The Annals of Probability
2003, Vol. 31, No. 4, 2003–2039

LARGE DEVIATIONS FOR THE ONE-DIMENSIONAL EDWARDS MODEL

BY REMCO VAN DER HOFSTAD, FRANK DEN HOLLANDER AND WOLFGANG KÖNIG

Eindhoven University of Technology, EURANDOM and TU Berlin

In this article, we prove a large deviation principle for the empirical drift of a one-dimensional Brownian motion with self-repellence called the Edwards model. Our results extend earlier work in which a law of large numbers and a central limit theorem were derived. In the Edwards model, a path of length T receives a penalty $e^{-\beta H_T}$, where H_T is the self-intersection local time of the path and $\beta \in (0,\infty)$ is a parameter called the strength of self-repellence. We identify the rate function in the large deviation principle for the endpoint of the path as $\beta^{2/3} I(\beta^{-1/3}\cdot)$, with $I(\cdot)$ given in terms of the principal eigenvalues of a one-parameter family of Sturm–Liouville operators. We show that there exist numbers $0 < b^{**} < b^* < \infty$ such that (1) I is linearly decreasing on $[0, b^{**}]$, (2) I is real-analytic and strictly convex on (b^{**}, ∞), (3) I is continuously differentiable at b^{**} and (4) I has a unique zero at b^*. (The latter fact identifies b^* as the asymptotic drift of the endpoint.) The critical drift b^{**} is associated with a crossover in the optimal strategy of the path: for $b \geq b^{**}$ the path assumes local drift b during the full time T, while for $0 \leq b < b^{**}$ it assumes local drift b^{**} during time $\frac{b^{**}+b}{2b^{**}}T$ and local drift $-b^{**}$ during the remaining time $\frac{b^{**}-b}{2b^{**}}T$. Thus, in the second regime the path makes an overshoot of size $\frac{b^{**}-b}{2}T$ so as to reduce its intersection local time.

1. Introduction and main results. A linear polymer is a long chain of atoms or molecules, often referred to as monomers, which have a tendency to repel each other. This self-repellence is due to the excluded-volume effect: two monomers cannot occupy the same space. The self-repellence causes the polymer to spread itself out more than it would do in the absence of self-repellence. The most widely used ways to describe a polymer are the *Domb–Joyce model* and the *Edwards model*, which start from random walk and Brownian motion, respectively, and build in an appropriate penalty for self-intersections. The main interest lies in the behavior of the expected end-to-end distance of the polymer when its length gets large.

For the Domb–Joyce model (which is sometimes called the *weakly self-avoiding walk*) there are many rigorous asymptotic results known in dimensions $d = 1$ and $d \geq 5$. However, dimensions $d = 2, 3, 4$ are very dificult and the asymptotic

Received March 2002; revised October 2002.
AMS 2000 subject classifications. 60F05, 60F10, 60J55, 82D60.
Key words and phrases. Self-repellent Brownian motion, intersection local time, Ray–Knight theorems, large deviations, Airy function.

behavior is still open. A standard reference on mathematical results for and computer simulations of polymers is [14], which also includes an introduction to the main tool in high dimensions, the lace expansion. A general background on polymers from a physics and chemistry point of view may be found in [17], while a survey of mathematical results for one-dimensional polymers appears in [9]. See [13] for some new and insightful heuristics for two-dimensional polymers.

In contrast to the discrete Domb–Joyce model, the definition of the continuous Edwards model requires substantial work in dimensions $d = 2, 3, 4$, which is due to the accumulation of self-intersections of Brownian motion (see [18, 19, 2]). Here too there are no rigorous asymptotic results known to date. In the present article, we study the one-dimensional Edwards model, which can be easily defined in terms of the Brownian local times. Our present work is a natural continuation of our earlier paper [10], where we derived a central limit theorem for the end-to-end distance. Our goal is to derive a large deviation principle.

In Section 1.1, we define the model and recall our earlier results. Our new results are stated in Section 1.2. Our strategy of proof is explained in Section 1.3. At the end of that section, we give an outline of the rest of the paper.

1.1. *The Edwards model.* Let $B = (B_t)_{t\geq 0}$ be standard Brownian motion on $\mathbb{R}$ starting at the origin ($B_0 = 0$). Let P be the Wiener measure and let E be expectation with respect to P. For $T > 0$ and $\beta \in (0, \infty)$, define a probability law $\mathbb{Q}_T^\beta$ on paths of length T by setting

$$\frac{d\mathbb{Q}_T^\beta}{dP}[\,\cdot\,] = \frac{1}{Z_T^\beta} e^{-\beta H_T[\,\cdot\,]}, \qquad Z_T^\beta = E(e^{-\beta H_T}), \tag{1.1}$$

where

$$H_T\big[(B_t)_{t\in[0,T]}\big] = \int_0^T du \int_0^T dv\, \delta(B_u - B_v) = \int_{\mathbb{R}} dx\, L(T,x)^2 \tag{1.2}$$

is the Brownian intersection local time up to time T. The first expression in (1.2) is formal only. In the second expression the Brownian local times $L(T, x)$, $x \in \mathbb{R}$, appear. The law $\mathbb{Q}_T^\beta$ is called the *T-polymer measure with strength of self-repellence β*. The Brownian scaling property implies that

$$\mathbb{Q}_T^\beta\big((B_t)_{t\in[0,T]} \in \cdot\,\big) = \mathbb{Q}^1_{\beta^{2/3}T}\big((\beta^{-1/3} B_{\beta^{2/3}t})_{t\in[0,T]} \in \cdot\,\big). \tag{1.3}$$

It is known that under the law $\mathbb{Q}_T^\beta$ the endpoint B_T satisfies the following central limit theorem:

THEOREM 1.1 (Central limit theorem). *There are numbers $a^*, b^*, c^* \in (0, \infty)$ such that for any $\beta \in (0, \infty)$:*

(i) *Under the law $\mathbb{Q}_T^\beta$, the distribution of the scaled endpoint $(|B_T| - b^*\beta^{1/3}T)/c^*\sqrt{T}$ converges weakly to the standard normal distribution.*

(ii) $\lim_{T\to\infty} \frac{1}{T} \log Z_T^\beta = -a^* \beta^{2/3}$.

Theorem 1.1 is contained in [10], Theorem 2 and Proposition 1. For the identification of a^*, b^* and c^*, see (2.4) below. Bounds on these numbers appeared in [7], Theorem 3. The numerical values are $a^* \approx 2.19$, $b^* \approx 1.11$ and $c^* \approx 0.63$. The law of large numbers corresponding to Theorem 1.1(i) was first obtained by Westwater [20] (see also [8], Section 0.6).

1.2. *Main results.* The main object of interest in the present article is the large deviation rate function J_β defined by

$$-J_\beta(b) = \lim_{T\to\infty} \frac{1}{T} \log \mathbb{Q}_T^\beta(B_T \approx bT), \qquad b \in \mathbb{R}, \tag{1.4}$$

where $B_T \approx bT$ is an abbreviation for

$$\begin{aligned} &|B_T - bT| \le \gamma_T \text{ for some } \gamma_T > 0 \text{ such that} \\ &\qquad\qquad \gamma_T/T \to 0 \text{ and } \gamma_T/\sqrt{T} \to \infty \text{ as } T \to \infty. \end{aligned} \tag{1.5}$$

[We will see that the limit in (1.4) does not depend on the choice of γ_T.] Actually, we prefer to work with the function I_β defined by

$$-I_\beta(b) = \lim_{T\to\infty} \frac{1}{T} \log E\big(e^{-\beta H_T} \mathbb{1}_{\{B_T \approx bT\}}\big), \qquad b \in \mathbb{R}, \tag{1.6}$$

which according to Theorem 1.1(ii) differs from J_β by a constant, namely, $I_\beta = J_\beta + a^* \beta^{2/3}$ [recall (1.1)]. It is clear from (1.3) that

$$\beta^{-2/3} I_\beta(\beta^{1/3} b) = I_1(b), \qquad b \ge 0, \tag{1.7}$$

provided the limit in (1.6) exists for $\beta = 1$ and $b \ge 0$. Moreover,

$$I_\beta(b) = I_\beta(-b), \qquad b \le 0. \tag{1.8}$$

Therefore, we may restrict ourselves to $\beta = 1$ and $b \ge 0$. In the following discussion we write $I = I_1$ and $J = J_1$.

Our first main result says that I exists and has the shape exhibited in Figure 1. (In [15], Corollary 2.6 and Remark 2.7, it was proved that $\lim_{T\to\infty} \frac{1}{T} \log E(e^{-H_T} \mid B_T = 0) = -a^{**}$, which essentially gives the existence of $I(0)$ with value a^{**}. Furthermore, the existence of $I(b^*)$ with value a^* follows from our earlier work [10], Proposition 1.)

THEOREM 1.2 (Large deviations). *Let* $\beta = 1$.

(i) *For any* $b \ge 0$, *the limit* $I(b)$ *in* (1.6) *exists and is finite.*

(ii) I *is continuous and convex on* $[0, \infty)$, *and continuously differentiable on* $(0, \infty)$.

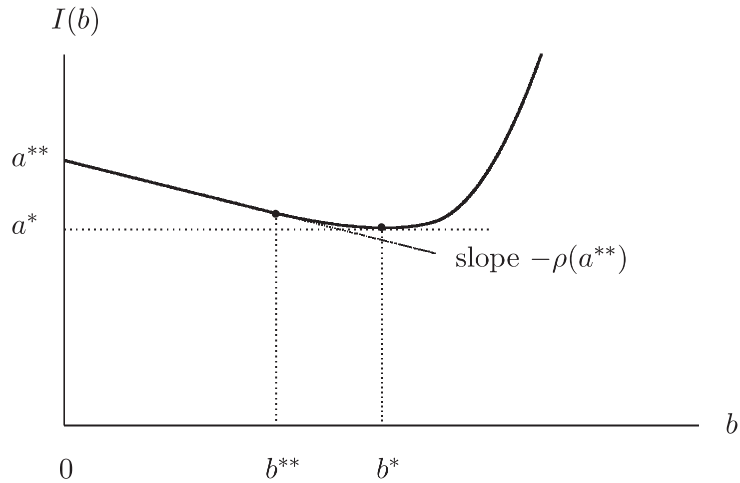

FIG. 1. *Qualitative picture of* $b \mapsto I(b) = J(b) + a^*$.

(iii) *There are numbers* $a^{**} \in (a^*, \infty)$, $b^{**} \in (0, b^*)$ *and* $\rho(a^{**}) \in (0, \infty)$ *such that* $I(0) = a^{**}$, *and* I *is linearly decreasing on* $[0, b^{**}]$ *with slope* $-\rho(a^{**})$, *is real-analytic and strictly convex on* (b^{**}, ∞), *and attains its unique minimum at* b^* *with* $I(b^*) = a^*$ *and* $I''(b^*) = 1/c^{*2}$.

(iv) $I(b) = \frac{1}{2}b^2 + \mathcal{O}(b^{-1})$ *as* $b \to \infty$.

The linear piece of the rate function has the following intuitive interpretation. If $b \geq b^{**}$, then the best strategy for the path to realize the large deviation event $\{B_T \approx bT\}$ is to assume local drift b during time T. In particular, the path makes no overshoot on scale T, and apparently this leads to the real-analyticity and strict convexity of I on (b^{**}, ∞). On the other hand, if $0 \leq b < b^{**}$, then this strategy is too expensive, since too small a drift leads to too large an intersection local time. Therefore the best strategy now is to assume local drift b^{**} during time $\frac{b^{**}+b}{2b^{**}}T$ and local drift $-b^{**}$ during the remaining time $\frac{b^{**}-b}{2b^{**}}T$. In particular, the path makes an overshoot on scale T, namely, $\frac{b^{**}-b}{2}T$, and this leads to the linearity of I on $[0, b^{**}]$. At the critical drift $b = b^{**}$, I is continuously differentiable.

For the identification of a^{**}, b^{**} and $\rho(a^{**})$, see (2.5). The numerical values are $a^{**} \approx 2.95$, $b^{**} \approx 0.85$ and $\rho(a^{**}) \approx 0.78$. These estimates can be obtained with the help of the method in [7]. For $b \to \infty$, $I(b)$ is determined by the Gaussian tail of B_T because the intersection local time H_T vanishes.

As a common feature in large deviation theory, there is an intimate relationship between the rate function I and the cumulant generating function Λ given by

$$\Lambda(\mu) = \lim_{T\to\infty} \frac{1}{T} \log E(e^{-H_T} e^{\mu B_T}), \qquad \mu \in \mathbb{R}. \tag{1.9}$$

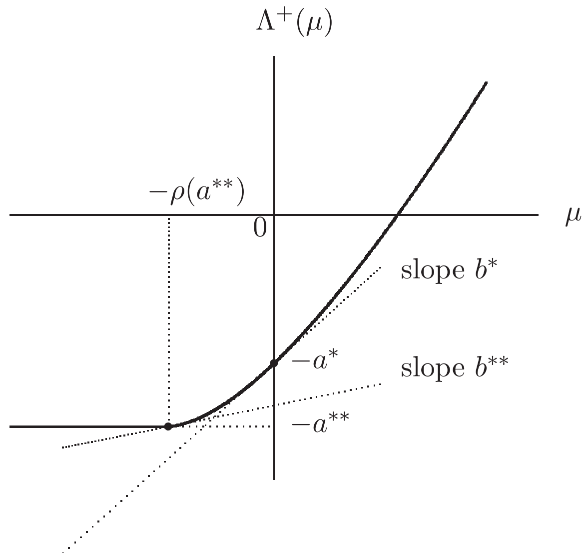

FIG. 2. *Qualitative picture of* $\mu \mapsto \Lambda^+(\mu)$.

