The Annals of Probability
1988, Vol. 16, No. 4, 1584–1595

NECESSARY CONDITIONS FOR SAMPLE BOUNDEDNESS OF p-STABLE PROCESSES[1]

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We extend to general p-stable processes the lower bound that Marcus and Pisier established for strongly stationary p-stable processes. This bound also extends the results of the author on general Gaussian processes.

1. Introduction. For a space T provided with a pseudometric d, denote by $N(T, d, \varepsilon)$ the smallest number of open balls of radius ε that cover T. Consider an index set T and a Gaussian process $(X_t)_{t \in T}$ indexed by T. Provide T with the pseudometric $d(t, u) = (E|X_t - X_u|^2)^{1/2}$. If X has a version with bounded paths, a result of Sudakov [6] shows that $\sup_{\varepsilon > 0} \varepsilon(\log N(T, d, \varepsilon))^{1/2} < \infty$. In the stationary case, Fernique [2] showed that Dudley's sufficient "metric entropy" condition [1]

$$\int_0^\infty (\log N(T, d, \varepsilon))^{1/2}\, d\varepsilon < \infty \tag{1}$$

is also necessary for the sample continuity of Gaussian processes. Marcus and Pisier [5] extended the Dudley–Fernique theorem to strongly stationary p-stable processes, $1 < p \le 2$, as well as Fernique's necessary condition to the case $p = 1$. They also extended Sudakov's result to general p-stable processes, $1 < p \le 2$.

Fernique [2] has given sufficient conditions weaker than (1) for sample continuity (resp. sample boundedness) of Gaussian processes. The author has shown [7] that these conditions are necessary, and the structure of abstract Gaussian processes is now well understood. In this paper, we extend the necessary conditions to p-stable processes, $1 \le p \le 2$. Our result extends the "necessary" part of the Marcus–Pisier result to the nonstationary case and also extends their own extension of the Sudakov result.

The condition we give is optimum of its type. The difference with the Gaussian case is that it is, however, not sufficient. This is due to the well-known fact that the properties of p-stable processes, for $p \ne 2$, are not entirely determined by the pseudodistance that they induce on the index set.

For $1 \le p \le 2$, a random variable θ_p is called (real-valued symmetric) p-stable if for each λ,

$$E \exp i\lambda\theta_p = \exp(-|\lambda|^p).$$

Then $E|\theta_p|^r < \infty$ for $r < p$.

A random vector $(X_1, \dots, X_n)$ is called (real-valued symmetric) p-stable if there exists a positive measure μ on $\mathbb{R}^n$ such that for all sequences $(\alpha_i)_{i=1}^n$ of $\mathbb{R}^n$,

Received June 1986; revised August 1987.
[1]This work was supported in part by NSF Grant DMS-86-03951.
AMS 1980 *subject classifications*. Primary 60G17; secondary 60G12.
Key words and phrases. p-stable, Gaussian, majorizing measures.

we have

$$E\exp\left(i\sum_{j=1}^{n}\alpha_j X_j\right)=\exp\left(-\int_{\mathbb{R}^n}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|^p d\mu(\beta)\right). \tag{2}$$

Hence $\sum_{i=1}^{n}\alpha_i X_i$ is distributed like $(\int|\sum_{i=1}^{n}\alpha_i\beta_i|^p\, d\mu(\beta))^{1/p}\theta_p$. In particular, for $1\le i<j\le n$, $r<1$, we have

$$\left(E|X_i-X_j|^r\right)^{1/r}=\left(E|\theta_p|^r\right)^{1/r}\left(\int|\beta_i-\beta_j|^p\, d\mu(\beta)\right)^{1/p}.$$

This shows that one can define for $r<1$ a pseudodistance d_r on the index set by

$$d_r(i,j)=\left(E|X_i-X_j|^r\right)^{1/r}$$

and that all these distances are equivalent.

A process $(X_t)_{t\in T}$ is called (real-valued symmetric) p-stable if for each finite sequence $t_1,\dots,t_n$, the random vector $(X_{t_1},\dots,X_{t_n})$ is p-stable. We fix once and for all $r<1$ (say $r=1/2$), and consider the pseudometric d on T given by

$$d(t,u)=\left(E|X_t-X_u|^r\right)^{1/r}. \tag{3}$$

We denote by D the diameter of (T,d). For $1<p\le 2$, we define q by $1/p+1/q=1$. For $p=1$, we set $q=\infty$. For $p>1$, $0<t\le 1$, we set $h_q(t)=(\log(1/t))^{1/q}$, and we set $h_\infty(t)=\log^+\log(1/t)$.

