A, while $F_A[s]$ is the set of numbers already used by the construction (i.e., having visited A during at least one stage) but still ***F**ree*, that is, nondumped. $I_A[s]$ is the set of (free) numbers ***I**nside* A and $O_A[s]$ is the set of (free) numbers ***O**utside* A. The intention here is that $I_A[s] \cap O_A[s] = \emptyset$, $F_A[s] = I_A[s] \cup O_A[s]$, $F_A[s] \cap D_A[s] = \emptyset$, and $F_A[s] \cup D_A[s] = \omega \restriction s$. The s stage approximation to A will be defined to be $A[s] = I_A[s] \cup D_A[s]$.

We say that a number $x \in \omega$ is *new* if it is greater than any number used in the construction so far.

To facilitate understanding of the construction, we suggest that the reader also consult the informal observations relative to stage $s + 1$ made on page 477.

Stage $s = 0$.

Set $A[s] = I_A[s] = O_A[s] = F_A[s] = D_A[s] = \emptyset$, and initialize all $\alpha \in \mathcal{T}$.

Stage $s + 1$.

This stage consists of substages $t \geq 0$ such that some strategy $\alpha \in \mathcal{T}$ acts (i.e., is processed) at substage $t + 1$ provided that $z(s + 1, t) \neq$ break. If so, α decides the value of $z(s+1,t+1)$ and $D(s+1,t+1)$, the value of its local parameters, and (accordingly), if $z(s+1,t+1) \in \omega$, which strategy $\alpha^\frown\langle n\rangle$ is *eligible to act next.*

Substage 0.

Set $z(s+1,0) = s$ and $D(s+1,0) = \emptyset$.

Substage $t+1$. (Under the assumption that $z(s+1,t) \in \omega$.)

We suppose that α is the strategy of length t which is eligible to act at this substage. We distinguish cases depending on the requirement R assigned to α.

Case 1. $\alpha \in \mathcal{R}_{\Psi,\Omega_0,\Omega_1}$. Process the first of the following cases applicable.

Reminder. We are using the notation Ψ and Ω_i as shorthand for Ψ_e and $\Omega_{e,i}$ for some index e, and Γ_i as shorthand for $\Gamma_{\alpha,i}$.

Case 1.1. There is a number $z \in S(\alpha^\frown\langle 1\rangle, s)$ such that $z \notin A[s]$, but $z \in \Gamma_1^{\Omega_1^A}[s]$.

Then set $z(s+1,t+1) = z$ for the least such z, and define

$$D(s+1,t+1) = D(s+1,t) \cup \{z(s+1,t)\} \cup \Big(\bigcup_{1\le i\le 2} S\big(\alpha^\frown\langle i\rangle, s\big) \setminus \{z(s+1,t+1)\}\Big)$$

and $\Gamma_1[s+1] = \emptyset$. Also reset $\Gamma_0[s+1] = \Gamma_0[s]$. Set $R(\alpha, s+1) = 0$.

Remark $R(\alpha, s+1) = 0$ indicates that $\alpha^\frown\langle 0\rangle$ will be eligible to act at substage $t+2$. (See procedure *Ending substage* $t+1$.) Note that the floating witness $z(s+1,t+1)$ will be passed to $\alpha^\frown\langle 0\rangle$.

Case 1.2. There is a number $z \in S(\alpha^\frown\langle 2\rangle, s)$ such that $z \in A[s] \cap \Psi^{\Omega_0^A \oplus \Omega_1^A}[s]$.

Then set $z(s+1,t+1) = z$ for the least such z, define

$$D(s+1,t+1) = D(s+1,t) \cup \{z(s+1,t)\} \cup \big(S\big(\alpha^\frown\langle 2\rangle, s\big) \setminus \{z(s+1,t+1)\}\big),$$

and, for $0 \le i \le 1$, define $\Gamma_i[s+1] = \Gamma_i[s] \cup \{\langle z(s+1,t+1), \Omega_i^A[s]\rangle\}$.

Set $R(\alpha, s+1) = 1$.

Case 1.3. Otherwise.

Then reset $z(s+1,t+1) = z(s+1,t)$, $D(s+1,t+1) = D(s+1,t)$, $\Gamma_i[s+1] = \Gamma_i[s]$ for $0 \le i \le 1$, and set $R(\alpha, s+1) = 2$.

Case 2. $\alpha \in \mathcal{N}_W$. Process the first of the following cases applicable.

Case 2.1. $N(\alpha, s) = 0$.

(Note that this means that $x(\alpha, s) \in O_A[s] \subseteq \omega \setminus A[s]$.) Set $z(s+1,t+1) = z(s+1,t)$ and $D(s+1,t+1) = D(s+1,t)$, and reset $x(\alpha, s+1) = x(\alpha, s)$ and $N(\alpha, s+1) = 0$.

Case 2.2. $N(\alpha, s) = 1$ and $x(\alpha, s) \in W[s]$.

Set $z(s+1,t+1) =$ break and

$$D(s+1,t+1) = D(s+1,t) \cup \{z(s+1,t)\} \cup S\big(\alpha^\frown\langle 1\rangle, s\big).$$

(Note that $S(\alpha^\frown\langle 1\rangle, s) = S(\alpha, s) \setminus \{x(\alpha, s)\}$ in this case.) Reset $x(\alpha, s+1) = x(\alpha, s)$, and set $N(\alpha, s+1) = 0$.

Case 2.3. $N(\alpha, s) = 1$ and $x(\alpha, s) \notin W[s]$.

Reset $z(s+1,t+1) = z(s+1,t)$ and $D(s+1,t+1) = D(s+1,t)$. Also reset $x(\alpha, s+1) = x(\alpha, s)$ and $N(\alpha, s+1) = 1$.

Case 2.4. $N(\alpha, s) =$ void and $z(s+1,t) \ge |\alpha|$.

Set $z(s+1,t+1) =$ break and $D(s+1,t+1) = D(s+1,t)$. Also set $x(\alpha, s+1) = z(s+1,t)$ and $N(\alpha, s+1) = 1$.

Case 2.5. Otherwise (i.e., $N(\alpha, s) =$ void and $z(s+1,t) < |\alpha|$).

Set $z(s+1,t+1) = \text{break}$ and $D(s+1,t+1) = D(s+1,t) \cup \{z(s+1,t)\}$. Also reset $x(\alpha, s+1) = -1$ and $N(\alpha, s+1) = \text{void}$.

Case 3. $\alpha \in \mathcal{P}_W$. Process the first of the following cases applicable.

Notation For the sake of Cases 3.2 and 3.3, we use the notation

$$\begin{aligned}\Omega_{\alpha,s+1} = &\big\{x(\beta,s) \bigm| x(\beta,s) \geq 0 \;\&\; N(\beta,s) = 1 \;\&\; \beta <_L \alpha\big\}\\ &\cup \big\{x(\beta,s+1) \bigm| x(\beta,s+1) \geq 0 \;\&\; N(\beta,s+1) = 1 \;\&\; \beta \subset \alpha\big\}.\end{aligned}$$

(Note that $N(\beta, s+1) = 1$ and $\beta \subset \alpha$ implies that $\beta^\frown\langle 1\rangle \subseteq \alpha$.)

Case 3.1. $P(\alpha, s) = 0$.

(Note that the implication here is that there is some $D \in W[s]$ such that $D \subseteq A[s]$.) Reset $z(s+1,t+1) = z(s+1,t)$, $D(s+1,t+1) = D(s+1,t)$ and reset $P(\alpha, s+1) = 0$.