More precisely, since I is convex on $[0, \infty)$ and on $(-\infty, 0]$, it is related to the two cumulant generating functions $\Lambda^+, \Lambda^- : \mathbb{R} \to \mathbb{R}$ given by

$$\Lambda^+(\mu) = \lim_{T\to\infty} \frac{1}{T} \log E\bigl(e^{-H_T} e^{\mu B_T} \mathbb{1}_{\{B_T \geq 0\}}\bigr), \tag{1.10}$$

$$\Lambda^-(\mu) = \lim_{T\to\infty} \frac{1}{T} \log E\bigl(e^{-H_T} e^{\mu B_T} \mathbb{1}_{\{B_T \leq 0\}}\bigr). \tag{1.11}$$

Obviously, $\Lambda^+(-\mu) = \Lambda^-(\mu)$ for any $\mu \in \mathbb{R}$, provided one limit exists, and

$$\Lambda(\mu) = \max\{\Lambda^+(\mu), \Lambda^-(\mu)\} = \Lambda^+(|\mu|), \qquad \mu \in \mathbb{R}. \tag{1.12}$$

Our second main result says that Λ^+ exists, has the shape exhibited in Figure 2 and its Legendre transform is equal to I on $[0, \infty)$.

THEOREM 1.3 (Exponential moments). *Let* $\beta = 1$.

(i) *For any* $\mu \in \mathbb{R}$, *the limit* $\Lambda^+(\mu)$ *in* (1.10) *exists and is finite*.
(ii) Λ^+ *equals* $-a^{**}$ *on* $(-\infty, -\rho(a^{**})]$, *is real-analytic and strictly convex on* $(-\rho(a^{**}), \infty)$, *and satisfies* $\lim_{\mu\downarrow-\rho(a^{**})}(\Lambda^+)'(\mu) = b^{**}$.
(iii) $\Lambda^+(\mu) = \frac{1}{2}\mu^2 + \mathcal{O}(\mu^{-1})$ *as* $\mu \to \infty$.
(iv) *The restriction of* I *to* $[0, \infty)$ *is the Legendre transform of* Λ^+, *that is*,

$$I(b) = \max_{\mu\in\mathbb{R}}[b\mu - \Lambda^+(\mu)], \qquad b \geq 0. \tag{1.13}$$

As a consequence of Theorem 1.3(ii), the maximum on the right-hand side of (1.13) is attained at some $\mu > -\rho(a^{**})$ if $b > b^{**}$ and at $\mu = -\rho(a^{**})$ if $0 \le b \le b^{**}$. Analogous assertions hold for Λ^-, in particular, the restriction of I to $(-\infty, 0]$ is the Legendre transform of Λ^-. Note that the cumulant generating function Λ in (1.9) is symmetric and strictly convex on $\mathbb{R}$, and nondifferentiable at 0, with $\Lambda(0) = -a^*$ and $\lim_{\mu\downarrow 0} \Lambda'(\mu) = b^*$.

1.3. *Strategy of the proof.* To give the reader some guidance to the proofs of Theorems 1.2 and 1.3, we now outline the approach that we follow. This approach heavily relies on the line of attack that we introduced in our earlier article [10].

The first basic tool is a description of the joint distribution of the local times and the endpoint at a fixed time $T > 0$ in terms of a combination of the two well-known Ray–Knight theorems. The main idea is that, conditional on B_T, $L(T, B_T)$ and $L(T, 0)$, the *middle part* of the local times,

$$X = \big(L(T, B_T - v)\big)_{v\in[0,B_T]}, \tag{1.14}$$

is a two-dimensional squared Bessel process, while the *two boundary parts*,

$$X^{\star,1} = \big(L(T, B_T + v)\big)_{v\in[0,\infty)} \quad \text{and} \quad X^{\star,2} = \big(L(T, -v)\big)_{v\in[0,\infty)}, \tag{1.15}$$

are two zero-dimensional squared Bessel processes (sometimes also called Feller's diffusions). Here, $y = B_T$ appears as the time horizon for X, while $h_1 = L(T, B_T)$ and $h_2 = L(T, 0)$ appear as the initial and terminal values for $X^{\star,1}$ and $X^{\star,2}$, respectively. The three processes are independent given h_1 and h_2, but are conditional on having total integral equal to T, since the sum of the integrals $\int_0^y X_v\,dv$, $t_1 = \int_0^\infty X_v^{\star,1}\,dv$ and $t_2 = \int_0^\infty X_v^{\star,2}\,dv$ is equal to T. Note that t_1 and t_2 are the time the Brownian motion spends in the intervals $[B_T, \infty)$ and $(-\infty, 0]$, respectively.

A full representation of the local time process $(L(T, x))_{x\in\mathbb{R}}$ and the endpoint B_T is achieved by integrating over the five variables y, h_1, h_2, t_1 and t_2. The intersection local time H_T equals the sum of the integrals of the *squares* of the processes. Hence, the density e^{-H_T} naturally splits into a product of the respective contributions coming from the middle part and the two boundary parts. The contribution coming from the two boundary parts is a certain function of the starting point $X_0^{\star,1} = X_0 = h_1$ and of t_1, and, respectively, of $X_0^{\star,2} = X_y = h_2$ and of t_2. Since y can be seen as the time at which the additive process of X, $A(t) = \int_0^t X_v\,dv$, reaches the value $T - t_1 - t_2$, the inverse A^{-1} and the time-changed process $Y = X \circ A^{-1}$ play an important role as well.

The second basic tool is a certain Girsanov transformation for the middle part. This transformation is chosen such that it absorbs the exponential of the integral over X^2 into the transition probability of the transformed process. The transformed process has much better recurrence properties than the free squared Bessel process, in particular, it has an invariant distribution. We rewrite the expectation under consideration in terms of an expected value over the transformed process, starting in the invariant distribution, and summarize the contributions coming from the two boundary parts in terms of expectations over the processes $X^{\star,1}$ and $X^{\star,2}$. In the description of the latter expectations the well-known Airy function appears in a natural way.

Since we integrate over all values t_1 and t_2 of the total integrals of $X^{\star,1}$ and $X^{\star,2}$, to derive the limit as $T \to \infty$ it is necessary to control the integrand by a bound that is integrable in t_1 and t_2, and does not depend on T, so that we can apply the dominated convergence theorem. This turns out to be a subtle issue. To solve this problem, we use a certain expansion in terms of the zeroes and certain L^2-normalized shifts of the Airy function, the latter of which turn out to form an L^2 orthonormal system. This fact is derived by showing that a certain operator, which is closely related to the Airy differential equation, possesses a compact resolvent.

An outline of the present paper is as follows. In Section 2 we introduce the preparatory material that is needed in the proofs: the squared Bessel processes, the Airy function, the Girsanov transformation, and the eigenvalue expansion in terms of the zeroes and the shifts of the Airy function. Two key propositions are presented in Section 3: a representation for the probabilities of a large class of events under the Edwards measure in terms of the Ray–Knight theorems and an integrable majorant under which the dominated convergence theorem can be applied. In Section 4 we carry out the proofs of Theorems 1.2 and 1.3. Some more refined results about the Edwards model (which will be needed in a forthcoming article [11]) appear in Section 5. Finally, Section 6 contains the proof of a technical result used in Section 5.

2. Preliminaries. In this section we provide the basic tools that are needed for the proofs of our main results stated in Section 1.2. These tools are taken from [8] and [10] and references cited therein. Recall the strategy of proof sketched in Section 1.3.

Section 2.1 introduces a certain family of Sturm–Liouville operators that is needed to define and describe the Girsanov transformation introduced in Section 2.2. These operators play the role of generators of the transformed process. Their spectral properties determine the constants a^*, b^* and c^* that appear in Theorem 1.1. Section 2.2 introduces the Girsanov transformation of the two-dimensional squared Bessel process and provides further ingredients that are needed for the formulation of the Ray–Knight theorems as well as a certain mixing property. Section 2.3 explains the relationship between the Airy function and the description of the boundary parts. Furthermore, it provides a spectral decomposition in terms of the zeroes and shifts of the Airy function, which turn out to be the eigenvalues and eigenfunctions of a certain operator that has a compact resolvent in an appropriate L^2 space.

2.1. *Sturm–Liouville operators and definition of the constants.* In [8], Section 0.4, we introduced and analyzed a family of Sturm–Liouville operators $\mathcal{K}^a : L^2[0,\infty) \cap C^2[0,\infty) \to C[0,\infty)$, indexed by $a \in \mathbb{R}$ and defined as

$$(\mathcal{K}^a x)(h) = 2hx''(h) + 2x'(h) + (ah - h^2)x(h), \qquad h \geq 0. \tag{2.1}$$

The operator $\mathcal{K}^a$ is symmetric and has a largest eigenvalue $\rho(a) \in \mathbb{R}$ with multiplicity 1. The corresponding strictly positive (and L^2-normalized) eigenfunction $x_a : [0,\infty) \to (0,\infty)$ is real-analytic and vanishes faster than exponential at infinity; more precisely,

$$\lim_{h\to\infty} h^{-3/2} \log x_a(h) = -\frac{\sqrt{2}}{3}. \tag{2.2}$$

The eigenvalue function $\rho : \mathbb{R} \to \mathbb{R}$ has the following properties:

(2.3)
(a) ρ is real-analytic;
(b) ρ is strictly log-convex, strictly convex and strictly increasing;
(c) $\lim_{a\downarrow -\infty} \rho(a) = -\infty$, $\rho(0) < 0$ and $\lim_{a\to\infty} \rho(a) = \infty$.

In terms of this object, the numbers a^*, b^* and c^* that appear in Theorem 1.1 are defined as

$$\rho(a^*) = 0, \qquad b^* = \frac{1}{\rho'(a^*)}, \qquad c^{*2} = \frac{\rho''(a^*)}{\rho'(a^*)^3}, \tag{2.4}$$

while the numbers a^{**} and b^{**} that appear in Theorem 1.2 are defined as

$$a^{**} = 2^{1/3}(-a_0), \qquad b^{**} = \frac{1}{\rho'(a^{**})}, \tag{2.5}$$

where a_0 (≈ -2.3381) is the largest zero of the Airy function:

(2.6) Ai is the unique solution of the Airy differential equation $y''(h) = hy(h)$ that vanishes at infinity.

From [10], Lemma 6, we know that $a^* < -a_0$. Therefore, $a^{**} > a^*$, which in turn implies that $b^{**} < b^*$.

2.2. *Squared Bessel processes, a Girsanov transformation and a mixing property.* The main pillars in our study of the Edwards model are the Ray–Knight theorems, which give a description of the joint distribution of the local time process $(L(T,x))_{x\in\mathbb{R}}$ and the endpoint B_T. These are summarized in Proposition 3.1 below. The key ingredients entering into this description are introduced here.

The first key ingredients are:

(i) a squared two-dimensional Bessel process (BESQ^2), $X = (X_v)_{v\geq 0}$,
(ii) a squared zero-dimensional Bessel process (BESQ^0), $X^\star = (X^\star_v)_{v\geq 0}$,

and their additive functionals

$$A(t) = \int_0^t X_v \, dv, \qquad A^\star(t) = \int_0^t X_v^\star \, dv, \qquad t \geq 0. \tag{2.7}$$

The respective (pre)generators of BESQ^2 and BESQ^0 are given by

$$Gf(h) = 2hf''(h) + 2f'(h), \qquad G^\star f(h) = 2hf''(h) \tag{2.8}$$

for twice continuously differentiable functions $f : [0, \infty) \to \mathbb{R}$. For $h \geq 0$, we write $\mathbb{P}_h$ and $\mathbb{P}_h^\star$ to denote the probability law of X and $X^\star$ given $X_0 = h$ and $X_0^\star = h$, respectively. BESQ^2 takes values in $C^+ = C^+[0, \infty)$, the set of nonnegative continuous functions on $[0, \infty)$. It has 0 as an entrance boundary, which is not visited in finite positive time with probability 1. BESQ^0 takes values in $C_0^+ = C_0^+[0, \infty)$, the subset of those functions in C^+ that hit zero and afterward stay at zero. It has 0 as an absorbing boundary, which is visited in finite time with probability 1.

The second key ingredient is a certain Girsanov transformation, which turns BESQ^2 into a diffusion with strong recurrence properties. Namely, the process $(D_y^{(a)})_{y \geq 0}$ defined by

$$D_y^{(a)} = \frac{x_a(X_y)}{x_a(X_0)} \exp\left\{ -\int_0^y \left[(X_v)^2 - aX_v + \rho(a) \right] dv \right\}, \qquad y \geq 0, \tag{2.9}$$

is a martingale under $\mathbb{P}_h$ for any $h \geq 0$ and hence serves as a density with respect to a new Markov process in the sense of a Girsanov transformation. More precisely, the transformed process, which we also denote by $X = (X_v)_{v \geq 0}$, has the transition density

$$\widehat{P}_y^a(h_1, h_2) \, dh_2 = \mathbb{E}_{h_1}\big(D_y^{(a)} \mathbb{1}_{\{X_y \in dh_2\}}\big), \qquad y, h_1, h_2 \geq 0. \tag{2.10}$$

We write $\widehat{\mathbb{P}}_h^a$ to denote the probability law of the transformed process X given $X_0 = h$. This transformed process possesses the invariant distribution $x_a(h)^2 \, dh$, and so

$$\widehat{\mathbb{P}}^a = \int_0^\infty dh \, x_a(h)^2 \, \mathbb{P}_h^a \tag{2.11}$$

is its probability law in equilibrium. The transformed process is reversible under $\widehat{\mathbb{P}}^a$, since BESQ^2 is reversible with respect to the Lebesgue measure. Hence, $x_a(h_1)^2 \widehat{P}_y^a(h_1, h_2)$ is symmetric in $h_1, h_2 \geq 0$ for any $y \geq 0$.