THEOREM A. *Let $1\le p\le 2$. Let $(X_t)_{t\in T}$ be a p-stable process. Suppose that $(X_t)_{t\in T}$ has a.s. bounded sample paths. Let $M>0$ be such that $P(\{\sup_{s,t\in T}|X_s-X_t|\ge M\})\le 1/2$. Then there is a probability measure m on (T,d) [where d is given by* (3)*] such that*

$$\forall x\in T,\qquad \int_0^D h_q\left(\sup\{m(\{t\});\, d(x,t)\le\varepsilon\}\right)d\varepsilon<K_pM, \tag{4}$$

where K_p depends on p only. If, moreover, $(X_t)_{t\in T}$ has a.s. continuous sample paths on (T,d), there exists a probability measure on m on (T,d) such that

$$\lim_{\varepsilon\to 0}\sup_{x\in T}\int_0^\varepsilon h_q\left(\sup\{m(\{t\});\, d(x,t)\le\varepsilon\}\right)d\varepsilon=0.$$

2. Tools. Marcus and Pisier [5], elaborating on earlier ideas of LePage [3], brought to light the essential fact that p-stable processes are mixtures of Gaussian processes. This fact will also be central here. Let Z be a positive random variable (r.v.) satisfying $P(Z>\lambda)=e^{-\lambda}$. Let $(Z_i)_{i=1}^{\infty}$ be independent identically distributed (i.i.d.) copies of Z and let $\Gamma_j=Z_1+\cdots+Z_j$. Let U be a finite set, $(X_t)_{t\in U}$ be a p-stable process on U and μ be a finite positive measure on $\mathbb{R}^U$ as in (2). Let Y be a r.v. valued in $\mathbb{R}^U$ and distributed according to $\mu/\|\mu\|$ and let $(Y_i)_{i=1}^{\infty}$ be i.i.d. copies of Y that are independent of the (Z_i). Denote by (Ω,Σ,P) the basic probability space. Consider another probability space (Ω',Σ',P'), on which is defined an i.i.d. sequence (g_i) of normalized Gaussian r.v. Denote by Pr the product probability on $\Omega\times\Omega'$.

LEMMA 1 ([5]). *For some constant $c(p)$, depending on p only, the $\mathbb{R}^U$-valued r.v. on $\Omega \times \Omega'$ given by*

$$c(p)\|\mu\|^{1/p} \sum_{i=1}^{\infty} g_i(\omega')(\Gamma_i(\omega))^{-1/p} Y_i(\omega) \tag{5}$$

has the same law as $(X_t)_{t \in U}$.

Given ω, (5) defines a Gaussian process (X_t^ω) given by

$$X_t^\omega = c(p)\|\mu\|^{1/p} \sum_{i=1}^{\infty} g_i(\Gamma_i(\omega))^{-1/p} Y_i(\omega)(t).$$

The canonical distance d_ω associated with (X_t^ω) is given by

$$d_\omega^2(s,t) = \left(c(p)\|\mu\|^{1/p}\right)^2 \sum_{i=1}^{\infty} (\Gamma_i(\omega))^{-2/p} (Y_i(\omega)(s) - Y_i(\omega)(t))^2. \tag{6}$$

For $p < 2$, let α given by $1/\alpha = 1/p - 1/2$.

LEMMA 2 ([4] and [5]). *For some constant $d(p)$, depending on p only, we have for s, t in U, $\varepsilon > 0$,*

$$P(\{\omega \in \Omega;\, d_\omega(s,t) \le \varepsilon d(s,t)\}) \le \exp(-d(p)\varepsilon^{-\alpha}). \tag{7}$$

PROOF. For $\lambda \ge 0$, (2) implies that for some constant $a(p)$ we have

$$E \exp(i\lambda(X_t - X_s)) = \exp(-\lambda^p a(p)\, d^p(s,t)).$$

On the other hand,

$$E_{\omega'} \exp(i\lambda(X_t - X_t)) = \exp\left(-\frac{\lambda^2}{2} d_\omega^2(s,t)\right),$$

so

$$\exp(-\lambda^p a(p)\, d^p(s,t)) = E_\omega \exp\left(-\frac{\lambda^2}{2} d_\omega^2(s,t)\right)$$

and (7) follows by Chebyshev's exponential inequality. □

The natural approach to Theorem A would be as follows (at least when T is finite). For $p = 2$, the result has been proved in [7], so for each ω we have a probability measure m_ω that satisfies (4) for d_ω and h_2, and one can try to take for m a mixture of the measures m_ω. Unfortunately, this simple approach seems to be doomed to failure because there is no way to ensure that the choice of the various measures m_ω is made in a coordinated way. Instead, we are going to use the machinery of [7] to reduce the proof of Theorem A to the proof of a simpler (but still nontrivial) statement.