Case 3.2. $P(\alpha, s) = 1$ and for some $D \in W[s]$,

$$D \subseteq \Omega_{\alpha,s+1} \cup D_A[s] \cup D(s+1,t) \cup \big\{z(s+1,t)\big\} \cup S(\alpha,s). \tag{5.1}$$

(Note that $S(\alpha,s) = S(\alpha^\frown\langle 1\rangle, s)$ in this case.) Set $z(s+1,t+1) = \text{break}$,

$$D(s+1,t+1) = D(s+1,t) \cup \big\{z(s+1,t)\big\} \cup S(\alpha,s),$$

and set $P(\alpha, s+1) = 0$.

Case 3.3. Otherwise (i.e., $P(\alpha,s) = \text{void}$ or $P(\alpha,s) = 1$ and (5.1) holds for no $D \in W[s]$.) Reset $z(s+1,t+1) = z(s+1,t)$, $D(s+1,t+1) = D(s+1,t)$, and set $P(\alpha, s+1) = 1$.

Ending substage $t+1$. Supposing that $\alpha \in \mathcal{Q}$ with $\mathcal{Q} \in \{\mathcal{R}, \mathcal{N}, \mathcal{P}\}$, if $z(s+1,t+1) \in \omega$, then define $\alpha^\frown\langle Q(\alpha,s+1)\rangle$ to be *eligible to act next* and go to substage $t+2$. Otherwise (i.e., if $z(s+1,t+1) = \text{break}$) go to *End of Stage $s+1$*.

Remark The last node eligible to act, and hence processed, at stage $s+1$ is either an N node via Case 2.2, 2.4, or 2.5 or otherwise a P node via Case 3.2.

End of stage $s+1$. Supposing that α of length t is the last strategy to be processed, define $\beta_{s+1} = \alpha$. Set $D(s+1) = D(s+1,t+1)$, and initialize all nodes in the set $G = \{\beta \mid \alpha < \beta\}$ (i.e., all nodes β such that $\alpha <_L \beta$ or $\alpha \subset \beta$). For every $\beta \in \mathcal{T}$ such that $\beta <_L \alpha$, reset β's parameters for stage $s+1$ to their value at stage s. Before proceeding note that, by initialization, for any $\beta \in \mathcal{N}$ such that $N(\beta, s+1) \in \{0,1\}$, $\beta \leq \alpha$. Define

$$\begin{aligned}I_A[s+1] &= \big\{x(\beta,s+1) \bigm| x(\beta,s+1) \geq 0 \;\&\; N(\beta,s+1) = 1\big\},\\ O_A[s+1] &= \big\{x(\beta,s+1) \bigm| x(\beta,s+1) \geq 0 \;\&\; N(\beta,s+1) = 0\big\},\\ F_A[s+1] &= I_A[s+1] \cup O_A[s+1],\\ D_A[s+1] &= D_A[s] \cup D(s+1),\end{aligned}$$

and

$$A[s+1] = I_A[s+1] \cup D_A[s+1].$$

(And note that $F_A[s+1] = \{x(\beta,s+1) \mid x(\beta,s+1) \geq 0\}$.) For every $\gamma \in \mathcal{T}$, redefine the stream for γ as follows: $S_A(\gamma, s+1) = \{x(\beta,s+1) \mid x(\beta,s+1) \in F_A[s+1] \;\&\; \gamma \subseteq \beta\}$.

Note that by resetting, if $\gamma <_L \alpha$, then $S(\gamma, s+1) = S(\gamma, s)$ whereas, by initialization, if $\alpha < \gamma$, then $S(\gamma, s+1) = \emptyset$.

Go to stage $s+2$.

Verification

The following informal observations clarify the mechanics of the construction and underline its inherent simplicity.

Some properties of stage $s+1$.

(i) $F_A[s+1]$ comprises precisely the set of witnesses $x(\gamma, s+1) \geq 0$ such that $\gamma \leq \beta_{s+1}$.

(ii) At most one number is removed from A at stage $s+1$. Indeed, this can only happen if Case 2.2 applies at substage $|\beta_{s+1}|+1$ and the witness $x = x(\beta_{s+1}, s)$ is extracted from A.[6]

(iii) $F_A[s+1] \setminus F_A[s] \subseteq \{s\}$. And if indeed $s \in F_A[s+1]$, then Case 2.4 applies at substage $|\beta_{s+1}|+1$ and $s = x(\beta_{s+1}, s+1)$. This also means that the floating witness $z(s+1, t)$ never changes value (i.e., $z(s+1,t) = s$ for all $0 \leq t \leq |\beta_{s+1}|$).

(iv) If $z(s+1, |\beta_{s+1}|) \neq s$ (i.e., if the floating witness changes value at least once), then $z(s+1, |\beta_{s+1}|) = x(\gamma, s)$ for some $\beta_{s+1} <_L \gamma$. Likewise, each intermediate value of the floating witness $z(s+1, t)$ is a witness $x(\gamma', s)$ for some $\beta_{s+1} <_L \gamma'$. Moreover, the only one of the values of the floating witness that (possibly) remains in $F_A[s+1]$ is $x(\gamma, s)$. Note that this happens if Case 2.4 applies at substage $|\beta_{s+1}|+1$ forcing $x(\beta_{s+1}, s+1) = x(\gamma, s)$. All other values (including s) of $z(s+1, t)$ are dumped into A.

(v) Nontrivial cases of (ii), (iii), and (iv) are mutually exclusive. In other words, extraction of a number from A (see (ii)) forces s and all $x(\gamma, s) \geq 0$ such that $\beta_{s+1} < \gamma$ to be dumped into A. On the other hand, $s \in F_A[s+1]$ (see (iii)) precludes any extraction from A and forces all $x(\gamma, s) \geq 0$ such that $\beta_{s+1} < \gamma$ to be dumped into A. Likewise, $x(\gamma, s) \in F_A[s+1]$ for some $\beta_{s+1} <_L \gamma$ (see (iv)) precludes any extraction from A and forces s (as well as all other $x(\hat{\gamma}, s) \geq 0$ such that $\beta_{s+1} < \hat{\gamma}$) to be dumped into A.

(vi) β_{s+1} is either in $\mathcal{N}$ and Case 2.2, 2.4, or 2.5 applies at substage $|\beta_{s+1}|+1$; otherwise β_{s+1} is in $\mathcal{P}$ and Case 3.2 applies at substage $|\beta_{s+1}|+1$.

We now verify the construction via the lemmas below. Note first that Lemmas 5.4–5.6 are proved by inspection (only for some of the statements involved) and straightforward induction arguments over the stages of the construction using the observations above. (Detailed proofs are given in Badillo [2].)

Lemma 5.4 *For all stages $s > 0$ and $x \in F_A[s]$, both (1) and (2) are true.*

(1) *One of the three following (mutually exclusive) cases applies for x:*

  (a) $x = s - 1$, $\beta_s \in \mathcal{N}$, *and* $x = x(\beta_s, s)$;

  (b) *there exists* $\gamma \in \mathcal{N}$ *such that* $\gamma \leq \beta_{s-1}$, $\beta_s <_L \gamma$, $x = x(\gamma, s-1)$, $x(\gamma, s) = \text{void}$, *and* $x(\beta_s, s) = x$;

  (c) *there exists* $\gamma \in \mathcal{N}$ *such that* $\gamma \leq \beta_{s-1}$, $\gamma \leq \beta_s$, *and* $x = x(\gamma, s-1) = x(\gamma, s)$.