The third key ingredient is the time-changed transformed process

$$Y = X \circ A^{-1} = (X_{A^{-1}(t)})_{t \geq 0}. \tag{2.12}$$

We write $\widetilde{\mathbb{P}}_h^a$ to denote the probability law of Y given $Y_0 = h$. This process possesses the invariant distribution $\frac{1}{\rho'(a)} h x_a(h)^2 \, dh$, and so

$$\widetilde{\mathbb{P}}^a = \frac{1}{\rho'(a)} \int_0^\infty dh \, h x_a(h)^2 \widetilde{\mathbb{P}}_h^a \tag{2.13}$$

is its probability law in equilibrium. Both transformed processes X and $Y = X \circ A^{-1}$ are ergodic.

The following mixing property will be used frequently in the sequel. By $\langle \cdot, \cdot \rangle$ we denote the inner product on $L^2 = L^2[0, \infty)$ and we write $\langle f, g \rangle_\circ = \int_0^\infty dh\, hf(h)g(h)$ for the inner product on L^2 weighted with the identity. The latter space is denoted by $L^{2,\circ} = L^{2,\circ}[0, \infty)$.

PROPOSITION 2.1. *Fix $a \in \mathbb{R}$ and fix measurable functions $f, g : [0, \infty) \to \mathbb{R}$ such that $f/\mathrm{id}, g \in L^{2,\circ}$. For any family of measurable functions $f_s, g_s : [0, \infty) \to \mathbb{R}$, $s \geq 0$, such that $f_s/\mathrm{id}, g_s \in L^{2,\circ}$, $s \geq 0$, and $f_s \to f, g_s \to g$ as $s \to \infty$ uniformly on compacts and in $L^{2,\circ}$, and for any family a_s, $s \geq 0$, such that $a_s \to a$ as $s \to \infty$,*

$$\lim_{s\to\infty} \widehat{\mathbb{E}}^{a_s}\left(\frac{f_s(X_0)}{x_{a_s}(X_0)} \frac{g_s(Y_s)}{x_{a_s}(Y_s)} \right) = \langle f, x_a \rangle \frac{1}{\rho'(a)} \langle g, x_a \rangle_\circ. \tag{2.14}$$

This proposition is only a slight extension of Proposition 3 in [10] and therefore we omit its proof.

2.3. *$BESQ^0$, the Airy function and a spectral decomposition.* In this section, which is technically involved, we derive certain integrability properties at $-\infty$ of the Airy function introduced in (2.6). This is necessary, since it turns out that the contribution to the random variable e^{-H_T} that comes from the two boundary parts can be summarized in an expression that is identical to the Airy function. Recall that $X^\star$ is a BESQ^0 process: it is used in Section 3.1 to represent the two boundary parts.

For $a < a^{**}$, introduce the function $y_a : [0, \infty) \to (0, \infty]$ defined by

$$y_a(h) = \mathbb{E}_h^\star\left(\exp\left\{ \int_0^\infty [aX_v^\star - (X_v^\star)^2]\, dv \right\} \right). \tag{2.15}$$

[As a consequence of (2.18) and Proposition 2.2 below, the expectation on the right-hand side is infinite for $a > a^{**}$.] It is known (see [10], Lemma 5) that y_a is equal to a normalized scaled shift of the Airy function (Ai):

$$y_a(h) = \frac{\mathrm{Ai}(2^{-1/3}(h - a))}{\mathrm{Ai}(-2^{-1/3}a)}, \qquad h \geq 0. \tag{2.16}$$

It is well known (see [5], page 43 and (6.2) below) that y_a vanishes faster than exponentially at infinity:

$$\lim_{h\to\infty} h^{-3/2} \log y_a(h) = -\frac{\sqrt{2}}{3}. \tag{2.17}$$

An important role is played in the sequel by the function $w : [0, \infty)^2 \to [0, \infty)$ defined by

$$w(h, t)\, dt = \mathbb{E}_h^\star\left(\exp\left\{ -\int_0^\infty (X_v^\star)^2\, dv \right\} \mathbb{1}_{\{A^\star(\infty) \in dt\}} \right). \tag{2.18}$$

Here we recall from (2.7) that $A^\star(\infty) = \int_0^\infty X_v^\star\,dv$. Informally, the function w and variants of it are used later to express the contribution coming from the two boundary parts when the Brownian motion spends time t in these parts (see the beginning of Section 3.1).

It is easily seen from (2.7) and (2.15) that $\int_0^\infty dt\, e^{at} w(h,t) = y_a(h)$ for $a < a^{**}$. We also have the representation for $w(h,t)$, derived in [10], Lemma 7,

$$w(h,t) = E_{h/2}\biggl(\exp\biggl\{-2\int_0^t B_s\,ds\biggr\}\bigg| T_0 = t\biggr)\varphi_h(t), \tag{2.19}$$

$$\varphi_h(t) = \frac{P_{h/2}(T_0 \in dt)}{dt} = (8\pi)^{-1/2} t^{-3/2} h e^{-h^2/8t}, \tag{2.20}$$

with $T_0 = \inf\{t > 0 : B_t = 0\}$ the first time B hits zero. (We write P_h and E_h for probability and expectation with respect to standard Brownian motion B starting at $h \geq 0$, so that $P = P_0$, $E = E_0$.)

We need the following expansion of the function w in terms of shifts of the Airy function:

PROPOSITION 2.2. (i) *For any* $\varepsilon > 0$,

$$w(h,t) = \sum_{k=0}^{\infty} \exp\{a^{(k)}(t-\varepsilon)\}\langle w(\cdot,\varepsilon), \mathrm{e}_k(\cdot)\rangle\, \mathrm{e}_k(h), \qquad h \geq 0,\ t \geq \varepsilon, \tag{2.21}$$

where

$$a^{(k)} = 2^{1/3} a_k, \qquad \mathrm{e}_k(h) = c_k \operatorname{Ai}\bigl(2^{-1/3}(h + a^{(k)})\bigr), \qquad h \geq 0, \tag{2.22}$$

with a_k the kth largest zero of Ai *and with c_k chosen such that* $\|\mathrm{e}_k\|_2 = 1$.

(ii) *There exist constants* $K_1, K_2, K_3 \in (0,\infty)$ *such that*

$$-a^{(k)} \sim K_1 k^{2/3}, \qquad k \to \infty, \tag{2.23}$$

$$\int_0^\infty h\mathrm{e}_k(h)^2\,dh \leq K_2 k^{2/3} \qquad \forall k, \tag{2.24}$$

$$\int_0^\infty \frac{1}{h}\mathrm{e}_k(h)^2\,dh \leq K_3 k^{1/3} \qquad \forall k. \tag{2.25}$$

[Note that $a^{(0)} = -a^{**}$ by (2.5).]

PROOF. (i) The proof comes in five steps. We write c for a generic constant in $(0,\infty)$ whose value may change from line to line.

Step 1. Let $\mathcal{K}^\star$ be the second-order differential operator on $C_0^\infty = C_0^\infty[0,\infty)$, the set of smooth functions $x : [0,\infty) \to \mathbb{R}$ that vanish at zero, defined by

$$(\mathcal{K}^\star x)(h) = \begin{cases} 2x''(h) - hx(h), & \text{if } h > 0, \\ 0, & \text{if } h = 0. \end{cases} \tag{2.26}$$

This operator is symmetric with respect to the L^2 inner product on $L_0^2 = L^2 \cap C_0^\infty$. Furthermore, we can identify all the eigenvalues and eigenfunctions of $\mathcal{K}^\star$ in L_0^2 in terms of scaled shifts of the Airy function. Namely, a comparison of (2.6) and (2.26) shows that the kth eigenspace is spanned by the eigenfunction $\mathrm{e}_k : [0,\infty) \to \mathbb{R}$ given in (2.22) and the kth eigenvalue is $a^{(k)}$, $k \in \mathbb{N}_0$.

Step 2. We next show that $\mathcal{K}^\star$ has a compact inverse on L^2. Therefore, this inverse has an orthonormal basis of eigenvectors in L^2 and, hence, the same is true for $\mathcal{K}^\star$ itself. Consequently, $(\mathrm{e}_k)_{k\in\mathbb{N}_0}$ is an orthonormal basis of L^2. This fact is needed later.

We begin by identifying the inverse of $\mathcal{K}^\star$. To do so, we follow [6]. Let

$$y_1(u) = \mathrm{Bi}(2^{1/3}u) - \mathrm{Bi}(0)\frac{\mathrm{Ai}(2^{1/3}u)}{\mathrm{Ai}(0)}, \qquad y_2(u) = \mathrm{Ai}(2^{1/3}u), \tag{2.27}$$

where Ai is the Airy function and Bi is another, linearly independent, solution to (2.6) (for the precise definitions of Ai and Bi, see [1], 10.4.1–10.4.3). Hence, both y_1 and y_2 solve $\mathcal{K}^\star y = 0$, y_1 satisfies the boundary condition at zero $[y_1(0) = 0]$, while y_2 satisfies the boundary condition at infinity ($y_2 \in L^2$). Let $G : [0,\infty)^2 \to \mathbb{R}$ (Green function) be defined by

$$G(u,v) = K y_1(u \wedge v) y_2(u \vee v) \qquad \text{with } K = -2y_1'(0)y_2(0). \tag{2.28}$$

Let Γ be the operator on L^2 defined by

$$(\Gamma y)(u) = \int_0^\infty G(u,v) y(v)\, dv. \tag{2.29}$$

According to [6], Proposition 2.15, $x = \Gamma y$ is a weak solution of the equation $\mathcal{K}^\star x = y$ with boundary condition $x(0) = 0$, for any $y \in L^2$. In fact, we can adapt the proof of [6], Proposition 9.12, to see that Γ is the inverse of $\mathcal{K}^\star$, since $\mathcal{K}^\star x = 0$ does not have solutions in L^2 that satisfy the boundary condition $x(0) = 0$. Hence, we are done once we show that Γ is a compact operator.

Step 3. By [6], Theorem 8.54, it suffices to show that Γ is a Hilbert–Schmidt operator, that is, G is square-integrable on $[0,\infty)^2$. To show this, we first note that (2.28) gives

$$\int_0^\infty du \int_0^\infty dv\, G^2(u,v) = 2K^2 \int_0^\infty du \int_0^u dv\, y_2(u)^2 y_1(v)^2. \tag{2.30}$$

Substitute (2.27) to see that, since $\mathrm{Ai} \in L^2$, it suffices to show that

$$\int_0^\infty du \int_0^u dv\ \mathrm{Ai}(u)^2\, \mathrm{Bi}(v)^2 < \infty. \tag{2.31}$$

Since Bi is locally bounded and $\mathrm{Ai} \in L^2$, the latter amounts to

$$\int_1^\infty du \int_1^u dv\ \mathrm{Ai}(u)^2\, \mathrm{Bi}(v)^2 < \infty. \tag{2.32}$$

We next use [1], 10.4.59 and 10.4.63, which show that

$$\begin{aligned} &\mathrm{Ai}(u) \le c u^{-1/4} \exp\{-\tfrac{2}{3} u^{3/2}\}, \\ &\mathrm{Bi}(v) \le c v^{-1/4} \exp\{\tfrac{2}{3} v^{3/2}\}, \qquad u, v \ge 1. \end{aligned} \tag{2.33}$$

Hence

$$\begin{aligned} &\int_1^\infty du \int_1^u dv\, \mathrm{Ai}(u)^2 \,\mathrm{Bi}(v)^2 \\ &\qquad \le c^4 \int_1^\infty du\, u^{-1/2} \int_1^u dv\, v^{-1/2} \exp\{-\tfrac{4}{3}(u^{3/2} - v^{3/2})\}. \end{aligned} \tag{2.34}$$

Use integration by parts to see that

$$\begin{aligned} &\int_1^u dv\, v^{-1/2} \exp\biggl\{-\frac{4}{3}(u^{3/2} - v^{3/2})\biggr\} \\ &\qquad = \frac{1}{2} \int_1^u dv\, v^{-1} \frac{d}{dv}\biggl(\exp\biggl\{-\frac{4}{3}(u^{3/2} - v^{3/2})\biggr\}\biggr) \\ &\qquad \le \frac{1}{2}\biggl[v^{-1} \exp\biggl\{-\frac{4}{3}(u^{3/2} - v^{3/2})\biggr\}\biggr]_{v=1}^u \le \frac{1}{2} u^{-1}, \qquad u \ge 1. \end{aligned} \tag{2.35}$$

Hence

$$\int_1^\infty du \int_1^u dv\, \mathrm{Ai}(u)^2 \,\mathrm{Bi}(v)^2 \le \tfrac{1}{2} c^4 \int_1^\infty du\, u^{-3/2} < \infty. \tag{2.36}$$

This proves that Γ is a compact operator, so that $(\mathrm{e}_k)_{k \in \mathbb{N}_0}$ is an orthonormal basis of L^2.

Step 4. To prove the expansion in (2.21), we now need the following lemma:

LEMMA 2.3. *For any $\varepsilon > 0$, the function w is a solution of the initial-boundary-value problem*

$$\begin{aligned} &\partial_t w(h, t) = \mathcal{K}^\star(w(\cdot, t))(h), \qquad h \ge 0, t > \varepsilon, \\ &w(0, t) \equiv 0, \qquad t \ge \varepsilon, \end{aligned} \tag{2.37}$$

and the initial value $w(\cdot, \varepsilon)$ lies in C_0^∞.