A finite metric space (U, δ) is called ultrametric if $\delta(s,u) \le \max(\delta(s,t), \delta(t,u))$ for s, t, u in U. So, two balls of U of the same radius are either identical or disjoint. We denote by $\mathscr{B}_i$ the family of closed balls of U of radius 4^{-i}. Let $i_0 \in \mathbb{Z}$ be the largest such that card $\mathscr{B}_{i_0} = 1$. We set $\gamma = 4^{-i_0}$. For x in U, we denote $k(x, i)$ the number of disjoint balls of $\mathscr{B}_{i+1}$ that are contained in $B(x, 4^{-i})$. We set

$$\xi_x^q(U) = \sum_{i \geq i_0} 4^{-i}(\log k(x, i))^{1/q}.$$

Let $LL(t) = \log_2^+(\log t)$ for $t > 1$ and $LL(1) = 0$, so we have $t \leq \exp(2^{LL(t)})$. We set

$$\xi_x^\infty(U) = \gamma + \sum_{i \geq i_0} 4^{-i} LL(k(x, i)).$$

[The term γ is needed for the technical reason that $LL(2) = 0$.] We set

$$\xi^q(U) = \inf_{x \in U} \xi_x^q(U), \qquad \xi^\infty(U) = \inf_{x \in U} \xi_x^\infty(U).$$

In Sections 3 and 4, we will prove the following fact.

THEOREM B. *Let $1 \leq p \leq 2$. Consider a finite ultrametric space (U, δ) and a p-stable process $(X_t)_{t \in U}$. Assume that the pseudo distance d induced by X is greater than δ. Let $M \geq 0$ be such that $P(\{\sup_{s,t \in T} |X_t - X_s| \geq M\}) \leq 1/2$. Then $\xi^q(U) \leq K_p M$, where K_p depends only on p.*

The bulk of Section 2 of [7] is devoted to prove, when $q = p = 2$, that Theorem A follows from Theorem B (which in that case is due to Fernique). The arguments make no use of any theory of stochastic processes, but only of constructions in abstract metric spaces; only minor modifications are needed to show that Theorem A follows from Theorem B for any $1 \leq p \leq 2$. To prove Theorem B, we have to show that $\sup_{s,t \in T} |X_t - X_s|$ is large whenever $\xi^q(U)$ is large. The information that $\xi^q(U)$ is large is very precise. This is what makes the proof possible.

3. Proof of Theorem B for $p > 1$. By hypothesis $P(\{\sup_{s,t \in U} |X_s - X_t| \geq M\}) \leq 1/2$. By Fubini's theorem, there exists a set $Z \subset \Omega$ with $P(Z) \geq 1/3$ such that for $\omega \in Z$ we have

$$P'\left(\left\{\sup_{s,t \in U} |X_s^\omega - X_t^\omega| \geq M\right\}\right) \leq 3/4.$$

Using the known fact that all the quantiles of the quantity $\sup_{s,t \in U} |X_s^\omega - X_t^\omega|$ are equivalent, the Gaussian case of Theorem A, that is Theorem 1 of [7], shows the following:

LEMMA 3. *For ω in Z, there exists a probability measure ν_ω on (U, d_ω) such that*

$$\forall x \in U, \qquad \int_0^\infty h_2\big(\nu_\omega(\{y \in U; d_\omega(x, y) \leq \varepsilon\})\big)\, d\varepsilon \leq KM,$$

where K is a universal constant.

The function $h_2(t)$ is convex for $t \le e^{-1/2}$, but is not convex for $0 < t \le 1$. For that reason, it will be more convenient to use the function $\varphi(t) = h_2(t/3)$ (the choice of 3 is rather arbitrary). This function is convex for $0 < t \le 1$. To prove Theorem B, we will exhibit a subset A of Ω, with $P(A) \ge 3/4$, and constants K_1, K_2, K_3, depending on p only, such that for ω in A, for each probability measure ν on U, we have

$$(8) \qquad \xi^q(U) \le K_1 \sup_{x \in U} \int_0^{K_2\gamma} \varphi\big(\nu(\{y \in U; d_\omega(x, y) \le \varepsilon\})\big)\, d\varepsilon + K_3\gamma.$$

We observe that $\varphi(t) \le h_2(t) + (\log 3)^{1/2}$, so for $\omega \in A \cap Z$ we have

$$\sup_{x \in U} \int_0^{K_2\gamma} \varphi\big(\nu(\{y \in U; d_\omega(x, y) \le \varepsilon\})\big)\, d\varepsilon \le KM + K_2\gamma(\log 3)^{1/2}.$$

Since (U, d) is of diameter greater than or equal to $\gamma/4$, we obviously have $\gamma \le K_4 M$, where K_4 depends on p only. Combining with (8), this proves Theorem B.