(2) *For all* $\gamma_1, \gamma_2 \in \mathcal{N}$ *such that* $x = x(\gamma_1, s) = x(\gamma_2, s)$, $\gamma_1 = \gamma_2$.

Remark By Lemma 5.4 and the definition of $F_A[s]$, we can now assume that $x \in F_A[s]$ if and only if there exists a unique (N strategy) $\gamma \leq \beta_s$ such that

$x = x(\gamma, s)$. Clearly also in this case for $(L, i) \in \{(I, 1), (O, 0)\}$, we have that $x \in L_A[s]$ if and only if $N(\gamma, s) = i$.

Lemma 5.5 *For all $s \geq 0$, the following statements are true:*

(1) $D(s) \subseteq D_A[s] \subseteq D_A[s+1]$,
(2) $F_A[s] = I_A[s] \cup O_A[s]$ *and* $I_A[s] \cap O_A[s] = \emptyset$,
(3) $D_A[s] \cap F_A[s] = \emptyset$,
(4) $\{n \mid 0 \leq n < s\} = F_A[s] \cup D_A[s]$,
(5) *for any* $\alpha \in \mathcal{T}$ *such that* $\alpha \subseteq \beta_s$,

$$F_A[s] = S(\alpha, s) \cup \{x(\gamma, s) \mid x(\gamma, s) \geq 0 \ \&\ \gamma < \alpha\}.$$

Lemma 5.6 *Suppose that $\beta \in \mathcal{T}$ is such that $x(\beta, s) \geq 0$. Then for all $\gamma \subseteq \beta$ such that $\gamma \in \mathcal{N}$, $N(\gamma, s) \in \{0, 1\}$ and $x(\gamma, s) \geq 0$.*

Lemma 5.7 *For any $\alpha \in \mathcal{T}$ and stage $s \geq 0$, $|S(\alpha, s+1) \setminus S(\alpha, s)| \leq 1$.*

Proof This follows by inspection of the construction at stage $s+1$. Indeed, if $z \in S(\alpha, s+1) \setminus S(\alpha, s)$, then for some substage t of stage $s+1$, $z(s+1, t) = z$. However, at most one such z survives without being dumped into $D(s+1)$. (And in this case $z = x(\beta_{s+1}, s+1)$.) □

Lemma 5.8 *For all stages $s \geq 0$ and any strategies $\alpha, \beta \in \mathcal{T}$ such that $S(\alpha, s) \neq \emptyset$ and $S(\beta, s) \neq \emptyset$, if $\alpha <_L \beta$, then* $\max S(\alpha, s) < \min S(\beta, s)$.

Proof This follows by induction over stages $s \geq 0$. The case $s = 0$ is trivially true. So consider case $s+1$. For the hypotheses of the lemma to be true at stage $s+1$, it must be the case that $\beta \leq \beta_{s+1}$ (otherwise, $S(\beta, s+1) = \emptyset$). If $\beta <_L \beta_{s+1}$, then $S(\alpha, s+1) = S(\alpha, s)$ and $S(\beta, s+1) = S(\beta, s)$ and the result follows by the induction hypothesis. Otherwise $\beta \subseteq \beta_{s+1}$. As seen in Lemma 5.7, if $D = S(\beta, s+1) \setminus S(\beta, s)$, then $|D| \leq 1$. If $|D| = 0$, then the result follows as above. Otherwise, suppose that z is the number contained in D. Then either $z = s$ and so $z > \max S(\alpha, s) \subseteq \{n \mid n < s\}$ or $z \in S(\gamma, s)$ for some $\beta <_L \gamma$ (via Case 1.1 or 1.2 applied at some substage $1 \leq t \leq |\beta|$ of stage $s+1$), in which case $z > \max S(\beta, s) > \max S(\alpha, s)$ by application of the induction hypothesis. □

From an inspection of Lemma 5.8 and its proof, we have the following corollary.

Corollary 5.9 *For any stage $s \geq 0$, strategy $\alpha \in \mathcal{T}$, and number z, if $z \in S(\alpha, s+1) \setminus S(\alpha, s)$, then* $z > \max S(\alpha, s)$.

Lemma 5.10 *For all $x, y \in \omega$, stages $0 \leq s < t$, and nodes $\alpha \in \mathcal{T}$, if $x \in S(\alpha, s) \cap I_A[s]$, $y \in S(\alpha, s+1) \setminus S(\alpha, s)$, and $\{x, y\} \subseteq S(\alpha, t)$, then $x \in I_A[t]$.*

Remark 1 Less formally, Lemma 5.10 says that if y enters a stream to which x already belongs as well as already belonging to A (at this point in the construction),[7] then for as long as both x and y remain in the stream, x remains in A.

Remark 2 Note that $x < y$ by Corollary 5.9.

Proof We reason by induction over stages $t \geq s+1$.

Case $t = s+1$. By inspection of the construction, we see that $y = x(\beta_{s+1}, s+1)$. Let $\beta \in \mathcal{N}$ be such that $x = x(\beta, s+1)$. From Lemma 5.6 and the definition of Case 2.4 of the construction, we can deduce that it is not the case that $\beta_{s+1} \subseteq \beta$. Moreover, $\beta_{s+1} \not<_L \beta$, since then β would be initialized at stage $s+1$ forcing

$x \in D(s+1) \subseteq D_A[s+1]$ and hence $x \notin S(\alpha, s+1) \subseteq F_A[s+1]$ by Lemma 5.5(3). Thus there are two subcases as follows.

Subcase $\beta <_L \beta_{s+1}$. Then $x = x(\beta, s+1)$ by Lemma 5.4(1)(c) and $N(\beta, s+1) = N(\beta, s)$ by resetting. Hence $x \in I_A[s+1]$ by definition.

Subcase $\beta \subset \beta_{s+1}$. Then, as above, $x = x(\beta, s+1)$. Moreover, note that if $x \in O_A[s+1]$, then Case 2.2 applies at substage $|\beta| + 1$ forcing $\beta_{s+1} = \beta$, which is a contradiction. Hence $x \in I_A[s+1]$.

Case $t > s+1$. We assume by the *extended induction hypothesis* that not only does the lemma hold for stage $t-1$, but also that the nodes $\beta, \gamma \in \mathcal{N}$ such that $x(\beta, t-1) = x$ and $x(\gamma, t-1) = y$ satisfy $\beta < \gamma$. (Note that we have already seen that the extended induction hypothesis is true when $t-1 = s+1$.) Again, we reason by subcases.

Subcase $\beta_t < \beta$. Note that $\beta_t \subset \beta$ can only happen via Case 3.2 of the construction, in which case β is initialized forcing $x \in D(t)$. So we can suppose that $\beta_t <_L \beta$. However, in this case, there is at most one strategy $\beta_t <_L \mu$ such that $x(\mu, t-1)$ is not forced into $D(t)$ by initialization.[8] However, $\beta_t <_L \beta < \gamma$, and we have $\{x, y\} \cap D(t) = \emptyset$ by hypothesis, which is a contradiction. Thus $\beta_t < \beta$ does not happen.

Remark We can now assume that $\beta \leq \beta_t$ and, by Lemma 5.4, that $x(\beta, t) = x(\beta, t-1)$.