PROOF. Use the Markov property at time $s > 0$ in (2.19) to see that, for any $h > 0$ and $t > s$,

$$w(h, t) = E_{h/2}\biggl(\exp\biggl\{-\int_0^s 2B_v\, dv\biggr\} \mathbb{1}_{\{T_0 > s\}} w(2B_s, t - s)\biggr). \tag{2.38}$$

Now differentiate with respect to s at $s = 0$, to obtain

$$\begin{aligned} 0 &= -h w(h, t) + 2(\partial_h)^2 w(h, t) - \partial_t w(h, t) \\ &= \mathcal{K}^\star(w(\cdot, t))(h) - \partial_t w(h, t). \end{aligned} \tag{2.39}$$

This shows that the partial differential equation in (2.37) is satisfied on $(0,\infty)^2$. It is clear that it is also satisfied at the boundary where $h=0$, since $w(0,t)=0$ for all $t>0$ [recall (2.18) and (2.19)]. □

Step 5. From (2.19) it follows that $w(\cdot,\varepsilon)\in C_0^\infty$ for any $\varepsilon>0$. A spectral decomposition in terms of the eigenvalues $(a^{(k)})_{k\in\mathbb{N}_0}$ and the eigenfunctions $(\mathrm{e}_k)_{k\in\mathbb{N}_0}$ of $\mathcal{K}^\star$ shows that (2.37) has the solution given in (2.21).

(ii) In [1], 10.4.94, 10.4.96, 10.4.97 and 10.4.105, the following asymptotics for the Airy function can be found. As $k\to\infty$,

$$\begin{aligned} -a_k &\sim ck^{2/3}, & a_{k-1}-a_k &\sim ck^{-1/3},\\ \max_{[a_k,a_{k-1}]}|\operatorname{Ai}| &\sim ck^{-1/6}, & |\operatorname{Ai}'(a_k)| &\sim ck^{1/6}. \end{aligned}\tag{2.40}$$

We use these in combination with the observation that, by (2.6), Ai is convex (concave) between any two successive zeroes where it is negative (positive).

The first assertion in (2.40) is (2.23). To prove (2.24) and (2.25), we write the recursion

$$c_k^{-2}=\int_0^\infty \operatorname{Ai}\big(2^{-1/3}(h+a^{(k)})\big)^2\,dh=c_{k-1}^{-2}+2^{1/3}\int_{a_k}^{a_{k-1}}\operatorname{Ai}(h)^2\,dh.\tag{2.41}$$

Using the second and third assertions in (2.40), we find that $\int_{a_k}^{a_{k-1}}\operatorname{Ai}(h)^2\,dh\asymp k^{-2/3}$ and hence that $c_k^{-2}\asymp k^{1/3}$. In a similar way, we find that

$$\begin{aligned} &\int_0^\infty h\operatorname{Ai}\big(2^{-1/3}(h+a^{(k)})\big)^2\,dh\le ck,\\ &\int_0^\infty \frac{1}{h}\operatorname{Ai}\big(2^{-1/3}(h+a^{(k)})\big)^2\,dh\le ck^{2/3}. \end{aligned}\tag{2.42}$$

Combining (2.42) with (2.22) and $c_k^{-2}\asymp k^{1/3}$, we obtain (2.24) and (2.25). □

3. Two key propositions. In this section we present the main pillars of our proofs. Section 3.1 introduces the Ray–Knight theorems, which give a flexible representation for the probabilities of a large class of events under the Edwards measure. Section 3.2 exhibits an integrable majorant under which limits may be interchanged with integrals (recall the strategy of proof sketched in Section 1.3). In Section 3.1 we employ the squared Bessel process and the Girsanov transformation introduced in Sections 2.1 and 2.2, while in Section 3.2 we rely on the spectral analysis involving the Airy function introduced in Section 2.3.

3.1. *Ray–Knight representation.* In this section we formulate the Ray–Knight theorems that were already outlined in Section 1.3. We do this in the compact form derived in [10], Section 1.2, which is best suited for the arguments in the sequel.

Recall that C_0^+, the set of continuous functions $[0,\infty) \to [0,\infty)$ that are absorbed in 0, is the state space of BESQ^0, $X^\star$. For any measurable set $G \subset C_0^+$, define $w_G : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$w_G(h,t)\,dt = \mathbb{E}_h^\star\biggl(\exp\biggl\{-\int_0^\infty (X_v^\star)^2\,dv\biggr\}\mathbb{1}_{\{X^\star \in G\}}\mathbb{1}_{\{A^\star(\infty)\in dt\}}\biggr). \tag{3.1}$$

It is clear that w_G is increasing in G. For $G = C_0^+$, $w_{C_0^+}$ is identical to w defined in (2.18). Informally, the function w_G summarizes the contribution to the expectation of e^{-H_T} coming from each of the two boundary parts [i.e., the local time processes $(L(T, B_T+v))_{v\in[0,\infty)}$ and $(L(T,-v))_{v\in[0,\infty)}$] when the path lies in the event G.

For $y \ge 0$, denote by $C^+[0,y]$ the set of non-negative continuous functions on $[0,y]$. Then the set $\mathcal{C}^+ = \bigcup_{y\ge 0}(\{y\}\times C^+[0,y])$ is the appropriate state space of the pair $(B_T, L(T, B_T - \cdot)|_{[0,B_T]})$ that consists of the endpoint B_T (≥ 0) and the middle piece, by which we mean the local time process between the endpoint B_T and the starting point 0.

PROPOSITION 3.1 (Ray–Knight representation). *Fix $a \in \mathbb{R}$. Then, for any $T > 0$ and any measurable sets $G^+, G^- \subset C_0^+$ and $F \subset \mathcal{C}^+$,*

$$\begin{aligned} &e^{aT} E\bigl(e^{-H_T} e^{-\rho(a)B_T}\mathbb{1}_{\{L(T,B_T+\cdot)\in G^+\}}\mathbb{1}_{\{(B_T, L(T,B_T-\cdot)|_{[0,B_T]})\in F\}}\mathbb{1}_{\{L(T,-\cdot)\in G^-\}}\bigr)\\ &\quad = \int_0^\infty dt_1 \int_0^\infty dt_2\, \mathbb{1}_{\{t_1+t_2\le T\}} e^{a(t_1+t_2)}\\ &\qquad \times \widehat{\mathbb{E}}^a\biggl(\mathbb{1}_{\{(A^{-1}(T-t_1-t_2), X|_{[0,A^{-1}(T-t_1-t_2)]})\in F\}}\\ &\qquad\qquad \times \frac{w_{G^+}(X_0,t_1)}{x_a(X_0)}\frac{w_{G^-}(Y_{T-t_1-t_2},t_2)}{x_a(Y_{T-t_1-t_2})}\biggr). \end{aligned} \tag{3.2}$$

PROOF. We briefly indicate how (3.2) comes about. Details can be found in [10], Section 1.2. Recall the notation in Section 2.2. Fix $T > 0$. Then, according to the Ray–Knight theorems, for any $t_1, t_2, h_1, h_2 \ge 0$ and $y > 0$, conditioned on the event

$$\begin{aligned} &\{B_T = y\}\cap\{L(T,B_T) = h_1\}\cap\{L(T,0) = h_2\}\\ &\quad \cap\biggl\{\int_{B_T}^\infty L(T,x)\,dx = t_1\biggr\}\cap\biggl\{\int_0^\infty L(T,-x)\,dx = t_2\biggr\}, \end{aligned} \tag{3.3}$$

the joint distribution of the processes

$$L(T, B_T+\cdot), \qquad L(B_T - \cdot)|_{[0,y]}, \qquad L(T,-\cdot) \tag{3.4}$$

on $C_0^+ \times C^+[0,y]\times C_0^+$ is equal to the joint distribution of the processes

$$X^{\star,1}(\cdot), \qquad X(\cdot)|_{[0,y]}, \qquad X^{\star,2}(\cdot) \tag{3.5}$$

under

$$\begin{aligned} &\mathbb{P}^{\star}_{h_1}\big(\cdot\,|A^{\star}(\infty)=t_1\big)\\ &\qquad\otimes\mathbb{P}_{h_1}\big(\cdot\,|A(y)=T-t_1-t_2, X_y=h_2\big)\otimes\mathbb{P}^{\star}_{h_2}\big(\cdot\,|A^{\star}(\infty)=t_2\big), \end{aligned}\tag{3.6}$$

where X is BESQ^2, and $X^{\star,1}$ and $X^{\star,2}$ are independent copies of BESQ^0. In particular, the intersection local time in (1.2) has the representation

$$H_T \stackrel{\text{law}}{=} \int_0^\infty (X_v^{\star,1})^2\,dv+\int_0^y (X_v)^2\,dv+\int_0^\infty (X_v^{\star,2})^2\,dv.\tag{3.7}$$

Use (2.10) for $y=A^{-1}(T-t_1-t_2)$ and note that, on the event $\{A(T-t_1-t_2)=y\}\cap\{X_0=h_1, X_y=h_2\}$, (2.9) becomes

$$D_y^{(a)}=\frac{x_a(h_2)}{x_a(h_1)}\exp\biggl\{-\int_0^y (X_v)^2\,dv\biggr\}e^{a(T-t_1-t_2)}e^{-\rho(a)y},\tag{3.8}$$

which implies that

$$\begin{aligned} e^{aT}e^{-H_T}e^{-\rho(a)B_T} &\stackrel{\text{law}}{=} \frac{x_a(h_1)}{x_a(h_2)}D_y^{(a)}\exp\{a(t_1+t_2)\}\\ &\qquad\times\exp\biggl\{-\int_0^\infty (X_v^{\star,1})^2\,dv\biggr\}\exp\biggl\{-\int_0^\infty (X_v^{\star,2})^2\,dv\biggr\}. \end{aligned}\tag{3.9}$$

Integrate the left-hand side with respect to P and the right-hand side with respect to the measure in (3.6), and absorb the term $D_y^{(a)}$ into the notation of the transformed diffusion. Integrate over $h_1,h_2\ge 0$ and note that X_0 has the distribution $x_a(h_1)^2\,dh_1$ under $\widehat{\mathbb{E}}^a$. Finally, use the notation in (3.1) to obtain (3.2). $\square$

3.2. *Domination.* To perform the limit $T\to\infty$ on the right-hand side of (3.2), we need the dominated convergence theorem to interchange this limit with the integrals over t_1 and t_2. The following proposition provides the required domination.

PROPOSITION 3.2 (Domination). *For any a_s, $s\ge 0$, in a compact subset of $(-\infty,a^{**})$, the map*

$$(t_1,t_2)\mapsto \sup_{s\ge 0} e^{a_s(t_1+t_2)}\widehat{\mathbb{E}}^{a_s}\biggl(\frac{w(X_0,t_1)}{x_{a_s}(X_0)}\frac{w(Y_s,t_2)}{x_{a_s}(Y_s)}\biggr)\tag{3.10}$$

is integrable over $(0,\infty)^2$.

PROOF. Under the expectation in (3.10) we make a change of measure from the invariant distribution of X to the invariant distribution of Y, that is, we replace $\widehat{\mathbb{E}}^{a_s}$ by $\widetilde{\mathbb{E}}^{a_s}$ and add a factor of $\rho'(a_s)/Y_0$. Fix $1<p\le q<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$, apply Hölder's inequality and use the stationarity of Y under $\widetilde{\mathbb{P}}^{a_s}$. This gives, for any $t_1, t_2 > 0$, the bound

$$\begin{aligned}
&\widetilde{\mathbb{E}}^{a_s}\left(\frac{w(X_0,t_1)}{x_{a_s}(X_0)}\frac{w(Y_s,t_2)}{x_{a_s}(Y_s)}\right)\\
&\quad= \widetilde{\mathbb{E}}^{a_s}\left(\frac{w(Y_0,t_1)}{Y_0 x_{a_s}(Y_0)}\frac{w(Y_s,t_2)}{x_{a_s}(Y_s)}\right) \le W_p^{(1)}(t_1)W_q^{(2)}(t_2),
\end{aligned}\tag{3.11}$$

where the functions $W_p^{(1)}, W_q^{(2)} : (0,\infty) \to (0,\infty)$ are defined by

$$\begin{aligned}
W_p^{(1)}(t) &= \widetilde{\mathbb{E}}^{a_s}\left(\left(\frac{w(Y_0,t)}{Y_0 x_{a_s}(Y_0)}\right)^p\right)^{1/p},\\
W_q^{(2)}(t) &= \widetilde{\mathbb{E}}^{a_s}\left(\left(\frac{w(Y_0,t)}{x_{a_s}(Y_0)}\right)^q\right)^{1/q}.
\end{aligned}\tag{3.12}$$

Hence, it suffices to show that the maps

$$t \mapsto e^{a_s t} W_p^{(1)}(t), \qquad t \mapsto e^{a_s t} W_q^{(2)}(t) \tag{3.13}$$

are integrable at zero and at infinity, uniformly in s, for a suitable choice of p and q. In the proof of Proposition 4 in [10] we showed that $W_p^{(1)}$ and $W_q^{(2)}$, with a_s replaced by a^*, are integrable at zero when $p < q$ with p, q sufficiently close to 2. An inspection of the proof shows that they are actually integrable at zero uniformly in s.

We show that $t \mapsto e^{a_s t} W_2^{(1)}(t)$ and $t \mapsto e^{a_s t} W_2^{(2)}(t)$ are integrable at infinity uniformly in s. This will complete the proof because the left-hand side of (3.11) does not depend on p, q.