The philosophy of the approach is that a large value of $\xi^q(U)$ means that (U, d) is very big in an appropriate sense. Since (7) means that $d_\omega(s, t)$ is, most of the time, not much smaller than $d(s, t)$, we can expect that U will be big with respect to d_ω for most values of ω. The construction is made rather delicate by the following feature: While (7) tells us precisely how $d_\omega(s, t)$ behaves compared to $d(s, t)$, if we take another couple (s', t'), we have no information about the joint behavior of $d_\omega(s, t)$, $d_\omega(s', t')$.

We need an auxiliary probability measure λ on (U, δ). This measure is homogeneous in the sense that the mass of any ball of radius 4^{-i} is divided evenly among all the balls of radius 4^{-i-1} that it contains. Equivalently, for any x in U, $\lambda(\{x\}) = (\Pi_{i \ge i_0} k(x, i))^{-1}$.

Let us now fix $i \ge i_0$. Let $B_1, B_2 \in \mathscr{B}_i$, with $B_1 \ne B_2$. Let $b, c > 0$. We define

$$A(B_1, B_2, b, c) = \big\{\omega \in \Omega; \lambda \otimes \lambda\big(\{(x, y) \in B_1 \times B_2; d_\omega(x, y) \le b4^{-i}\}\big) \ge c\lambda(B_1)\lambda(B_2)\big\}.$$

Lemma 4. $P(A(B_1, B_2, b, c)) \le (1/c)\exp(-d(p)b^{-\alpha})$.

Proof. For x in B_1, y in B_2, the ultrametricity shows that $\delta(x, y) > 4^{-i}$, so $d(x, y) > 4^{-i}$. Let

$$A(x, y) = \big\{\omega \in \Omega; d_\omega(x, y) \le b4^{-i}\big\},$$

so by (7) we have

$$E\big(1_{A(x, y)}\big) = P(A(x, y)) \le \exp(-d(p)b^{-\alpha}).$$

So we have

$$E\left(\int\int_{B_1 \times B_2} 1_{A(x, y)}\, d\lambda(x)\, d\lambda(y)\right) \le \lambda(B_1)\lambda(B_2)\exp(-d(p)b^{-\alpha}).$$

The result follows since

$$A(B_1, B_2, b, c) = \left\{\omega \in \Omega; \iint_{B_1 \times B_2} 1_{A(x,y)}(\omega)\, d\lambda(x)\, d\lambda(y) \le c\lambda(B_1)\lambda(B_2)\right\}.$$

□

For $B \in \mathscr{B}_i$, and $j \le i$, there is a unique $B' \in \mathscr{B}_j$ that contains B. We denote by $k(B, j)$ the number of elements of $\mathscr{B}_{j+1}$ contained in B', so $k(B, j) = k(x, j)$ whenever x belongs to B. In particular $k(B, i)$ is the number of elements of $\mathscr{B}_{i+1}$ that are contained in B. Also, if $j \le i' \le i$, and $B \in \mathscr{B}_i$, $B' \in B_{i'}$, $B \subset B'$, we have $k(B, j) = k(B', j)$. We denote by $\mathscr{B}_i'$ the subset of $\mathscr{B}_i$ that consists of the balls B in $\mathscr{B}_i$ for which

$$k(B, i) \ge 2\prod_{j<i} k(B, j).$$

Set $\tau = 6/d(p)$. Consider B in $\mathscr{B}_i'$, so that $k(B, i) > 1$.

Given two balls $B_1, B_2 \in \mathscr{B}_{i+1}$, $B_1, B_2 \subset B$, $B_1 \ne B_2$, we consider the event

$$C(B, B_1, B_2) = A\Big(B_1, B_2, \big(\tau \log(2k(B, i))\big)^{-1/\alpha}, \big(2k(B, i)\big)^{-2}\Big).$$

It follows from Lemma 4 that we have

$$P\big(C(B, B_1, B_2)\big) \le 2^{-4} k(B, i)^{-4}. \tag{9}$$

For D in $\mathscr{B}_i$, we consider the event

$$\bar{A}(D) = \bigcup C(B, B_1, B_2), \tag{10}$$

where the union is taken over all choices of $j \ge i$, B in $\mathscr{B}_j'$, $B \subset D$, B_1, B_2 in $\mathscr{B}_{j+1}$, $B_1 \ne B_2$, $B_1, B_2 \subset B$. [If no such choice is possible, we set $\bar{A}(D) = \varnothing$.]

LEMMA 5. $P(\bar{A}(D)) \le (2\prod_{j<i} k(D, j))^{-2}$.

PROOF. The proof goes by decreasing induction over i. If i is large enough that D has only one point, then $\bar{A}(D) = \varnothing$, so that $P(\bar{A}(D)) = 0$. Assume now that we have proved the lemma for $i + 1$, and let D be in $\mathscr{B}_i$.