Subcase $\beta_t < \gamma$. As above, we can suppose that $\beta_t <_L \gamma$. As $y = x(\gamma, t-1)$, for y to survive in $S(\alpha, t) \subseteq F_A[t]$ it must be the case that $y = x(\beta_t, t)$ (since otherwise $y \in D(t)$) via Case 2.4 applied to $z(t, |\beta_t|) = y$ at substage $|\beta_t| + 1$. Thus Case 2.2 does not occur at any substage of stage t.[9] In particular (under the inductive assumption that $x \in I_A[t-1]$), this means that $x \in I_A[t]$. Also $\beta_t \neq \beta$ (as $x(\beta_t, t-1) =$ void by definition of Case 2.4). Hence $\beta < \beta_t$.

Subcase $\beta_t \geq \gamma$. In this subcase, Case 2.2 of the construction does not apply to node β during stage t since this would force $\beta_t = \beta < \gamma$. Moreover, $x(\beta, t) = x(\beta, t-1) = x$ and $x(\gamma, t) = x(\gamma, t-1) = y$ (by Lemma 5.4(1)(c)). Combining these two observations, we see that $x \in I_A[t]$ and that the extended induction hypothesis is again satisfied. □

Notation, assumptions, and definitions. For $n \geq 0$, we define

$$\mathrm{True}_{\infty,n} = \{\alpha \mid |\alpha| = n \;\&\; \forall t (\exists s \geq t)[\alpha \subseteq \beta_s]\}.$$

If $\mathrm{True}_{\infty,n} \neq \emptyset$, and letting $\beta = \min_{<_L} \mathrm{True}_{\infty,n}$ (i.e., the least strategy of length n under $<_L$), we define $\delta_n = \beta$ if there exists s_β such that, for all $s \geq s_\beta$, β is not initialized at stage s.[10] Otherwise, δ_n is undefined.

For any $\gamma \in \mathcal{T}$ and parameter $p(\gamma, s)$, if $\lim_{s\to\infty} p(\gamma, s)$ exists, then we define $p(\gamma)$ to be this value (otherwise we say that $p(\gamma)$ is *undefined*). We define

$$D_A = \bigcup_{s\in\omega} D_A[s],$$
$$F_A = \{n \mid \exists s (\forall t \geq s)[n \in F_A[t]]\},$$

and we define I_A and O_A likewise (so that $F_A = I_A \cup O_A$). Define

$$A = \{n \mid \exists s (\forall t \geq s)[n \in A[t]]\}.$$

Also, for all $\alpha \in \mathcal{T}$, define

$$S(\alpha) = \big\{n \bigm| \exists s(\forall t \geq s)\big[n \in S(\alpha, t)\big]\big\}.$$

Lemma 5.11 *For all $n \geq 0$, δ_n is defined.*

Proof This is by induction on n. The case $n = 0$ is obvious. So suppose that $\alpha = \delta_n$ is defined, and let s_n be a stage such that $\beta_s \geq \alpha$ for all $s \geq s_n$. There are three cases to consider.

Case $\alpha \in \mathcal{R}$. By construction, at each α-true stage s, $|\beta_s| > \alpha$. Hence, by the induction hypothesis, $\beta^\frown\langle i\rangle \in \mathrm{True}_{\infty,n+1}$ for some $i \in \{0, 1, 2\}$. Thus δ_{n+1} is defined.

Case $\alpha \in \mathcal{N}$. Inspection of the construction shows that, for any α-true stage $s > 0$, if $m = z(s, |\alpha|)$, then either $m = s - 1$ or $m = x(\gamma, s - 1)$ for some $\alpha <_L \gamma$, whereas for all $t \geq s$, either $m \in D_A[t]$ or $m = x(\beta, t)$ for some $\alpha \subseteq \beta$.[11] It follows that for all α-true stages $s_n < p < r$, $z(p, |\alpha|) \neq z(r, |\alpha|)$ (and in fact $z(p, |\alpha|) < z(r, |\alpha|)$). Hence at one such α-true stage s (if $N(\alpha, s-1) = \text{void}$), Case 2.4 of the construction will apply, so that $x(\alpha, s) = z(s, |\alpha|)$. Moreover, clearly for all $t \geq s$, $x(\alpha, t) = x(\alpha, s)$. Note also that Case 2.2 can apply at most once after stage s. In other words, there is a stage s' such that at every α-true stage $t \geq s'$, $|\beta_t| > |\alpha|$. Thus (as in the first case) δ_{n+1} is defined to be $\alpha^\frown\langle i\rangle$ for some $i \in \{0, 1\}$.

Case $\alpha \in \mathcal{P}$. Clearly Case 3.2 applies at most once after stage s_n. Thus, as above, δ_{n+1} is defined to be $\alpha^\frown\langle i\rangle$ for some $i \in \{0, 1\}$.

Note that to each case there corresponds a stage s_{n+1} as in the induction hypothesis. Thus the latter is validated. This concludes the proof of the lemma. □

Corollary 5.12 *For all $n \geq 0$, $S(\delta_n)$ is infinite.*

Proof It follows from Lemma 5.11 that, for all n such that $\delta_n \in \mathcal{N}$, $x(\delta_n)$ is defined (with value in ω). Moreover, a straightforward argument by induction using Lemma 5.4(2) implies that, for all such $p \neq m$, $x(\delta_p) \neq x(\delta_m)$. It now suffices to note that $\{x(\delta_m) \mid \delta_m \in \mathcal{N} \;\&\; m > n\} \subseteq S(\delta_n)$. □

Lemma 5.13 *The following statements are true:*

(1) $A = D_A \cup I_A$,
(2) $F_A = I_A \cup O_A$ *and* $I_A \cap O_A = \emptyset$,
(3) $D_A \cap F_A = \emptyset$,
(4) $\omega = F_A \cup D_A$,
(5) *for any* $\alpha \in \mathcal{T}$ *such that* $\alpha \subseteq \delta$,

$$F_A = S(\alpha) \cup \big\{x(\gamma) \bigm| x(\gamma) \geq 0 \;\&\; \gamma < \alpha\big\}.$$

Proof Statements (1) and (2) are obvious by definition, whereas (3), (4), and (5) follow by application of Lemma 5.5 using induction over the stages of the construction. □

Notation For $G \in \{F, I, O\}$ and $\alpha \in \mathcal{T}$, we use the notation $G_A^{<\alpha}[s]$ to denote the set $G_A[s] \cap \{x(\gamma, s) \mid \beta < \alpha\}$.

Lemma 5.14 *For $G \in \{F, I, O\}$, any $\alpha \subseteq \delta$ and stage s_α such that $\alpha \leq \beta_s$ for all $s \geq s_\alpha$, $G_A^{<\alpha}[s] = G_A^{<\alpha}[s_\alpha]$.*

Proof The proof is by a straightforward induction over $s \geq s_\alpha$. □

By definition of $\mathcal{T}$ and δ, for any requirement Q there is precisely one strategy α associated with Q such that $\alpha \subseteq \delta$. Accordingly, we consider each such α by cases.

Lemma 5.15 $\alpha \in \mathcal{R}_{\Psi,\Omega_0,\Omega_1}$. *If* $A = \Psi^{\Omega_0^A \oplus \Omega_1^A}$ *and* A *is not c.e., then* $A \leq_{\mathrm{e}} \Omega_i^A$ *for some* $i \in \{0, 1\}$.

Proof Define $\Lambda = \{\langle z, \emptyset\rangle \mid z \in D_A\}$. There are three cases to consider.