We use Proposition 2.2 with $\varepsilon = 1$ together with the representations [recall (2.13)]

$$\begin{aligned}
W_2^{(1)}(t) &= \frac{1}{\sqrt{\rho'(a_s)}}\left(\int_0^\infty dh\,\frac{1}{h} w(h,t)^2\right)^{1/2},\\
W_2^{(2)}(t) &= \frac{1}{\sqrt{\rho'(a_s)}}\left(\int_0^\infty dh\, h w(h,t)^2\right)^{1/2}.
\end{aligned}\tag{3.14}$$

Using (2.21), the Cauchy–Schwarz inequality and the fact that $\|\mathrm{e}_k\|_2 = 1$, we estimate

$$\begin{aligned}
W_2^{(1)}(t) \le \frac{1}{\sqrt{\rho'(a_s)}}\Bigg(&\|w(\cdot,1)\|_2^2 \sum_{k_1,k_2=0}^{\infty} \exp\{(a^{(k_1)} + a^{(k_2)})(t-1)\}\\
&\times \int_0^\infty \frac{1}{h} |\mathrm{e}_{k_1}(h)||\mathrm{e}_{k_2}(h)|\,dh\Bigg)^{1/2},\\
&\qquad t \ge 1.
\end{aligned}\tag{3.15}$$

Using the Cauchy–Schwarz inequality for the last integral, we obtain the bound

$$W_2^{(1)}(t) \le \frac{\|w(\cdot,1)\|_2}{\sqrt{\rho'(a_s)}} \sum_{k=0}^{\infty} \exp\{a^{(k)}(t-1)\} \biggl(\int_0^\infty \frac{1}{h} \mathrm{e}_k(h)^2\,dh\biggr)^{1/2}, \qquad t \ge 1. \tag{3.16}$$

In the same way, we find that

$$W_2^{(2)}(t) \le \frac{\|w(\cdot,1)\|_2}{\sqrt{\rho'(a_s)}} \sum_{k=0}^{\infty} \exp\{a^{(k)}(t-1)\} \biggl(\int_0^\infty h \mathrm{e}_k(h)^2\,dh\biggr)^{1/2}, \qquad t \ge 1. \tag{3.17}$$

Substitute (2.24) and (2.25) into (3.16) and (3.17), and use that $a^{(k)} \le a^{(0)} = -a^{**}$ to estimate

$$W_2^{(1)}(t) \vee W_2^{(2)}(t) \le c e^{-a^{**}(t-2)} \sum_{k=0}^{\infty} \exp\{a^{(k)}\} k^{1/3}, \qquad t \ge 2. \tag{3.18}$$

By (2.23), the sum in the right-hand side converges. Since $a_s < a^{**}$, $s \ge 0$, is bounded away from a^{**}, it is now obvious that the maps $t \mapsto e^{a_s t} W_2^{(1)}(t)$ and $t \mapsto e^{a_s t} W_2^{(2)}(t)$ are integrable at infinity uniformly in s. $\square$

4. Proving Theorems 1.2 and 1.3. In Sections 4.3 and 4.3 we give the proofs of Theorems 1.2 and 1.3 with the help of Propositions 3.1 and 3.2. In Section 4.1 we derive a technical proposition that is needed along the way.

4.1. *Growth rate of a restricted moment generating function.* Abbreviate $B_{[0,T]} = \{B_t : t \in [0,T]\}$ for the range of the path up to time T. For $T > 0$ and $\delta, C \in (0,\infty]$, define events

$$\mathcal{E}(\delta; T) = \{B_{[0,T]} \subset [-\delta, B_T + \delta]\}, \tag{4.1}$$

$$\mathcal{E}^{\le}(\delta, C; T) = \Bigl\{\max_{x \in [-\delta,\delta]} L(T,x) \le C, \max_{x \in [B_T-\delta, B_T+\delta]} L(T,x) \le C\Bigr\}. \tag{4.2}$$

In words, on $\mathcal{E}(\delta; T)$ the path does not visit more than the δ neighborhood of the interval between its starting point 0 and its endpoint B_T, while on $\mathcal{E}^{\le}(\delta, C; T)$ its local times in the δ neighborhoods of these two points are bounded by C. Note that both $\mathcal{E}(\infty; T)$ and $\mathcal{E}^{\le}(\delta, \infty; T)$ are the full space.

Recall the eigenvalue function $\rho : \mathbb{R} \to \mathbb{R}$ introduced in Section 2.1 and denote by $\rho^{-1} : \mathbb{R} \to \mathbb{R}$ its inverse function. Proposition 4.1 below identifies the exponential rate of decay of the expectation of $e^{-H_T} e^{\mu B_T}$ on $\{B_T \ge 0\}$ for μ large as $-\rho^{-1}(-\mu)$. Moreover, it shows that an insertion of the indicators of the events $\mathcal{E}$ and $\mathcal{E}^{\le}$ does not change the exponential rate, but only the next order term, which turns out to converge. For the last statement of Proposition 4.1, recall the definition of $B_T \approx bT$ below (1.4).

PROPOSITION 4.1. *Fix $\mu > -\rho(a^{**})$. Then, for any $\delta, C \in (0,\infty]$ there exists a constant $K_1(\delta, C) \in (0,\infty)$ such that, for any $\mu_T \to \mu$ as $T \to \infty$,*

$$\begin{aligned}\exp\{\rho^{-1}(-\mu_T)T\}E\big(e^{-H_T}e^{\mu_T B_T}\mathbb{1}_{\mathcal{E}(\delta,T)}\mathbb{1}_{\mathcal{E}^{\le}(\delta,C;T)}\mathbb{1}_{\{B_T\ge 0\}}\big)\\ = K_1(\delta,C)+o(1).\end{aligned} \tag{4.3}$$

Moreover, if $\mu = \mu_b$ solves $I(b) = \mu b - \Lambda^+(\mu)$, then the same is true when $\mathbb{1}_{\{B_T\ge 0\}}$ is replaced by $\mathbb{1}_{\{B_T\approx bT\}}$.

PROOF. The proof is divided into seven steps, which we outline now. In step 1 we use Proposition 3.1 to rewrite the left-hand side of (4.3) in terms of two integrals over expectations with respect to squared Bessel processes. In step 2 we use Proposition 3.2 to handle the case $C = \infty$, which is easier than the case $C < \infty$. In step 3 we turn to the case $C < \infty$ and apply the (strong) Markov property to prepare for an application of Proposition 2.1. Since a new integral appears via the application of the Markov property, it is necessary to argue in step 4 for the boundedness of the integrand of this integral. In step 5 we identify the limit of the integrand, and in step 6 we finish the identification of the limit in (4.3), again relying on Proposition 3.2. The last statement of Proposition 4.1 is proved in step 7.

We may assume that $\mu_T > -\rho(a^{**})$ for all T. Fix $\delta, C \in (0,\infty]$ and choose a_T such that $\mu_T + \rho(a_T) = 0$, that is, $a_T = \rho^{-1}(-\mu_T) < a^{**}$. Clearly, $\lim_{T\to\infty} a_T = \rho^{-1}(-\mu) < a^{**}$. Since, on $\mathcal{E}(\delta; T) \cap \{B_T \le 2\delta\}$, we can estimate

$$H_T = 4\delta\int_{-\delta}^{3\delta}\frac{dx}{4\delta}L(T,x)^2 \ge 4\delta\bigg(\int_{-\delta}^{3\delta}\frac{dx}{4\delta}L(T,x)\bigg)^2 = \frac{T^2}{4\delta}, \tag{4.4}$$

we may insert the indicator of $\{B_T \ge 2\delta\}$ in the expectation on the left-hand side of (4.3), paying only a factor $1 + o(1)$ as $T \to \infty$.

Step 1. Introduce the following subsets of C_0^+ and $\mathcal{C}^+$, respectively [see below (3.1)]:

$$G^{\le}_{\delta,C} = \{g \in C_0^+ : g(\delta) = 0, \max g \le C\}, \tag{4.5}$$

$$F^{\le}_{\delta,C} = \Big\{(y,f) \in \mathcal{C}^+ : y \ge 2\delta, \max_{[0,\delta]} f \le C, \max_{[y-\delta,y]} f \le C\Big\}. \tag{4.6}$$

Note that

$$\begin{aligned}&\mathcal{E}(\delta;T)\cap\mathcal{E}^{\le}(\delta,C;T)\cap\{B_T\ge 2\delta\}\\ &\quad= \{L(T,B_T+\cdot)\in G^{\le}_{\delta,C}\}\cap\{L(T,-\cdot)\in G^{\le}_{\delta,C}\}\\ &\qquad\cap\{(B_T, L(T,B_T-\cdot)|_{[0,B_T]})\in F^{\le}_{\delta,C}\}.\end{aligned} \tag{4.7}$$

Apply Proposition 3.1 for $a = a_T$ with $F = F^{\le}_{\delta,C}$ and $G^+ = G^- = G^{\le}_{\delta,C}$, to get

$$
\begin{aligned}
&\text{l.h.s. of (4.3)}\\
&\quad = \big(1+o(1)\big)\int_0^\infty dt_1 \int_0^\infty dt_2\, \mathbb{1}_{\{t_1+t_2\le T\}} e^{a_T(t_1+t_2)}\\
&\qquad \times \widehat{\mathbb{E}}^{a_T}\bigg(\frac{w_{G^{\le}_{\delta,C}}(X_0,t_1)}{x_{a_T}(X_0)} \mathbb{1}_{\{A^{-1}(T-t_1-t_2)\ge 2\delta\}} \mathbb{1}_{\{\max_{[0,\delta]} X \le C\}}\\
&\qquad\qquad \times \mathbb{1}_{\{\max_{[A^{-1}(T-t_1-t_2)-\delta, A^{-1}(T-t_1-t_2)]} X\le C\}} \frac{w_{G^{\le}_{\delta,C}}(Y_{T-t_1-t_2},t_2)}{x_{a_T}(Y_{T-t_1-t_2})}\bigg).
\end{aligned}
\tag{4.8}
$$

Step 2. In the case $C = \infty$, the last two indicators vanish and we can identify the limit of the integrand as $T \to \infty$ with the help of Lemma 2.1. Indeed, apply Lemma 2.1 for $f(\cdot) = w_{G_\delta}(\cdot, t_1)$ and $g(\cdot) = w_{G_\delta}(\cdot, t_2)$, where we put $G_\delta = G^{\le}_{\delta,\infty} = \{g \in C_0^+ : g(\delta) = 0\}$. Then we obtain that the integrand converges to

$$
e^{a(t_1+t_2)}\langle w_{G_\delta}(\cdot,t_1), x_a\rangle \frac{1}{\rho'(a)} \langle w_{G_\delta}(\cdot,t_2), x_a\rangle_\circ,
\tag{4.9}
$$

where we also use that $A^{-1}(\infty) = \infty$ because X never hits 0 [recall (2.7)]. According to Proposition 3.2, we are allowed to interchange the limit $T \to \infty$ with the two integrals over t_1 and t_2. This implies that (4.3) holds with $K_1(\delta,\infty)$ identified as

$$
K_1(\delta,\infty) = \langle y_a^{(\delta)}, x_a\rangle \frac{1}{\rho'(a)} \langle y_a^{(\delta)}, x_a\rangle_\circ,
\tag{4.10}
$$

where $y_a^{(\delta)}(h)$ is defined as [recall (3.1)]

$$
\begin{aligned}
y_a^{(\delta)}(h) &= \int_0^\infty dt\, e^{at} w_{G_\delta}(h,t)\\
&= \mathbb{E}_h^\star\bigg(\exp\bigg\{\int_0^\infty [aX_v^\star - (X_v^\star)^2]\,dv\bigg\}\mathbb{1}_{\{X_\delta^\star=0\}}\bigg).
\end{aligned}
\tag{4.11}
$$

Trivially, $K_1(\delta,\infty) > 0$. Since $y_a^{(\delta)} \le y_a$, it follows from (2.2) and (2.17) that $K_1(\delta,\infty) < \infty$.

Step 3. Next, we return to (4.8) and consider the case $C \in (0,\infty)$. Note that the integrals over t_1 and t_2 can both be restricted to $[0, C\delta]$, since $w_{G^{\le}_{\delta,C}}(h,t) = 0$ for $t > C\delta$ as is seen from (3.1) and (4.5).