CASE 1. Assume $D \in \mathscr{B}_i'$, so $k(D, i) \ge 2$. Let

$$\bar{A}_1 = \bigcup\{\bar{A}(D'); D' \in \mathscr{B}_{i+1}, D' \subset D\},$$

$$\bar{A}_2 = \bigcup\{C(D, B_1, B_2); B_1, B_2 \in \mathscr{B}_{i+1}, B_1 \ne B_2, B_1, B_2 \subset D\}.$$

We have $\bar{A}(D) = \bar{A}_1 \cup \bar{A}_2$. We have, by the induction hypothesis, and since $k(D, i) \ge 2$,

$$P(\bar{A}_1) \le k(D, i)\Big(2\prod_{j \le i} k(D, j)\Big)^{-2} \le \tfrac{1}{2}\Big(2\prod_{j<i} k(D, j)\Big)^{-2}.$$

We have, using (9),

$$P(\bar{A}_2) \le \tfrac{1}{2}k(D, i)^2 \cdot 2^{-4} k(D, i)^{-4} \le \tfrac{1}{2}k(D, i)^{-2} \le \tfrac{1}{2}\Big(2\prod_{j<i} k(D, j)\Big)^{-2}$$

and $P(\bar{A}(D)) \le (2\prod_{j<i} k(D, j))^{-2}$.

CASE 2. We have $D \notin \mathscr{B}_i'$; with the same notation as before, $\bar{A}(D) = \bar{A}_1$, so, by the induction hypothesis,

$$P(\bar{A}(D)) \leq k(D,i)\Big(2\prod_{j\leq i} k(D,j)\Big)^{-2} \leq \Big(2\prod_{j<i} k(D,j)\Big)^{-2}.$$

This completes the proof. □

Let $A = \Omega \setminus \bar{A}(U)$, so $P(A) \geq 3/4$. We fix $\omega \in A$ and we proceed to prove (8). For $B \in \mathscr{B}_i'$, we set

$$\alpha(B,i) = 4^{-i-2}(\tau \log(2k(B,i)))^{-1/\alpha}.$$

LEMMA 6. *Fix $B \in \mathscr{B}_i'$. For $y \in B$, let*

$$H_y = \{x \in U;\, d_\omega(x,y) \leq \alpha(B,i)\}.$$

Then for each y in U,

$$\lambda(H_y \cap B) \leq 3\lambda(B)/2k(B,i).$$

PROOF. Suppose otherwise, and let y be in U with

$$\lambda(H_y \cap B) > 3\lambda(B)/2k(B,i).$$

For C in $\mathscr{B}_{i+1}$, $C \subset B$, we have $\lambda(C) = \lambda(B)/k(B,i)$. It follows that there exist at least two balls B_1, B_2 of $\mathscr{B}_{i+1}$, $B_1, B_2 \subset B$ such that for $l = 1,2$ we have

$$\lambda(H_y \cap B_l) \geq \lambda(B_l)/2k(B,i).$$

For x_1, x_2 in H_y, we have $d_\omega(x_1,x_2) \leq 2\alpha(B,i)$, so we have

$$\begin{aligned}\lambda \otimes \lambda(\{(x_1,x_2) \in B_1 \times B_2;\, d_\omega(x_1,x_2) \leq 2\alpha(B,i)\})\\ \geq \lambda(B_1)\lambda(B_2)/4k(B,i)^2.\end{aligned}$$

This, however, contradicts the fact that $\omega \notin C(B, B_1, B_2)$ and concludes the proof. □

Let x be in B. Since the function $\varphi(t)$ is convex, we have

$$\begin{aligned} I_B &:= \lambda(B)^{-1}\int_B \varphi(\nu(H_x))\,d\lambda(x)\\ &\geq \varphi\Big(\lambda(B)^{-1}\int_B \nu(H_x)\,d\lambda(x)\Big) \qquad (11)\\ &= \varphi\Big(\int g\,d\nu\Big),\end{aligned}$$

where

$$g(y) = \lambda(B)^{-1}\lambda(H_y \cap B).$$

It follows from Lemma 6 that $0 \leq g \leq 3/(2k(B,i))$, so we have

$$I_B \geq (\log(2k(B,i)))^{1/2}.$$

We get, since $1/2 - 1/\alpha = 1/q$,

$$a(B,i)I_B \geq 4^{-i-2}\tau^{-1/\alpha}(\log(2k(B,i)))^{1/q}. \tag{12}$$

For x in U, let us enumerate as

$$i_1(x) < \cdots < i_{l(x)}(x)$$

the indexes j such that $B(x,4^{-j}) \in \mathscr{B}_j'$. Note that $i_1(x) = i_0$. For $l \leq l(x)$, let

$$\begin{aligned} b(x,l) &= a\big(B\big(x,4^{-i_l(x)}\big), i_l(x)\big) \\ &= 4^{-i_l(x)-2}\big(\tau\log\big(2k(x,i_l(x))\big)\big)^{-1/\alpha}. \end{aligned}$$