Case $\alpha^\frown\langle 2\rangle \subset \delta$. Consider $x \in S(\alpha^\frown\langle 2\rangle)$. Clearly $x \notin A$. Indeed, it cannot be the case that $x \in A \cap \Psi^{\Omega_0^A \oplus \Omega_1^A}$, since then x would have been removed from $\alpha^\frown\langle 2\rangle$'s stream via Case 1.2 of the construction. Moreover, if $x \in A \setminus \Psi^{\Omega_0^A \oplus \Omega_1^A}$, then $A \neq \Psi^{\Omega_0^A \oplus \Omega_1^A}$, which is a contradiction. We see therefore that $\alpha^\frown\langle 2\rangle \subseteq \delta$ implies that $A =^* D_A$, that is, that A is c.e. Hence $\alpha^\frown\langle 2\rangle \subset \delta$ cannot apply (under the assumptions of the lemma).

Case $\alpha^\frown\langle 1\rangle \subset \delta$. Consider $x \in S(\alpha^\frown\langle 1\rangle)$. By construction, there exists a unique stage s_x and, for $i \in \{0, 1\}$, a unique axiom $\langle x, F_{i,x}\rangle$ such that $F_{i,x} = \Omega_i^A[s_x]$ was enumerated into Γ_i at stage $s_x + 1$ via Case 1.2 of the construction. Now, it follows from Lemma 5.8, Corollary 5.9, and the dumping activity at stage $s_x + 1$ that $\{z \mid z > x \;\&\; z \in A[s_x]\} \subseteq D_A \subseteq A$. On the other hand, we can also deduce from Lemma 5.14, Lemma 5.10, and the dumping activity at stage $s_x + 1$ that $\{z \mid z < x \;\&\; z \in A[s_x]\} \subseteq A$. Notice now that these observations imply that, for each $i \in \{0, 1\}$,

$$F_{i,x} \subseteq \Omega_i^{A \cup \{x\}}, \tag{5.2}$$

whereas the definition of Case 1.2 implies that

$$x \in \Psi^{F_{0,x} \oplus F_{1,x}}. \tag{5.3}$$

- Suppose that $x \in A$. Then by (5.2), $F_{1,x} \subseteq \Omega_1^A$, and so $x \in \Gamma_1^{\Omega_1^A}$.
- Now suppose that $x \notin A$. Then $x \notin \Gamma_1^{\Omega_1^A}$. Indeed $x \in \Gamma_1^{\Omega_1^A}$ would imply the transfer of x from $\alpha^\frown\langle 1\rangle$'s stream to $\alpha^\frown\langle 0\rangle$'s stream at some stage $s > s_x$ (via Case 1.1).

We see therefore that $\alpha^\frown\langle 1\rangle \subseteq \delta$ implies (by Lemma 5.13) that $A =^* \Phi_1^{\Omega_1^A}$, where $\Phi_1 = \Gamma_1 \cup \Lambda$.

Case $\alpha^\frown\langle 0\rangle \subset \delta$. Consider $x \in S(\alpha^\frown\langle 0\rangle)$, and (for $i \in \{0, 1\}$) let s_x and $F_{i,x}$ be defined as above. Also let $t_x + 1$ be the stage at which the application of Case 1.1 caused x to be transferred from $\alpha^\frown\langle 1\rangle$'s stream to $\alpha^\frown\langle 0\rangle$'s stream. Note that, by an argument similar to the one used in the last case, it follows from Lemmas 5.8, 5.10, and 5.14, Corollary 5.9, and the dumping activity at stage $t_x + 1$ that

$$F_{1,x} \subseteq \Omega_1^A \tag{5.4}$$

(i.e., whether or not $x \in A$).

- Suppose that $x \in A$. Then by (5.2), $F_{0,x} \subseteq \Omega_0^A$, and so $x \in \Gamma_0^{\Omega_0^A}$.
- Now suppose that $x \notin A$. Then $x \notin \Gamma_0^{\Omega_0^A}$. Indeed, if $x \in \Gamma_0^{\Omega_0^A}$, then $F_{0,x} \subseteq \Omega_0^A$. However, by (5.4), $F_{1,x} \subseteq \Omega_1^A$ and by (5.3) $x \in \Psi^{F_{0,x} \oplus F_{1,x}}$. Thus $x \in \Psi^{\Omega_0^A \oplus \Omega_1^A} \setminus A$, which is a contradiction.

We see therefore that $\alpha^\frown\langle 0\rangle \subseteq \delta$ implies (by Lemma 5.13) that $A =^* \Phi_0^{\Omega_0^A}$, where $\Phi_0 = \Gamma_0 \cup \Lambda$. $\square$

Lemma 5.16 *$\alpha \in \mathcal{N}_W$. Then $x(\alpha) \in A$ if and only if $x(\alpha) \notin W$.*

Proof Inspection of the construction shows that if $\alpha^\frown\langle 1\rangle \subseteq \delta$, then $x(\alpha) \in A \setminus W$, whereas if $\alpha^\frown\langle 0\rangle \subseteq \delta$, then $x(\alpha) \in W \setminus A$. □

Notation For $G \in \{F, I, O\}$ and $\alpha \subseteq \delta$, we define (on the strength of Lemma 5.14) $G_A^{<\alpha} = \lim_{s\to\infty} G_A^{<\alpha}[s]$.

Note that, for any $\alpha \subseteq \delta$, $O_A^{<\alpha} \subseteq \overline{A}$.

Lemma 5.17 *$\alpha \in \mathcal{P}_W$. Let $E = O_A^{<\alpha}$. If there is no $D \in W$ such that $D \subseteq A$, then for all $D \in W$, $D \cap E \neq \emptyset$.*

Proof Let s_α be the least stage such that $\beta_s \geq \alpha$ for all $s \geq s_\alpha$.

Case $\alpha^\frown\langle 0\rangle \subseteq \delta$. Then Case 3.2 is applied relative to α at some stage $s \geq s_\alpha$, and it follows by Lemma 5.14 and the dumping activity at stage s that there is a finite set $D \in W$ such that $D \subseteq A$.

Case $\alpha^\frown\langle 1\rangle \subseteq \delta$. Then Case 3.2 applies at no stage $s \geq s_\alpha$, and we can deduce from Lemmas 5.5 and 5.13 in conjunction with Lemma 5.14 that $D \cap E \neq \emptyset$ for all $D \in W$. □

Lemma 5.18 *All the requirements are satisfied.*

Proof For the N and P requirements, this is immediate by Lemmas 5.16 and 5.17. Satisfaction of each R requirement follows from the conjunction of Lemma 5.15 with the fact that all the N requirements are satisfied (and hence A is not c.e.). □

Lemma 5.19 *A is low.*

Proof Consider $n \in \omega$. Note that, by construction, n can only be extracted from A by N strategies of length at most n and, moreover, that each such strategy extracts n at most once. It follows that n can be extracted from A at most $2^n + 1$ times. Since this is true for all $n \in \omega$, the construction defines a Δ_2^0 approximation to A. Since A is also enumeration 1-generic, A is low (by Corollary 3.11). □

We can now conclude the proof of Theorem 5.3 by setting $a = \deg_e(A)$. □

Theorem 5.20 *There exists a properly Σ_2^0 enumeration 1-generic nonsplittable degree.*

Proof We proceed as in the proof of Theorem 5.3, except that we replace requirements N_W by requirements of the form

$$N_{B,\Phi,\Psi}: \quad B = \Phi^A \ \&\ A = \Psi^B \Rightarrow (\exists x \in B)\big[\lim_{s\to\infty} B_s(x)\uparrow\big],$$

where $(B, \Phi, \Psi) \in \{(B_e, \Phi_e, \Psi_e)\}_{e\in\omega}$ and where the latter is a standard listing of triples of Σ_2^0 sets and enumeration operators with associated uniform approximations (Σ_2^0 for the sets B_e and c.e. for the Φ and Ψ operators) under the standard proviso that for every Δ_2^0 set C, there is an index i such that $\{B_{i,s}\}_{s\in\omega}$ is a Δ_2^0 approximation to C.