Let us abbreviate $s = T - t_1 - t_2$. We first apply the Markov property for the process X at time δ and integrate over all values $z = A(\delta)$. Because of the appearance of the indicator of $\{\max_{[0,\delta]} X \le C\}$, we may restrict to $z \in [0, C\delta]$ [recall (2.7)]. We note that the additive functional of the process $(X_{\delta+t})_{t\ge 0}$ given that $A(\delta)=z$, denoted by $\widetilde{A}=(\widetilde{A}(t))_{t\geq 0}$, is given by $\widetilde{A}(t)=A(t+\delta)-z$. Making the change of variables $s=\widetilde{A}(t)+z$, we see that $A^{-1}(s)=\widetilde{A}^{-1}(s-z)+\delta$ for any $s\geq 0$. Defining $f^{t_1}_{s,T}:(0,\infty)^2\to[0,\infty)$ by

$$
\begin{aligned}
&f^{t_1}_{s,T}(h,z)\,dh\,dz\\
&\quad= x_{a_T}(h)\widehat{\mathbb{E}}^{a_T}\biggl(\frac{w_{G^{\leq}_{\delta,C}}(X_0,t_1)}{x_{a_T}(X_0)}\\
&\qquad\qquad\times \mathbb{1}_{\{A^{-1}(s)\geq 2\delta\}}\mathbb{1}_{\{\max_{[0,\delta]}X\leq C\}}\mathbb{1}_{\{X_\delta\in dh\}}\mathbb{1}_{\{A(\delta)\in dz\}}\biggr),
\end{aligned}
\tag{4.12}
$$

we thus obtain that the expectation under the integral in (4.8) can be written as

$$
\begin{aligned}
&\widehat{\mathbb{E}}^{a_T}\biggl(\frac{w_{G^{\leq}_{\delta,C}}(X_0,t_1)}{x_{a_T}(X_0)}\mathbb{1}_{\{A^{-1}(s)\geq 2\delta\}}\mathbb{1}_{\{\max_{[0,\delta]}X\leq C\}}\\
&\qquad\times \mathbb{1}_{\{\max_{[A^{-1}(s)-\delta,A^{-1}(s)]}X\leq C\}}\frac{w_{G^{\leq}_{\delta,C}}(Y_s,t_2)}{x_{a_T}(Y_s)}\biggr)\\
&\quad=\int_0^{C\delta}dz\,\widehat{\mathbb{E}}^{a_T}\biggl(\frac{f^{t_1}_{s,T}(X_0,z)}{x_{a_T}(X_0)}\mathbb{1}_{\{\max_{[\widetilde{A}^{-1}(s-z),\widetilde{A}^{-1}(s-z)+\delta]}X\leq C\}}\\
&\qquad\qquad\times\frac{w_{G^{\leq}_{\delta,C}}(X_{\widetilde{A}^{-1}(s-z)+\delta},t_2)}{x_{a_T}(X_{\widetilde{A}^{-1}(s-z)+\delta})}\biggr).
\end{aligned}
\tag{4.13}
$$

The tilde can now be removed. We next apply the Markov property for the process Y at time $s-z$ [respectively, the strong Markov property for the process X at time $A^{-1}(s-z)$], to write

$$
\text{r.h.s. of (4.13)}=\int_0^{C\delta}dz\,\widehat{\mathbb{E}}^{a_T}\biggl(\frac{f^{t_1}_{s,T}(X_0,z)}{x_{a_T}(X_0)}\frac{g^{t_2}_T(Y_{s-z})}{x_{a_T}(Y_{s-z})}\biggr),
\tag{4.14}
$$

where $g^{t_2}_T$ is defined by

$$
g^{t_2}_T(h)=x_{a_T}(h)\widehat{\mathbb{E}}^{a_T}_h\biggl(\mathbb{1}_{\{\max_{[0,\delta]}X\leq C\}}\frac{w_{G^{\leq}_{\delta,C}}(X_\delta,t_2)}{x_{a_T}(X_\delta)}\biggr).
\tag{4.15}
$$

Step 4. We want to take the limit $s\to\infty$ in (4.14) (recall that $s=T-t_1-t_2$) and use Proposition 2.1. Therefore we need dominated convergence. To establish this, we note that

$$
\sup_{h\in[0,C]}\sup_{t\in[0,C\delta]}\sup_{T\geq 1}\frac{w(h,t)}{x_{a_T}(h)}=K<\infty
\tag{4.16}
$$

[see (2.18)–(2.20) and recall that x_a is bounded away from zero on $[0,C]$ and continuous in a]. By (4.15) and (4.16), the last quotient in the right-hand side of (4.14) is bounded above by K. Substituting (4.12) into (4.14) and using that $w_{G^{\leq}_{\delta,C}} \leq w_{C_0^+} = w$, we therefore obtain

$$
\begin{aligned}
&\text{integrand of r.h.s. of (4.14)}\\
&\quad \leq K \widehat{\mathbb{E}}^{a_T}\left(\frac{f^{t_1}_{s,T}(X_0,z)}{x_{a_T}(X_0)}\right)\\
&\quad \leq K \frac{\widehat{\mathbb{E}}^{a_T}((w(X_0,t_1)/x_{a_T}(X_0))\mathbb{1}_{\{\max_{[0,\delta]} X\leq C\}}\mathbb{1}_{\{A(\delta)\in dz\}})}{dz}\\
&\quad \leq K^2 \frac{\widehat{\mathbb{P}}^{a_T}(A(\delta)\in dz)}{dz}.
\end{aligned}
\tag{4.17}
$$

It is easy to see from (2.9) that the right-hand side of (4.17) is bounded uniformly in $T \geq 1$ and $z \in [0, C\delta]$. Therefore we have an integrable majorant for (4.14), which allows us to interchange the limit $s \to \infty$ with the integral over z.

Step 5. To identify the limit as $s \to \infty$ of the integrand on the right-hand side of (4.14), we apply Lemma 2.1 to see that this integrand converges to $\langle f^{t_1}(\cdot,z), x_a(\cdot)\rangle \frac{1}{\rho'(a)}\langle g^{t_2}, x_a\rangle_\circ$, with f^{t_1} and g^{t_2} the pointwise limits of $f^{t_1}_{s,T}$ and $g^{t_2}_T$, respectively:

$$
\begin{aligned}
f^{t_1}(h,z)\,dh\,dz = x_a(h)\widehat{\mathbb{E}}^a\Bigg(&\frac{w_{G^{\leq}_{\delta,C}}(X_0,t_1)}{x_a(X_0)}\\
&\times \mathbb{1}_{\{\max_{[0,\delta]} X\leq C\}}\mathbb{1}_{\{X_\delta\in dh\}}\mathbb{1}_{\{A(\delta)\in dz\}}\Bigg),
\end{aligned}
\tag{4.18}
$$

$$
g^{t_2}(h)\,dh = x_a(h)\widehat{\mathbb{E}}^a_h\left(\mathbb{1}_{\{\max_{[0,\delta]} X\leq C\}}\frac{w_{G^{\leq}_{\delta,C}}(X_0,t_2)}{x_a(X_\delta)}\right).
\tag{4.19}
$$

Using this in (4.14) and interchanging the integral over z with the limit $s \to \infty$, we obtain that

$$
\lim_{s\to\infty}\big(\text{l.h.s. of (4.13)}\big) = \langle f^{t_1}, x_a\rangle \frac{1}{\rho'(a)}\langle g^{t_2}, x_a\rangle_\circ
\tag{4.20}
$$

with $f^{t_1}(h) = \int_0^{C\delta} dz\, f^{t_1}(h,z)$.

Step 6. Finally, recall that $s = T - t_1 - t_2$ and that $e^{a_T(t_1+t_2)}$ times the left-hand side of (4.13) is equal to the integrand on the right-hand side of (4.8). According to Proposition 3.2, we are allowed to interchange the limit $T \to \infty$ with the two integrals over t_1 and t_2. Hence we obtain that (4.3) holds with $K_1(\delta, C)$ identified as the integral over t_1, t_2 of the right-hand side of (4.20), which is a strictly positive finite number. This proves the statement with the indicator on $\mathbb{1}_{\{B_T\geq 0\}}$.

Step 7. To prove the statement with $\mathbb{1}_{\{B_T\geq 0\}}$ replaced by $\mathbb{1}_{\{B_T\approx bT\}}$, we let $\mu = \mu_b$ solve $I(b) = \mu b - \Lambda^+(\mu)$. The statement follows when we show that for every $\eta \in \mathbb{R}$, we have that

$$\begin{aligned}\exp\{\rho^{-1}(-\mu)T\}E\left(\exp\left\{\eta\frac{B_T-bT}{\sqrt{T}}\right\}e^{-H_T}e^{\mu B_T}\mathbb{1}_{\mathcal{E}(\delta,T)}\mathbb{1}_{\mathcal{E}^{\le}(\delta,C;T)}\mathbb{1}_{\{B_T\ge 0\}}\right)\\ =\exp\left\{\frac{\eta^2}{2}\sigma_b^2\right\}K_1(\delta,C)+o(1)\end{aligned}\tag{4.21}$$

for some $\sigma_b^2\in(0,\infty)$. Indeed, (4.21) shows that $\mathbb{1}_{\{|B_T-bT|>\gamma_T,B_T\ge 0\}}$ is asymptotically negligible for any γ_T such that $\gamma_T/\sqrt{T}\to\infty$.

To prove (4.21), we rewrite the left-hand side as

$$\begin{aligned}\exp\{[\rho^{-1}(-\mu)-\rho^{-1}(-\mu_{\eta,T})]T-\eta b\sqrt{T}\}\exp\{\rho^{-1}(-\mu_{\eta,T})T\}\\ \times E\big(e^{-H_T}e^{\mu_{\eta,T}B_T}\mathbb{1}_{\mathcal{E}(\delta,T)}\mathbb{1}_{\mathcal{E}^{\le}(\delta,C;T)}\mathbb{1}_{\{B_T\ge 0\}}\big),\end{aligned}\tag{4.22}$$

where $\mu_{\eta,T}=\mu+\eta/\sqrt{T}$. Clearly, $\mu_{\eta,T}\to\mu$, so that the second factor converges to $K_1(\delta,C)$. We are therefore left to compute the exponential. We note that since $\mu=\mu_b$ solves $I(b)=\mu b-\Lambda^+(\mu)$, we have that $\rho'(-\mu_b)=1/b$. Therefore,

$$\rho^{-1}(-\mu_{\eta,T})=\rho^{-1}(-\mu)-\frac{\eta}{\sqrt{T}}\frac{1}{\rho'(-\mu)}+\frac{\eta^2}{2T}\frac{d^2}{d\mu^2}\rho^{-1}(-\mu)+o(T^{-1}).\tag{4.23}$$

Therefore,

$$\begin{aligned}\exp\{[\rho^{-1}(-\mu)-\rho^{-1}(-\mu_{\eta,T})]T-\eta b\sqrt{T}\}\\ =\exp\left\{\frac{\eta^2}{2}\frac{d^2}{d\mu^2}\rho^{-1}(-\mu)\right\}(1+o(1)),\end{aligned}\tag{4.24}$$

which completes the proof with $\sigma_b^2=-\frac{d^2}{d\mu^2}\rho^{-1}(-\mu_b)$. □

4.2. *Proof of Theorem* 1.3(i)–(iii). In this section we prove the existence and the properties of the cumulant generating function Λ^+ on $\mathbb{R}$ as depicted in Figure 2. The strictly increasing piece in $[-\rho(a^{**}),\infty)$ is handled in step 1, the proof which is a direct application of Proposition 4.1. Step 2 is a technical step toward the identification of the linear piece in $(-\infty,-\rho(a^{**})]$ [and of the value of $I(0)$ as well]. Step 3 handles the linear piece, while step 4 handles the behavior at $+\infty$.

Step 1. For any $\mu>-\rho(a^{**})$, the limit in (1.10) exists and equals $\Lambda^+(\mu)=-\rho^{-1}(-\mu)$. On $(-\rho(a^{**}),\infty)$, the function Λ^+ is real-analytic and strictly convex, and satisfies $\lim_{\mu\downarrow-\rho(a^{**})}(\Lambda^+)'(\mu)=b^{**}$.

PROOF. Fix $\mu>-\rho(a^{**})$, apply Proposition 4.1 with $\delta=C=\infty$ and use the continuity of ρ to obtain that the limit in the definition of $\Lambda^+(\mu)$ in (1.10) exists and equals $-\rho^{-1}(-\mu)$. This proves the first assertion. The remaining assertions follow from (2.3)–(2.5). □

In the following step, we consider the density e^{-H_T} on the event $\{B_T \approx 0\}$ and make a technical step toward the identification of $I(0)$ and $\Lambda^+(\mu)$ for small μ. We derive a lower bound for the expectation for those paths that never go below $-\delta$ and have local times that are bounded by C in the δ neighborhood of the starting point 0. Recall that γ_T is a function that satisfies $\gamma_T/T \to 0$ and $\gamma_T/\sqrt{T} \to \infty$ as $T \to \infty$.

Step 2. For any $\delta \in (0, \infty)$ and $C \in (0, \infty]$,

$$
\begin{aligned}
&E\big(e^{-H_T} \mathbb{1}_{\{B_T \in [0,\gamma_T]\}} \mathbb{1}_{\{\min_{[0,T]} B \geq -\delta\}} \mathbb{1}_{\{\max_{[-\delta,\delta]} L(T,\cdot) \leq C\}}\big) \\
&\qquad \geq e^{-a^{**}T + o(T)}, \qquad T \to \infty.
\end{aligned} \tag{4.25}
$$

PROOF. Pick $a = a^{**}$ and apply Proposition 3.1 for

$$
\begin{aligned}
F = F_{\delta,C} = \Big\{(y, f) \in \mathcal{C}^+ : y \leq \delta, \max_{[y-\delta,y]} f \leq C\Big\}, \\
G^+ = C_0^+, \; G^- = G^{\leq}_{\delta,C}
\end{aligned} \tag{4.26}
$$

[recall (4.5)]. Note that the event under the expectation on the left-hand side of (4.25) contains the event

$$
\begin{aligned}
&\{L(T, B_T + \cdot) \in C_0^+\} \\
&\qquad \cap \{(B_T, L(T, B_T - \cdot)) \in F_{\delta,C}\} \cap \{L(T, -\cdot) \in G^{\leq}_{\delta,C}\}.
\end{aligned} \tag{4.27}
$$

Also note that $e^{-\rho(a^{**})B_T} \leq 1$ when $B_T \geq 0$ because $\rho(a^{**}) > 0$. Therefore we find

$$
\begin{aligned}
&\text{l.h.s. of (4.25)} \\
&\quad \geq \int_0^\infty dt_1 \int_0^\infty dt_2 \, \mathbb{1}_{\{t_1+t_2 \leq T\}} e^{-a^{**}s} \\
&\qquad \times \widehat{\mathbb{E}}^{a^{**}} \left(\frac{w(X_0, t_1)}{x_{a^{**}}(X_0)} \mathbb{1}_{\{A^{-1}(s) \leq \delta\}} \mathbb{1}_{\{\max_{[A^{-1}(s)-\delta, A^{-1}(s)]} X \leq C\}} \frac{w_{G^{\leq}_{\delta,C}}(Y_s, t_2)}{x_{a^{**}}(Y_s)} \right),
\end{aligned} \tag{4.28}
$$

where we again abbreviate $s = T - t_1 - t_2$. Next we interchange the two integrals, restrict the t_2 integral to $[0, \delta]$ and the t_1 integral to $[T - t_2 - \delta, T - t_2]$, estimate $A^{-1}(s) \leq A^{-1}(\delta)$ for $s \leq \delta$, and integrate over $s = T - t_1 - t_2$ to get

$$
\begin{aligned}
&\text{l.h.s. of (4.25)} \\
&\quad \geq \int_0^\delta dt_2 \int_0^\delta ds \, \widehat{\mathbb{E}}^{a^{**}} \bigg(\frac{w(X_0, T - t_2 - s)}{x_{a^{**}}(X_0)} \\
&\qquad\qquad \times \mathbb{1}_{\{A^{-1}(\delta) \leq \delta\}} \mathbb{1}_{\{\max_{[0,\delta]} X \leq C\}} \frac{w_{G^{\leq}_{\delta,C}}(Y_s, t_2)}{x_{a^{**}}(Y_s)} \bigg).
\end{aligned} \tag{4.29}
$$

Now we use Proposition 2.2(i) to estimate $w(X_0, T-s-t_2) \geq e^{-a^{**}T+o(T)}$, uniformly on the domain of integration. The remaining expectation on the right-hand side no longer depends on T and is strictly positive for any $\delta \in (0,\infty)$ and $C \in (0,\infty]$. $\square$

Step 3. Λ^+ equals $-a^{**}$ on $(-\infty, -\rho(a^{**})]$.