For x in U, let

$$\eta_x(U) = \sum_{l \leq l(x)} 4^{-i_l(x)}\big(\log\big(2k(x,i_l(x))\big)\big)^{-1/\alpha}.$$

We have

$$\begin{aligned} &\int_U \sum_{l \leq l(x)} b(x,l)\varphi\big(\nu\big(\{y \in U; d_\omega(x,y) \leq b(x,l)\}\big)\big)\, d\lambda(x) \\ &\quad = \sum \int_B a(B,i)\varphi\big(\nu\big(\{y \in U; d_\omega(x,y) \leq a(B,i)\}\big)\big)\, d\lambda(x), \end{aligned}$$

where the summation is taken over each value of i and each $B \in \mathscr{B}_i'$. Using (12), we see that this latter quantity dominates

$$4^{-2}\tau^{-1/\alpha}\sum 4^{-i}(\log(2k(B,i)))^{1/q}\lambda(B),$$

where the summation is over the same range. But this quantity is

$$4^{-2}\tau^{-1/\alpha}\int_U \eta_x(U)\, d\lambda(x).$$

We now observe that for $l < l(x)$, we have $b(x,l+1) \leq b(x,l)/4$. Also, $b(x,1) \leq \tau^{-1/\alpha}\gamma$, since $k(x,i_0) \geq 2$. It follows that if we let $K_2 = \tau^{-1/\alpha}$, we have

$$\begin{aligned} &\sum_{l \leq l(x)} b(x,l)\varphi\big(\nu\big(\{y \in U; d_\omega(x,y) \leq b(x,l)\}\big)\big) \\ &\quad \leq 2\int_0^{K_2\gamma}\varphi\big(\nu\big(\{y \in U; d_\omega(x,y) \leq \varepsilon\}\big)\big)\, d\varepsilon. \end{aligned}$$

In conclusion, for ω in A, we have shown that for each probability measure ν on U, we have

$$\begin{aligned} &\sup_{x \in U}\int_0^{K_2\gamma}\varphi\big(\nu\big(\{y \in U; d_\omega(x,y) \leq \varepsilon\}\big)\big)\, d\varepsilon \\ &\quad \geq \int_U\left(\int_0^{K_2\gamma}\varphi\big(\nu\big(\{y \in U; d_\omega(x,y) \leq \varepsilon\}\big)\big)\, d\varepsilon\right) d\lambda(x) \\ &\quad \geq 2^{-5}\tau^{-1/\alpha}\int_U \eta_x(U)\, d\lambda(x). \end{aligned} \tag{13}$$

To prove (8), we have to evaluate this latter integral. For x in U, $s < l(x)$ and

$$i_0 = i_1(x) < \cdots < i_s(x) \leq j < i_{s+1}(x),$$

we have

$$k(x,j) \leq 2^{2^{j-i_0}} k(x, i_s(x))^{2^{j-i_s(x)}} \cdots k(x, i_1(x))^{2^{j-i_1(x)}}$$

as is shown by immediate induction over j. It follows that

$$(\log k(x,j))^{1/q} \leq (2^{j-i_0}\log 2)^{1/q} + \sum_{1\leq l\leq s} (2^{j-i_l(x)}\log k(x, i_l(x)))^{1/q}.$$

A simple computation shows that there are constants K_5, K_6 such that $\xi_x(U) \leq K_5\eta_x(U) + K_6\gamma$. Since $\xi(U) \leq \int_U \xi(U)\,d\lambda(x)$, this and (13) prove (8) and conclude the proof. □

4. Proof of Theorem B for $p = 1$. The overall approach is similar to the case $p > 1$, but the details are quite different. We again try to find $A \subset \Omega$ with $P(A) \geq 3/4$ such that (8) holds. We define λ as in Section 3. For x in U, $i \geq i_0$, we set $m(x,i) = [LL(k(x,i))/4]$, where $[t]$ denotes the integer part of t. So we have $LL(k(x,i)) \leq 4m(x,i) + 4$, and

$$k(x,i) \leq \exp(2^{4m(x,i)+4}).$$

For B in $\mathscr{B}_j$, $j \geq i$, we define $m(B,i)$ as $m(x,i)$ for any x in B. For a ball B in $\mathscr{B}_i$, let

$$\hat{B} = \{(x,y) \in B \times B;\ \delta(x,y) > 4^{-i-1}\},$$

so $(x,y) \in B^2 \setminus \hat{B}$ if and only if x and y belong to some ball $B' \in \mathscr{B}_{i+1}$, $B' \subset B$. It follows that $\lambda \otimes \lambda(\hat{B}) = \lambda^2(B)(1 - 1/k(B,i))$.