Strategy for node $\alpha \in \mathcal{N}_{B,\Phi,\Psi}$ (Cooper and Copestake [5])

In the following outline, we use z for the momentary value of the floating witness passed to α (so that the value of z in (1) below is different from its value in (2)), $S(\alpha)$ for the momentary value of α's stream, and so on. Each strategy α will have two parameters whose roles are as a witness $x(\alpha)$ and an oracle witness $F(\alpha)$.

(1) Set $x(\alpha) = z$, and put $x(\alpha)$ in A.
(2) Wait for a minimal finite set F such that

$$\big\langle x(\alpha), F\big\rangle \in \Psi \qquad \text{and} \qquad F \subseteq B \cap \Phi^{A \cup S(\alpha) \cup \{z\}}.$$

(3) Set $F(\alpha) = F$, and dump $(S(\alpha) \setminus \{x(\alpha)\}) \cup \{z\}$ into A.[12]
(4) Remove $x(\alpha)$ from A.
(5) Wait for $x(\alpha) \notin \Psi^B$.
(6) Put $x(\alpha)$ into A.
(7) Wait for $F(\alpha) \subseteq B$.
(8) Go back to Step (4).

Finitary outcomes. There are three finitary outcomes to the strategy, each corresponding to either $A \neq \Psi^B$ or $B \neq \Phi^A$.

(i) The strategy gets stuck at Step 2. In this case $x(\alpha) \in A$ and either $x(\alpha) \notin \Psi^B$ (so that $x(\alpha) \in A \setminus \Psi^B$) or otherwise $x(\alpha) \in \Psi^B$, but for every finite set F such that $\langle x(\alpha), F\rangle \in \Psi$ and $F \subseteq B$, $F \nsubseteq \Phi^A$ (so that, for some d, $d \in B \setminus \Phi^A$).

(ii) The strategy gets stuck at Step 5. In this case $x(\alpha) \in \Psi^B \setminus A$.

(iii) The strategy gets stuck at Step 7. In this case $F(\alpha) \subseteq \Phi^A$, so there is some $d \in F(\alpha)$ such that $d \in \Phi^A \setminus B$.

Infinitary outcome. There is one infinitary outcome as follows.

(iv) The strategy loops through Step 4 via Step 8 infinitely often. In this case, during each loop the passage from Step 5 to 6 corresponds to some $x \in F(\alpha)$ having left B, and the passage from Step 7 to 8 corresponds to x having reentered B. Thus there is some $x \in D(\alpha)$ such that $\lim_{s\to\infty} B_s(x)\uparrow$.

In the *tree of strategies* (see page 473), for $\alpha \in \mathcal{N}$, we define $\alpha^\frown\langle n\rangle \in \mathcal{T}$ for all $n \in \{0, 1, 2\}$.

The strategy $\alpha \in \mathcal{N}_{B,\Psi,\Phi}$ has several local parameters as follows. $N(\alpha, s) \in \{0, 1, 2, \text{void}\}$ is the *outcome* parameter, $x(\alpha, s) \in \{-1\} \cup \omega$ is the *witness* parameter, and $F(\alpha, s) \in \{-1\} \cup \mathcal{F}$ the *oracle witness* parameter associated with α. Strategy α also has a *pause switch* parameter $p(\alpha, s) \in \{\text{continue}, \text{pause}\}$. When α is in its initialized state, $N(\alpha, s) = \text{void}$, $x(\alpha, s) = F(\alpha, s) = -1$, and $p(\alpha, s) = \text{continue}$. The temporary parameter $l(\alpha, s+1)$ is defined if $s+1$ is α-true and denotes the last α-true stage $t \leq s$, provided that α has not been initialized at any stage $t < r \leq s$. If there is no such stage, then $l(\alpha, s+1) = 0$. Strategy α works with its own relativized approximation $\{B[\alpha, s]\}_{s\in\omega}$ defined in a similar way to approximations used in [13], as follows. For all $s \in \omega$, set

$$B[\alpha, s] = \begin{cases} \emptyset & \text{if } s = 0, \\ B[\alpha, s-1] & \text{if } s > 0 \text{ and } s \text{ is not } \alpha\text{-true}, \\ \bigcap_{l(\alpha,s)\leq t<s} B_t & \text{if } s > 0 \text{ and } s \text{ is } \alpha\text{-true}, \end{cases} \tag{5.5}$$

and (for $s > 0$), define

$$\Psi^B[\alpha, s] = \big\{x \;\big|\; \exists D\big[\langle x, D\rangle \in \Psi[s-1] \;\&\; D \subseteq B[\alpha, s]\big]\big\}.$$

At substage $t+1$ of stage $s+1$ of the construction, *Case 2* of the construction on page 475 is replaced by the following.

<u>*Case 2.*</u> $\alpha \in \mathcal{N}_{B,\Phi,\Psi}$. Process the first of the following cases applicable.

Remark For clarity, notes have been added below following each case. In these notes we use the shorthand $x(\alpha)$ to denote $x(\alpha, s+1)$ provided that

$x(\alpha, s+1) = x(\alpha, s)$, and z to denote $z(s+1, t+1)$ provided that $z(s+1, t+1) = z(s+1, t)$. Moreover for $0 \leq n \leq 8$, "Step n" refers to the strategy for α described above.

<u>*Case 2.1.*</u> $N(\alpha, s) = 0$, $p(\alpha, s) =$ continue, and $x(\alpha, s) \in \Psi^B[\alpha, s+1]$.

Set $z(s+1, t+1) = z(s+1, t)$, $D(s+1, t+1) = D(s+1, t)$. For $q \in \{N, p, x, F\}$, reset $q(\alpha, s+1) = q(\alpha, s)$.

Note. The strategy $\alpha^\frown\langle 0\rangle$ is eligible to act at substage $t+2$ and floating witness z is passed to $\alpha^\frown\langle 0\rangle$. Then $N(\alpha, s+1) = N(\alpha, s) = 0$ means that $x(\alpha)$ remains outside A. This case corresponds to waiting at Step 5.

<u>*Case 2.2.*</u> $N(\alpha, s) = 0$, $p(\alpha, s) =$ continue, and $x(\alpha, s) \notin \Psi^B[\alpha, s+1]$.

Set $z(s+1, t+1) = z(s+1, t)$, $D(s+1, t+1) = D(s+1, t)$. Set $N(\alpha, s+1) = 1$, and for $q \in \{p, x, F\}$, reset $q(\alpha, s+1) = q(\alpha, s)$.

Note. The strategy $\alpha^\frown\langle 1\rangle$ is eligible to act at substage $t+2$ and floating witness z is passed to $\alpha^\frown\langle 1\rangle$. Then $N(\alpha, s) = 0$ and $N(\alpha, s+1) = 1$ means that $x(\alpha)$ is put back into A. This case corresponds to moving from Step 5 to Step 7.

<u>*Case 2.3.*</u> $N(\alpha, s) = 0$ and $p(\alpha, s) =$ pause.

Set $z(s+1, t+1) = z(s+1, t)$, $D(s+1, t+1) = D(s+1, t)$. Set $p(\alpha, s+1) =$ continue, and for $q \in \{N, x, F\}$, reset $q(\alpha, s+1) = q(\alpha, s)$.