PROOF. For $\mu \leq -\rho(a^{**})$, define $\Lambda^+_-(\mu)$ and $\Lambda^+_+(\mu)$ as in (1.10) with lim replaced by lim inf and lim sup, respectively. Since Λ^+_+ is obviously nondecreasing, we have $\Lambda^+_+(\mu) \leq \Lambda^+(-\rho(a^{**})+\varepsilon)$ for $\mu \leq -\rho(a^{**})$ and any $\varepsilon > 0$. Using step 1 and the continuity of ρ, we see that $\lim_{\varepsilon\downarrow 0}\Lambda^+(-\rho(a^{**})+\varepsilon) = -\rho^{-1}(\rho(a^{**})) = -a^{**}$, which shows that $\Lambda^+_+(\mu) \leq -a^{**}$. To get the reversed inequality for $\Lambda^+_-(\mu)$, bound

$$\begin{aligned} &E\big(e^{-H_T}e^{\mu B_T}\mathbb{1}_{\{B_T\geq 0\}}\big) \\ &\quad \geq E\big(e^{-H_T}e^{\mu B_T}\mathbb{1}_{\{B_T\in[0,\gamma_T]\}}\big) \geq e^{\mu\gamma_T}E\big(e^{-H_T}\mathbb{1}_{\{B_T\in[0,\gamma_T]\}}\big), \end{aligned} \tag{4.30}$$

take logs, divide by T, let $T\to\infty$ and use step 2 to obtain that $\Lambda^+_-(\mu) \geq -a^{**}$. Since $\Lambda^+_- \leq \Lambda^+_+$, this implies the assertion. $\square$

Step 4. $\Lambda^+(\mu) = \frac{1}{2}\mu^2 + \mathcal{O}(\mu^{-1})$ as $\mu\to\infty$.

PROOF. According to step 1, we have $\Lambda^+(\mu) = -\rho^{-1}(-\mu)$ for $\mu > -\rho(a^{**})$. Hence, to obtain the asymptotics for $\Lambda^+(\mu)$ as $\mu\to\infty$, we need to obtain the asymptotics for $\rho(a)$ as $a\to-\infty$. In the following analysis we consider $a<0$.

We use Rayleigh's principle (see [6], Proposition 10.10) to write [recall (2.1)]

$$\begin{aligned} \rho(a) &= \sup_{x\in L^2\cap C^2:\,\|x\|_2=1}\langle\mathcal{K}^a x, x\rangle \\ &= \sup_{x\in L^2\cap C^2:\,\|x\|_2=1}\int_0^\infty \big[-2hx'(h)^2 + (ah-h^2)x(h)^2\big]\,dh. \end{aligned} \tag{4.31}$$

Substituting $x(h) = (-a)^{1/4}y((-a)^{1/2}h)$, we get

$$\begin{aligned} \rho(a) &= (-a)^{1/2} \\ &\quad\times \sup_{y\in L^2\cap C^2:\,\|y\|_2=1}\int_0^\infty \big[-2hy'(h)^2 - \big(h+h^2(-a)^{-3/2}\big)y(h)^2\big]\,dh. \end{aligned} \tag{4.32}$$

Hence, we have the upper bound $\rho(a) \leq V(-a)^{1/2}$ with

$$V = \sup_{y\in L^2\cap C^2:\,\|y\|_2=1}\int_0^\infty \big[-2hy'(h)^2 - hy(h)^2\big]\,dh. \tag{4.33}$$

By completing the square under the integral and partially integrating the cross term, we easily see that $y^*(h) = \frac{1}{\sqrt{2}} e^{-h/\sqrt{2}}$ is the maximizer of (4.33) and $V = -\sqrt{2}$. Substituting y^* into (4.32), we can also bound $\rho(a)$ from below:

$$\rho(a) \geq -\sqrt{2}(-a)^{1/2} - (-a)^{-1} \int_0^\infty h^2 y^*(h)^2 \, dh. \tag{4.34}$$

Therefore,

$$\rho(a) = -\sqrt{2}(-a)^{1/2} + \mathcal{O}(|a|^{-1}), \qquad a \to -\infty. \tag{4.35}$$

Consequently,

$$\Lambda^+(\mu) = -\rho^{-1}(-\mu) = \tfrac{1}{2}\mu^2 + \mathcal{O}(\mu^{-1}), \qquad \mu \to \infty. \tag{4.36}$$

□

Steps 1, 3 and 4 complete the proof of Theorem 1.3(i)–(iii).

4.3. *Proofs of Theorems* 1.2 *and* 1.3(iv). In this section we prove the existence and properties of the rate function I on $[0, \infty)$ as depicted in Figure 1 and its relationship to the cumulant generating function Λ^+. Step 5 identifies I on $[0, \infty)$ as the Legendre transform of the restriction of Λ^+ to $[-\rho(a^{**}), \infty)$. This is done along the standard lines of the proof of the well-known Gärtner–Ellis theorem (see [4], Theorem 2.3.6). In steps 6 and 7 we prove the lower and upper bound in the linear piece on $[0, a^{**}]$. In step 8 we finally complete the proofs of Theorems 1.2 and 1.3(iv).

For $b \in \mathbb{R}$, define $I_-(b)$ and $I_+(b)$ as in (1.6) with lim replaced by lim sup and lim inf, respectively.

Step 5. For any $b > b^{**}$, the limit in (1.6) exists and (1.13) holds.

PROOF. Fix $b > b^{**}$. In step 1 we prove the lower bound in (1.13) via the exponential Chebyshev inequality. In steps 2–4 we prove the upper bound via an exponential change of measure argument.

Step 1. To derive $\geq$ in (1.13) for I_- instead of I, bound, for any $\mu \in \mathbb{R}$,

$$\begin{aligned} E\big(e^{-H_T} \mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}\big) &\leq e^{-\mu b T + |\mu|\gamma_T} E\big(e^{-H_T} e^{\mu B_T} \mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}\big) \\ &\leq e^{-\mu b T + |\mu|\gamma_T} E\big(e^{-H_T} e^{\mu B_T} \mathbb{1}_{\{B_T \geq 0\}}\big), \end{aligned} \tag{4.37}$$

where the last inequality holds for any T sufficiently large because $\gamma_T / T \to 0$ as $T \to \infty$. Take logs, divide by T, let $T \to \infty$, use (1.10) and minimize over $\mu \in \mathbb{R}$ to obtain

$$-I_-(b) \leq \min_{\mu \in \mathbb{R}} [-\mu b + \Lambda^+(\mu)]. \tag{4.38}$$

This shows that $\geq$ holds in (1.13) for I replaced by I_-.

Step 2. To derive $\le$ in (1.13) for I_+ instead of I, bound, for any $\mu \in \mathbb{R}$,

$$
\begin{aligned}
&E\big(e^{-H_T}\mathbb{1}_{\{|B_T-bT|\le\gamma_T\}}\big)\\
&\qquad \ge E\big(e^{-H_T}\mathbb{1}_{\mathcal{E}(\delta,T)}\mathbb{1}_{\{|B_T-bT|\le\gamma_T\}}\mathbb{1}_{\{B_T\ge 0\}}\big)\\
&\qquad \ge e^{-\mu bT-|\mu|\gamma_T}P^{\mu,\delta,T}(|B_T-bT|\le\gamma_T)E\big(e^{-H_T}e^{\mu B_T}\mathbb{1}_{\mathcal{E}(\delta,T)}\mathbb{1}_{\{B_T\ge 0\}}\big),
\end{aligned}
\tag{4.39}
$$

where $P^{\mu,\delta,T}$ denotes the probability law whose density with respect to P is proportional to $e^{-H_T}e^{\mu B_T}\mathbb{1}_{\mathcal{E}(\delta,T)}\mathbb{1}_{\{B_T\ge 0\}}$.

Step 3. Let μ_b be the maximizer of the map $\mu \mapsto \mu b - \Lambda^+(\mu)$. [Note that, by step 1, the maximizer is unique and is characterized by $(\Lambda^+)'(\mu_b) = b$.] Next we argue that

$$
\lim_{T\to\infty} P^{\mu_b,\delta,T}(|B_T-bT|\le\gamma_T) = 1. \tag{4.40}
$$

Indeed, pick $\varepsilon_T = \gamma_T/cT > 0$ (with $c > 0$ to be specified later) and estimate

$$
\mathbb{1}_{\{B_T\ge bT+\gamma_T\}} \le e^{\varepsilon_T[B_T-bT-\gamma_T]}. \tag{4.41}
$$

This implies, with the help of step 1 and Proposition 4.1 with $\mu_T = \mu + \varepsilon_T$, $C = \infty$, that

$$
\begin{aligned}
&P^{\mu_b,\delta,T}(B_T \ge bT+\gamma_T)\\
&\qquad \le e^{-\varepsilon_T[bT+\gamma_T]}e^{[\Lambda^+(\mu_b+\varepsilon_T)-\Lambda^+(\mu_b)]T}\big(1+o(1)\big), \qquad T\to\infty.
\end{aligned}
\tag{4.42}
$$

A Taylor expansion of Λ^+ around μ_b, in combination with the observation that $(\Lambda^+)'(\mu_b) = b$ and $c = (\Lambda^+)''(\mu_b) > 0$, yields that the right-hand side of (4.42) is equal to

$$
\begin{aligned}
&\exp\left\{\frac{c}{2}\varepsilon_T^2 T[1+\mathcal{O}(\varepsilon_T)] - \varepsilon_T\gamma_T\right\}\\
&\qquad = \exp\left\{-\frac{\gamma_T^2}{2cT}\left[1+\mathcal{O}\left(\frac{\gamma_T}{T}\right)\right]\right\}, \qquad T\to\infty.
\end{aligned}
\tag{4.43}
$$

The right-hand side vanishes as $T \to \infty$ because $\gamma_T/T \to 0$ and $\gamma_T/\sqrt{T} \to \infty$. This shows that $\lim_{T\to\infty} P^{\mu_b,\delta,T}(B_T \ge bT+\gamma_T) = 0$. Analogously, replacing ε_T by $-\varepsilon_T$, we can prove that $\lim_{T\to\infty} P^{\mu_b,\delta,T}(B_T \le bT-\gamma_T) = 0$. Hence, (4.40) holds.

Step 4. Use (4.40) in (4.39) for $\mu = \mu_b$, take logs, divide by T, let $T \to \infty$, and use step 1 and Proposition 4.1 to obtain

$$
-I_+(b) \ge -\mu_b b + \Lambda^+(\mu_b) = -\max_{\mu\in\mathbb{R}}[\mu b - \Lambda^+(\mu)]. \tag{4.44}
$$

This shows that $\le$ holds in (1.13) for I replaced by I_+. Combine (4.38) and (4.44) to obtain that $I_- = I = I_+$ and that (1.13) holds on (b^{**}, ∞). $\square$

Step 6. For any $b \geq 0$, $I_-(b) \geq -b\rho(a^{**}) + a^{**}$.

PROOF. Estimate

$$\mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}} \leq \mathbb{1}_{\{B_T \leq bT + \gamma_T\}} \leq e^{-\rho(a^{**})[B_T - bT - \gamma_T]} \tag{4.45}$$

to obtain, for T sufficiently large,

$$\begin{aligned} &E\big(e^{-H_T}\mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}\big) \\ &\quad \leq 2E\big(e^{-H_T}\mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}\mathbb{1}_{\{B_T \geq 0\}}\big) \\ &\quad \leq 2e^{b\rho(a^{**})T + \gamma_T\rho(a^{**})}E\big(e^{-H_T}e^{-\rho(a^{**})B_T}\mathbb{1}_{\{B_T \geq 0\}}\big). \end{aligned} \tag{4.46}$$

According to the definition of Λ^+ in (1.10), the expectation in the right-hand side is equal to $e^{\Lambda^+(-\rho(a^{**}))T + o(T)}$. We therefore obtain that $I(b) \geq -b\rho(a^{**}) - \Lambda^+(-\rho(a^{**}))$. Now step 3 concludes the proof. $\square$

Step 7. For any $0 \leq b \leq b^{**}$, $I_+(b) \leq -b\rho(a^{**}) + a^{**}$.