For B in $\mathscr{B}_i$, and $m(B,i) < r \leq 2m(B,i) + 1$, define

$$\begin{aligned} C(B,r) = \{\omega \in \Omega;\ \lambda \otimes \lambda(\{(x,y) \in \hat{B};\ d_\omega(x,y) \leq \tau 2^{-r}4^{-i-1}\}) \\ \geq (\exp(-2^{2r}))\lambda \otimes \lambda(\hat{B})\}, \end{aligned}$$

where τ is given by $\tau = (d(1)/2)^{1/2}$. The proof of Lemma 4 shows that $P(C(B,r)) \leq \exp(-2^{2r})$. Let

$$C(B) = \bigcup C(B,r),$$

where the union is taken over all choices of r, $m(B,i) < r \leq 2m(B,i) + 1$. So we have $P(C(B)) \leq 2\exp(-2^{2m(B,i)+2})$.

We denote by $\mathscr{B}_i'$ the collection of those B in $\mathscr{B}_i$ that satisfy

$$m(B,i) \geq 2\sum_{i_0\leq j<i} m(B,j) + 3(i - i_0 + 1). \tag{14}$$

[The reason for using $i - i_0 + 1$ instead of $i - i_0$ is to ensure that $m(B,i) \geq 3$.]

We denote, for D in $\mathscr{B}_i$,

$$\bar{A}(D) = \bigcup\Big\{C(B);\ B \subset D,\ B \in \bigcup_{j\geq i}\mathscr{B}_j'\Big\}.$$

[If no such B exists, we set $\bar{A}(D) = \varnothing$.]

LEMMA 7. *For D in $\mathscr{B}_i$, we have* $P(\bar{A}(D)) \le (4\prod_{i_0 \le j < i} k(D, j))^{-2}$.

PROOF. The proof is very similar to that of Lemma 5 and goes by decreasing induction over i. Since $\bar{A}(D) = \varnothing$ if D contains only one point, the formula holds for i large enough. Assuming now that it has been proved for $i + 1$, we prove it for i.

CASE 1. $D \in \mathscr{B}_i'$. We have $\bar{A}(D) = \bar{A}_1 \cup C(D)$, where

$$\bar{A}_1 = \bigcup\{\bar{A}(B);\, B \in \mathscr{B}_{i+1},\, B \subset D\}.$$

By the induction hypothesis,

$$P(\bar{A}_1) \le k(D, i)\Big(4 \prod_{i_0 \le j \le i} k(D, i)\Big)^{-2}.$$

Since $D \in \mathscr{B}_i'$, we have $k(D, i) \ge 2$, so that

$$P(\bar{A}_1) \le \tfrac{1}{2}\Big(4 \prod_{i_0 \le j < i} k(D, j)\Big)^{-2}.$$

We also have $P(C(D)) \le 2\exp(-2^{2m(D, i)+2})$. Now

$$m(D, i) \ge 2 \prod_{i_0 \le j < i} m(D, j) + 3(i - i_0 + 1),$$

so

$$2^{2m(D, i)} \ge 2^{i - i_0 + 1} 2^{\max_{j<i}(4m(D, j)+4)} \ge \sum_{i_0 \le j < i} 2^{4m(D, j)+4}.$$

On the other hand, since $m(D, i) \ge 1$, we have $2^6 \exp(-2^{2m(D, i)+1}) \le 1$, and since $k(D, j) \le \exp(2^{4m(D, j)+4})$, we get

$$2\exp(-2^{2m(D, i)+2}) \le \tfrac{1}{2}\Big(4 \prod_{i_0 \le j < i} k(D, j)\Big)^{-2}$$

and this completes the induction.

CASE 2. $D \notin \mathscr{B}_i'$. Then

$$P(\bar{A}(D)) \le k(D, i)\Big(4 \prod_{i_0 \le j \le i} k(D, j)\Big)^{-2} \le \Big(4 \prod_{i_0 \le j < i} k(D, j)\Big)^{-2}.$$

The proof is complete. □

We set $A = \Omega \setminus \bar{A}(U)$, so $P(A) \ge 1 - 1/16$. We fix ω in A, and we proceed to prove (8). For x in U, we denote by $i_1(x) < \cdots < i_{l(x)}(x)$ the indexes i such that $B(x, i) \in \mathscr{B}_i'$. (It is possible that no such index exists.) For $l \le l(x)$, we set

$$t_l(x) = 2i_l(x) + m(x, i_l(x)),$$
$$u_l(x) = 2i_l(x) + 2m(x, i_l(x)) + 1.$$

By definition of $\mathscr{B}_i'$, we have $m(x, i_l(x)) > 2m(x, i)$ whenever $i < i_l(x)$; so we have $u_{l'}(x) \le t_l(x)$ whenever $l' < l \le l(x)$. Let us fix $x \in U$ and $l \le l(x)$, and let $B = B(x, 4^{-i_l(x)})$. Let us fix r such that

$$m(x, i_l(x)) < r \le 2m(x, i_l(x)) + 1.$$

Fix $y \in U$, and define a by

$$\lambda(\{z \in B;\, d_\omega(y, z) \le \tau 2^{-r-1} 4^{-i_l(x)-1}\}) = a\lambda(B).$$