Note. The strategy $\alpha^\frown\langle 0\rangle$ is eligible to act at substage $t+2$ and floating witness z is passed to $\alpha^\frown\langle 0\rangle$. Then $N(\alpha, s+1) = N(\alpha, s) = 0$ means that $x(\alpha)$ remains outside A. In this case, α's strategy was paused at Step 4 at the previous α-true stage, but now resumes and moves to Step 5.

<u>*Case 2.4.*</u> $N(\alpha, s) = 1$ and $F(\alpha, s) \subseteq B[\alpha, s+1]$.

Set $z(s+1, t+1) =$ break and

$$D(s+1, t+1) = D(s+1, t) \cup S\big(\alpha^\frown\langle 1\rangle, s\big) \cup \big\{z(s+1, t)\big\}.$$

(Note that $S(\alpha^\frown\langle 2\rangle, s) = \emptyset$ in this case.) Set $N(\alpha, s+1) = 0$, $p(\alpha, s+1) =$ pause, and for $q \in \{x, F\}$, reset $q(\alpha, s+1) = q(\alpha, s)$.

Note. The strategy $\alpha = \beta_{s+1}$ and $S(\alpha^\frown\langle 1\rangle, s) \cup \{z(s+1, t)\}$ is dumped into A. Then $N(\alpha, s) = 1$ and $N(\alpha, s+1) = 0$ means that $x(\alpha)$ is removed from A. In this case, α's strategy moved from Step 7 via Step 8 to Step 4 and its processing is paused.

<u>*Case 2.5.*</u> $N(\alpha, s) = 1$ and $F(\alpha, s) \not\subseteq B[\alpha, s+1]$.

Set $z(s+1, t+1) = z(s+1, t)$, $D(s+1, t+1) = D(s+1, t)$. For $q \in \{N, p, x, F\}$, reset $q(\alpha, s+1) = q(\alpha, s)$.

Note. The strategy $\alpha^\frown\langle 1\rangle$ is eligible to act at substage $t+2$ and floating witness z is passed to $\alpha^\frown\langle 1\rangle$. Then $N(\alpha, s+1) = N(\alpha, s) = 1$ means that $x(\alpha)$ remains inside A. This case corresponds to waiting at Step 7.

Notation For the sake of Cases 2.6 and 2.7, we use the notation

$$\begin{aligned}\Omega_{\alpha,s+1} = {} & \big\{x(\beta, s) \bigm| x(\beta, s) \geq 0 \;\&\; N(\beta, s) \geq 1 \;\&\; \beta <_L \alpha\big\} \\ & \cup \big\{x(\beta, s+1) \bigm| x(\beta, s+1) \geq 0 \;\&\; N(\beta, s+1) \geq 1 \;\&\; \beta \subset \alpha\big\}.\end{aligned}$$

(Note that $N(\beta, s+1) = i$ and that $\beta \subset \alpha$ implies that $\beta^\frown\langle i\rangle \subseteq \alpha$.)

<u>*Case 2.6.*</u> $N(\alpha, s) = 2$ and for some finite set F, $\langle x(\alpha, s), F\rangle \in \Psi[s]$ and

$$F \subseteq B[\alpha, s+1] \cap \big(\Phi[s]\big)^{\Omega_{\alpha,s+1} \cup D_A[s] \cup D(s+1,t) \cup \{z(s+1,t)\} \cup S(\alpha^\frown\langle 2\rangle, s)}. \tag{5.6}$$

Set $z(s+1, t+1) = \text{break}$ and

$$D(s+1, t+1) = D(s+1, t) \cup \{z(s+1, t)\} \cup S(\alpha^\frown\langle 2\rangle, s).$$

Set $N(\alpha, s+1) = 0$ and $F(\alpha, s+1) = F$ for the least F satisfying (5.6). For $q \in \{p, x\}$, reset $q(\alpha, s+1) = q(\alpha, s)$.

Note. The strategy $\alpha = \beta_{s+1}$ and $\{z(s+1, t)\} \cup S(\alpha^\frown\langle 2\rangle, s)$ is dumped into A. Then $N(\alpha, s) = 2$ and $N(\alpha, s+1) = 0$ means that $x(\alpha)$ is removed from A. This case corresponds to moving from Step 2 to Step 5.

<u>Case 2.7.</u> $N(\alpha, s) = 2$, but there is no such finite set F.

Set $z(s+1, t+1) = z(s+1, t)$, $D(s+1, t+1) = D(s+1, t)$. For $q \in \{N, p, x, F\}$, reset $q(\alpha, s+1) = q(\alpha, s)$. (Note that $F(\alpha, s) = -1$ in this case.)

Note. The strategy $\alpha^\frown\langle 2\rangle$ is eligible to act at substage $t+2$ and floating witness z is passed to $\alpha^\frown\langle 2\rangle$. Then $N(\alpha, s+1) = N(\alpha, s) = 2$ means that $x(\alpha)$ remains inside A. This case corresponds to waiting at Step 2.

<u>Case 2.8.</u> $N(\alpha, s) = \text{void}$.

Set $z(s+1, t+1) = \text{break}$, $D(s+1, t+1) = D(s+1, t)$. Set $N(\alpha, s) = 2$, $x(\alpha, s+1) = z(s+1, t)$, and for $q \in \{p, F\}$, reset $q(\alpha, s+1) = q(\alpha, s)$.

Note. The strategy $\alpha = \beta_{s+1}$ and α appropriates the floating witness $z(s+1, t)$ as its local witness $x(\alpha, s+1)$. Then $N(\alpha, s+1) = 2$ means that this new witness $x(\alpha, s+1)$ is put into A. This case corresponds to application of Step 1 and moving to Step 2.

End of stage $s+1$ is the same as in the proof of Theorem 5.3 with two small adjustments.

The first involves modifying the set of strategies initialized. Indeed, in the present context, letting $\alpha = \beta_{s+1}$ it is the set $\widehat{G} = \{\beta \mid \alpha^\frown\langle 0\rangle <_L \beta\}$ that is initialized. Note that when α is a P node, then the initialization defined here has the same effect as initializing the set $G = \{\beta \mid \alpha < \beta\}$ (since any $\alpha^\frown\langle 0\rangle \subseteq \beta$ is in its initial state anyway in this case). Likewise, the same can be said if α is an N node and Case 2.6 or 2.8 is applied (at the last substage $|\alpha| + 1$). However, in Case 2.4, the effect of restricting initialization to the set G' means that the subtree $\{\gamma \mid \alpha^\frown\langle 0\rangle \subseteq \gamma\}$ is protected against initialization. This is important as the proof may need to construct an infinite path through this subtree in order to define its true path δ.

The second adjustment relates to the fact that in the present construction, for any N node α and stage s such that $x(\alpha, s) \geq 0$, $x(\alpha, s) \in A[s]$ if and only if $N(\alpha, s) \in \{1, 2\}$ (and not just $N(\alpha, s) = 1$). Hence the construction defines

$$I_A[s] = \{x(\beta, s+1) \mid x(\beta, s+1) \geq 0 \;\&\; N(\beta, s+1) \geq 1\}.$$

Verification. Checking that $N_{B,\Phi,\Psi}$ is satisfied is carried out by making the assumption that there exists $\alpha \subseteq \delta$ such that $\alpha \in \mathcal{N}_{B,\Phi,\Psi}$ and considering the outcomes of the strategy for α. This involves a straightforward argument which can be derived from the description of the steps of the strategy for α in conjunction with the specification of how strategy α is processed (i.e., via Cases 2.1–2.8 above), by taking into account the notes added to each case. Note, however, that the assumption that $\alpha \subseteq \delta$ exists requires us to also prove that, for some $i \in \{0, 1, 2\}$, $\alpha^\frown\langle i\rangle \subseteq \delta$, that is, implicitly that δ is infinite (which we do directly via Lemma 5.11 in the proof of Theorem 5.3).