Then we have

$$\lambda \otimes \lambda(\{(u, v) \in B^2;\, d_\omega(u, v) \le \tau 2^{-r} 4^{-i_l(x)-1}\}) \ge a^2\lambda^2(B).$$

So we have

$$\begin{aligned} \lambda \otimes \lambda(\{(u, v) \in \hat{B};\, d_\omega(u, v) \le \tau 2^{-r} 4^{-i_l(x)-1}\}) \\ \ge \lambda^2(B)(a^2 - 1/k(x, i_l(x))) \\ \ge \lambda \otimes \lambda(\hat{B})(a^2 - 1/k(x, i_l(x))). \end{aligned}$$

Since $\omega \notin \bar{A}(B)$, we see that

$$a^2 \le 1/k(x, i_l(x)) + \exp(-2^{2r}).$$

Since $r - 1 \le 2m(x, i_l(x))$ and $k(x, i_l(x)) \ge \exp(2^{4m(x, i_l(x))})$, we get

$$a^2 \le \exp(-2^{2r}) + \exp(-2^{2(r-1)}) \le 2\exp(-2^{2(r-1)}).$$

So, if we set

$$g(y) = \frac{1}{\lambda(B)}\lambda(\{z \in B,\, d_\omega(y, z) \le \tau 2^{-r-1} 4^{-i_l(x)-1}\}),$$

we have shown that $0 \le g \le 2\exp(-2^{2(r-1)})$. For a probability measure ν on U, the computation of (11) shows that

$$\lambda(B)^{-1} \int_B \varphi(\nu(\{y;\, d_\omega(x, y) \le \tau 2^{-r-1} 4^{-i_l(x)-1}\}))\, d\lambda(x) \ge 2^{r-1}. \tag{15}$$

On the other hand,

$$\begin{aligned} \int_{2^{-u_l(x)-4}}^{2^{-t_l(x)-4}} \varphi(\nu(\{y;\, d_\omega(x, y) \le \tau\varepsilon\}))\, d\varepsilon \\ \ge \sum_{i=t_l(x)+4}^{u_l(x)+3} 2^{-i-1}\varphi(\nu(\{y;\, d_\omega(x, y) \le \tau 2^{-i}\})), \end{aligned}$$

so combining with (15) [where we take $r = i - 3 - 2i_l(x)$], we get

$$\begin{aligned} \int_B d\lambda(x)\left(\int_{2^{-u_l(x)-4}}^{2^{-t_l(x)-4}} \varphi(\nu(\{y;\, d_\omega(x, y) \le \tau\varepsilon\}))\, d\varepsilon\right) \\ \ge 2^{-5} 4^{-i_l(x)} m(x, i_l(x))\lambda(B) \\ = 2^{-5}\int_B 4^{-i_l(x)} m(x, i_l(x))\, d\lambda(x). \end{aligned} \tag{16}$$

Using the fact that $u_{l'}(x) \le t_l(x)$ for $l' < l$, and that $2^{-t_l(x)} \le \gamma$, we get by summation from (16)

$$\int_U d\lambda(x) \int_0^{K_7\gamma} \varphi(\nu(\{y; d_\omega(x, y) \le \varepsilon\}))\, d\varepsilon \tag{17}$$
$$\ge \tau 2^{-5} \int_U \eta_x(U)\, d\lambda(x),$$

where $\eta_x(U) = \sum_{l \le l(x)} 4^{-i_l(x)} m(x, i_l(x))$ and where K_7 is a universal constant.

For x in U, $s < l(x)$ and $i_1(x) < \cdots < i_s(x) \le j < i_{s+1}(x)$, we have, by immediate induction over j, that

$$m(x, j) \le 3(j - i_0 + 1)3^{j - i_0} + \sum_{l \le s} 3^{j - i_l(x)} m(x, i_l(x)).$$

For this, we deduce by summation that for some universal constants K_8, K_9, we have

$$\xi_x(U) \le K_8\gamma + K_9\eta_x(U),$$

so $\xi(U) \le \int \xi_x(U)\, d\lambda(x) \le K_8\gamma + K_9 \int_U \eta_x(U)\, d\lambda(x)$.

Combining with (17), this proves (8) and finishes the proof. □

Acknowledgment. My warmest thanks are due to Professor G. Pisier who suggested this direction of research.

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