This is the reason for the introduction of a *pause mechanism*, that is, the use of the pause parameter $p(\alpha, s)$, since it forces δ to be infinite. To see this, we first consider the role of Case 2.4 by supposing that it is applied to α at stage $s+1$. We also assume that α is on the true path and that our work is subsequent to a stage s_α after which α is never in its initial state so that, in particular, $x(\alpha, s)$ has already stabilized at its final value. We denote (as usual) this value as $x(\alpha)$. Note first that, for any R strategy β and outcome $i \in \{0, 1, 2\}$ such that $\beta^\frown\langle i\rangle \subseteq \alpha$, $x(\alpha) \in S(\beta^\frown\langle i\rangle, s)$, and that every $z \in S(\alpha^\frown\langle 1\rangle, s)$ except at most one number (via Case 2.2) entered stream $\beta^\frown\langle i\rangle$'s stream (and $\alpha^\frown\langle 1\rangle$'s stream) at some stage $t < s$ when $x(\alpha)$ was already in $\beta^\frown\langle i\rangle$'s stream as well as already being in A. Thus the removal of $x(\alpha)$ from A by Case 2.4 would invalidate Lemma 5.10 applied to $\beta^\frown\langle i\rangle$ thus violating β's strategy, unless all such z are dumped into A. Likewise, the floating witness $z = z(s+1, t)$ (with $t = |\alpha|$) was processed (perhaps trivially) by β earlier in the stage under the assumption that $x(\alpha) \in A$, and so it also cannot enter $\beta^\frown\langle i\rangle$'s stream at this stage without invalidating Lemma 5.10 applied to $\beta^\frown\langle i\rangle$ thus violating β's strategy. Thus we see that the dumping of all of $S(\alpha^\frown\langle 1\rangle, s) \cup \{z(s+1, t)\}$ by Case 2.4 is fundamental to the preservation of β's strategy. Moreover, since there is now no floating witness left to pass to the strategy $\alpha^\frown\langle 0\rangle$, the inherent role attributed to the floating witness by the construction dictates that the stage must be broken at this point. (There is also a principle of simplicity underlying this approach.)

Suppose now that the pause mechanism is absent, so that in effect $p(\alpha, s) =$ continue for all s. This means that Case 2.3 will never apply. Notice also that outcome (iv) of α's strategy entails that Case 2.2 and Case 2.4 each apply to α at infinitely many α-true stages. Moreover, this outcome does not exclude the situation in which there is some stage s^* such that every instance of Case 2.4 being applied to α is followed, at the next α-true stage, by an instance of Case 2.2 being applied to α. Observe that, under these conditions, the true path $\delta = \alpha$ (i.e., is finite). However, if we now reintroduce the pause mechanism, we see that Case 2.3 applies subsequent to each instance of Case 2.4 (over the set of α-true stages), meaning that not only is $\alpha^\frown\langle 0\rangle \subseteq \beta_s$ for infinitely many s (assuming that Case 2.4 applies infinitely often), but also that at each such stage a different floating witness is passed to $\alpha^\frown\langle 0\rangle$ (and so into its stream/the subtree below it). Moreover, when Case 2.3 applies, $x(\alpha) \notin A$ and so the floating witness z that Case 2.3 passes to $\alpha^\frown\langle 0\rangle$ can safely enter $\alpha^\frown\langle 0\rangle$'s stream without violating the strategy of any N node β such that $\beta^\frown\langle i\rangle \subseteq \alpha$, since any such β processed z under the assumption that $x(\alpha) \notin A$. (Remember here that $\alpha^\frown\langle 0\rangle$'s stream is a subset of $\beta^\frown\langle i\rangle$'s stream.)

We conclude from this discussion that the pause mechanism enables us to apply Lemma 5.11 in the present context and so deduce that δ is infinite. We are then able to prove that all the requirements are satisfied in a similar manner to that undertaken in Lemmas 5.15–5.18. Note that, in particular, for any $\alpha \in \mathcal{N}_{B,\Phi,\Psi}$ such that $\alpha \subseteq \delta$, the pause mechanism does not affect the outcome of α's strategy. Indeed, we find that $\alpha^\frown\langle 2\rangle \subseteq \delta$ corresponds to outcome (i) of α's strategy and that $\alpha^\frown\langle 1\rangle \subseteq \delta$ corresponds to outcome (iii), whereas $\alpha^\frown\langle 0\rangle \subseteq \delta$ corresponds to (finitary) outcome (i) or (infinitary) outcome (iv). □

Corollary 5.21 *There exist both a low and a properly Σ_2^0 enumeration 1-generic degree $0_e < a < 0'_e$ that is not 1-generic.*

Proof Apply Theorem 5.3 and Theorem 5.20 with the fact that every 1-generic enumeration degree is splittable. □

Notes

1. The function c_A clearly depends on the approximation $\{A_s\}_{s\in\omega}$ and not just on A.

2. For example, if $m = \max D$, let γ be the string of length $\max\{2x, 2m+1\}+1$ defined such that $\gamma(n) = \sigma(n)$ for all $n < |\sigma|$ and such that for all $|\sigma| \leq m < |\gamma|$, $\gamma(m) = 0$ if m is even, and $\gamma(m) = 1$ if m is odd.

3. For inclusion, $\subset$ is only used when strictness is important.

4. Note that termination of a stage is determined by the value of $z(s+1, t)$ only, not by the length of the strategies eligible to act.

5. This first case (i.e., $x(\alpha, s+1) = z(s+1, t)$) happens only if $\alpha \in \mathcal{N}$.

6. In this case $x = x(\beta_{s+1}, s) = x(\beta_{s+1}, s+1)$.

7. By Lemma 5.7, y is the unique such number.

8. We have that $x(\mu, t-1) \notin D(t)$ if and only if (i) Case 1.1 or 1.2 applies at some stage $r \leq |\beta_t|$, (ii) $z(t, p) = x(\mu, t-1)$ for all $r \leq p \leq |\beta_t|$, and (iii) $x(\beta_t, t) = x(\mu, t-1)$ via Case 2.4 at substage $|\beta_t| + 1$.

9. Case 2.2 can only happen at substage $|\beta_t| + 1$ since it induces $z(t, |\beta_t| + 1) = \text{break}$.

10. That is, such that for all $s \geq s_\beta$, $\beta_s \not<_L \beta$ and, if $|\beta_s| < |\beta|$, then $\beta <_L \beta_s$. Note that this observation does *not* apply to the tree construction of Theorem 5.20 where it may be the case that $\beta_s \subset \delta_n$ for infinitely many s. (In this case, in the tree construction of Theorem 5.20 any such β_s is an N node, and it is in fact the case that $\beta_s{}^\frown\langle 0\rangle \subseteq \delta_n$.)

11. Note that $\alpha \subset \beta$ implies that $x(\alpha, t-1) \geq 0$, that is, is already defined (see Lemma 5.6).

12. Note that in the construction $S(\alpha{}^\frown\langle 2\rangle) = S(\alpha) \setminus \{x(\alpha)\}$.

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