PROOF. Fix $0 \leq b \leq b^{**}$, pick $b' > b^{**}$ and put $\alpha = b/b' \in [0, 1)$. We split the path $(B_s)_{s \in [0,T]}$ into two pieces: $s \in [0, \alpha T]$ and $s \in [\alpha T, T]$. First we bound from below by inserting several indicators:

$$\begin{aligned} &E\big(e^{-H_T}\mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}\big) \\ &\quad \geq E\big(e^{-H_T}\mathbb{1}_{\{|B_{\alpha T} - b'\alpha T| \leq \gamma_T/2\}}\mathbb{1}_{\{\max_{[0,\alpha T]} B \leq B_{\alpha T} + \delta\}} \\ &\qquad \times \mathbb{1}_{\{\max_{[B_{\alpha T} - \delta, B_{\alpha T} + \delta]} L(\alpha T, \cdot) \leq C\}}\mathbb{1}_{\{|\widetilde{B}_{(1-\alpha)T}| \leq \gamma_T/2\}} \\ &\qquad \times \mathbb{1}_{\{\min_{[0,(1-\alpha)T]} \widetilde{B} \geq -\delta]\}}\mathbb{1}_{\{\max_{[B_{\alpha T} - \delta, B_{\alpha T} + \delta]} \widetilde{L}((1-\alpha)T, \cdot) \leq C\}}\big). \end{aligned} \tag{4.47}$$

Here, $(\widetilde{B}_s)_{s \in [0,(1-\alpha)T]}$ is the Brownian motion with $\widetilde{B}_s = B_{\alpha T + s} - B_{\alpha T}$, and $\widetilde{L}((1-\alpha)T, x) = L(T, x) - L(\alpha T, x)$, $x \in \mathbb{R}$, are its local times.

On the event under the expectation in the right-hand side, we may estimate

$$\begin{aligned} H_T &= H_{\alpha T} + \widetilde{H}_{(1-\alpha)T} + 2\int_{B_{\alpha T} - \delta}^{B_{\alpha T} + \delta} L(\alpha T, x)\widetilde{L}\big((1-\alpha)T, x\big)\,dx \\ &\leq H_{\alpha T} + \widetilde{H}_{(1-\alpha)T} + 4\delta C^2, \end{aligned} \tag{4.48}$$

where $\widetilde{H}_{(1-\alpha)T}$ denotes the intersection local time for the second piece. Using the Markov property at time αT, we therefore obtain the estimate

$$\begin{aligned} &E\big(e^{-H_T}\mathbb{1}_{\{|B_T - bT| \leq \gamma_T\}}\big) \\ &\quad \geq e^{-4\delta C^2}E\big(e^{-H_{\alpha T}}\mathbb{1}_{\{|B_{\alpha T} - b'\alpha T| \leq \gamma_T/2\}} \\ &\qquad\qquad \times \mathbb{1}_{\{\max_{[0,\alpha T]} B \leq B_{\alpha T} + \delta\}}\mathbb{1}_{\{\max_{[B_{\alpha T} - \delta, B_{\alpha T} + \delta]} L(\alpha T, \cdot) \leq C\}}\big) \\ &\qquad \times E\big(e^{-H_{(1-\alpha)T}}\mathbb{1}_{\{|B_{(1-\alpha)T}| \leq \gamma_T/2\}} \\ &\qquad\qquad \times \mathbb{1}_{\{\min_{[0,(1-\alpha)T]} B \geq -\delta\}}\mathbb{1}_{\{\max_{[-\delta,\delta]} L((1-\alpha)T, \cdot) \leq C\}}\big). \end{aligned} \tag{4.49}$$

(The tilde can be removed afterward.) Now use Proposition 4.1 (in combination with an argument like in parts 2 and 3 of the proof of step 5) for the first term (with T replaced by αT) and use step 2 for the second term [with T replaced by $(1-\alpha)T$] to conclude that

$$I(b) \le \alpha I(b') + (1-\alpha)a^{**} = \frac{b}{b'}\big(I(b') - a^{**}\big) + a^{**}. \tag{4.50}$$

Let $b' \downarrow b^{**}$, use the continuity of I in b^{**} and note that $I(b^{**}) - a^{**} = -b^{**}\rho(a^{**})$ by step 5 to conclude the proof. □

Step 8. Theorems 1.2 and 1.3(iv) hold.

PROOF. Steps 1 and 5 allow us to identify I on (b^{**}, ∞) as $I(b) = -b \times \rho(a_b) + a_b$, where a_b solves $\rho'(a_b) = 1/b$ [the maximum in (1.13) is attained at $\mu = -\rho(a_b)$]. From this and (2.3)–(2.5) it follows that

$$\begin{aligned} &I'(b) = -\rho(a_b), \\ &I''(b) = -\rho'(a_b)\frac{d}{db}a_b = \frac{[\rho'(a_b)]^3}{\rho''(a_b)} > 0, \qquad b > b^{**}. \end{aligned} \tag{4.51}$$

In particular, I is real-analytic and strictly convex on (b^{**}, ∞). Since $a_{b^{**}} = a^{**}$, it in turn follows that

$$\min_{b \ge 0} I(b) = \min_{b > b^{**}} I(b) = I(b^*) = a^*, \tag{4.52}$$

where a^* solves $\rho(a^*) = 0$ [the minimum is attained at $b^* = 1/\rho'(a^*)$]. This, together with steps 5–7, proves Theorem 1.2(i)–(iii).

Step 5 shows that (1.13) holds on (b^{**}, ∞). To show that it also holds on $[0, b^{**}]$, use step 3 to get

$$-b\rho(a^{**}) + a^{**} = \max_{\mu \in \mathbb{R}}[b\mu - \Lambda^+(\mu)], \qquad 0 \le b \le b^{**}, \tag{4.53}$$

since the maximum is attained at $\mu = -\rho(a^{**})$. Recall from steps 6 and 7 that the left-hand side is equal to $I(b)$. Thus we have proved Theorem 1.3(iv).

Finally, Theorem 1.2(iv) is an immediate consequence of Theorem 1.3(iii)–(iv). □

5. Addendum: an extension of Proposition 4.1. At this point we have completed the proof of the main results in Section 1. In Sections 5 and 6 we derive an extension of Proposition 4.1 that will be needed in a forthcoming article [11]. In that article we show that several one-dimensional polymer models in discrete space and time, such as the weakly self-avoiding walk, converge to the Edwards model, after appropriate scaling, in the limit of vanishing self-repellence or diverging step variance. The proof is based on a coarse-graining argument, for which we need Proposition 5.1 below.

Recall the events in (4.1) and (4.2). For $\delta \in (0,\infty)$, $\alpha \in [0,\infty)$, define the event

$$\mathcal{E}^{\geq}(\delta,\alpha;T) = \Big\{ \max_{x\in[B_T-\delta,B_T+\delta]} L(T,x) \geq \alpha\delta^{-1/2} \Big\}. \tag{5.1}$$

Note that $\mathcal{E}^{\geq}(\delta,0;T)$ is the full space.

Proposition 5.1(i) below is the analogue of Proposition 4.1 for the event $\mathcal{E}^{\geq}(\delta,\alpha;T)$ instead of $\mathcal{E}^{\leq}(\delta,C;T)$ (which is essentially its complement). Proposition 5.1(ii) below shows that the contribution coming from $\mathcal{E}^{\geq}$ is negligible with respect to the contribution coming from $\mathcal{E}^{\leq}$ in the limit as $\delta \downarrow 0$.

PROPOSITION 5.1. *Fix $\mu > -\rho(a^{**})$. Then*:

(i) *For any $\delta \in (0,\infty)$ and $\alpha \in [0,\infty)$ there exists a $K_2(\delta,\alpha) \in (0,\infty)$ such that*

$$\begin{aligned} &\exp\{\rho^{-1}(-\mu)T\} E\big(e^{-H_T} e^{\mu B_T} \mathbb{1}_{\mathcal{E}(\delta,T)} \mathbb{1}_{\mathcal{E}^{\geq}(\delta,\alpha;T)} \mathbb{1}_{\{B_T\geq 0\}}\big) \\ &\qquad = K_2(\delta,\alpha) + o(1), \qquad T\to\infty. \end{aligned} \tag{5.2}$$

(ii) *For any $\alpha \in (0,\infty)$,*

$$\lim_{\delta\downarrow 0} \frac{K_2(\delta,\alpha)}{K_1(\delta,\infty)} = 0, \tag{5.3}$$

where $K_1(\delta,\infty)$ is the constant in Proposition 4.1 [*recall* (4.10)].

PROOF. (i) As in the proof of Proposition 4.1, we may insert the indicator on $\{B_T \geq 2\delta\}$ in the expectation on the left-hand side of (5.2) and add a factor of $1+o(1)$.

Introduce the measurable subsets of C_0^+ and $\mathcal{C}^+$, respectively,

$$G^{\geq}_{\delta,\alpha} = \{g \in C_0^+ : g(\delta) = 0, \max g \geq \alpha\delta^{-1/2}\}, \tag{5.4}$$

$$F^{\geq}_{\delta,\alpha} = \Big\{(y,f) \in \mathcal{C}^+ : y \geq 2\delta, \max_{[0,\delta]} f \geq \alpha\delta^{-1/2}\Big\}. \tag{5.5}$$

Note from (4.1) and (5.1) that

$$\begin{aligned} &\mathcal{E}(\delta;T) \cap \mathcal{E}^{\geq}(\delta,\alpha;T) \cap \{B_T \geq 2\delta\} \\ &\quad = \{L(T,-\cdot) \in G_\delta\} \\ &\qquad \cap \Big(\{L(T,B_T+\cdot) \in G^{\geq}_{\delta,\alpha}\} \\ &\qquad\quad \cup \big[\{(B_T, L(T,B_T-\cdot)|_{[0,B_T]}) \in F^{\geq}_{\delta,\alpha}\} \cap \{L(T,B_T+\cdot) \in G_\delta\}\big]\Big) \end{aligned} \tag{5.6}$$

with $G_\delta = \{g \in C_0^+ : g(\delta) = 0\}$.

Pick $a \in \mathbb{R}$ such that $\mu + \rho(a) = 0$, that is, $a = \rho^{-1}(-\mu) < a^{**}$. Apply Proposition 3.1 twice for $G^- = G_\delta$ and the two choices: (1) $F = F^{\geq}_{\delta,\alpha}$, $G^+ = G_\delta$ and (2) $F = \mathcal{C}^+$, $G^+ = G^{\geq}_{\delta,\alpha}$. Sum the two resulting equations to obtain

$$\begin{aligned}
&\text{l.h.s. of (5.2)}\\
&\quad = \big(1+o(1)\big) \int_0^\infty dt_1 \int_0^\infty dt_2\, \mathbb{1}_{\{t_1+t_2 \leq T\}} e^{a(t_1+t_2)}\\
&\qquad \times \widehat{\mathbb{E}}^a\bigg(\bigg[\frac{w_{G_\delta}(X_0,t_1)}{x_a(X_0)} \mathbb{1}_{\{A^{-1}(T-t_1-t_2)\geq 2\delta\}}\\
&\qquad\qquad \times \mathbb{1}_{\{\max_{[0,\delta]} X \geq \alpha\delta^{-1/2}\}} + \frac{w_{G^{\geq}_{\delta,\alpha}}(X_0,t_1)}{x_a(X_0)}\bigg] \frac{w_{G_\delta}(Y_{T-t_1-t_2},t_2)}{x_a(Y_{T-t_1-t_2})}\bigg).
\end{aligned} \tag{5.7}$$

In the same way as in the proof of Proposition 4.1, we obtain that [recall (4.10) and (4.11)]

$$\lim_{T\to\infty} \big(\text{r.h.s. of (5.7)}\big) = K_2(\delta,\alpha) \tag{5.8}$$

with

$$\begin{aligned}
&K_2(\delta,\alpha)\\
&\quad = \bigg[\widehat{\mathbb{E}}^a_h\bigg(\frac{y_a^{(\delta)}(X_0)}{x_a(X_0)} \mathbb{1}_{\{\max_{[0,\delta]} X \geq \alpha\delta^{-1/2}\}}\bigg) + \langle x_a, y_a^{(\delta,\alpha)}\rangle\bigg] \frac{1}{\rho'(a)} \langle x_a, y_a^{(\delta)}\rangle_\circ,
\end{aligned} \tag{5.9}$$

where $y_a^{(\delta)}$ is defined in (4.11) and $y_a^{(\delta,\alpha)}$ is defined as [recall (3.1)]

$$\begin{aligned}
y_a^{(\delta,\alpha)}(h) &= \int_0^\infty dt\, e^{at} w_{G^{\geq}_{\delta,\alpha}}(h,t)\\
&= \mathbb{E}^\star_h\bigg(\exp\bigg\{\int_0^\infty [aX^\star_v - (X^\star_v)^2]\,dv\bigg\} \mathbb{1}_{\{X^\star_\delta = 0\}} \mathbb{1}_{\{\max_{[0,\delta]} X^\star \geq \alpha\delta^{-1/2}\}}\bigg).
\end{aligned} \tag{5.10}$$

The right-hand side of (5.9) is a strictly positive finite number.

(ii) Fix $\alpha \in (0,\infty)$. From (4.10) and (5.8) we see that $K_2(\delta,\alpha)/K_1(\delta,\infty) = K^{(1)}(\delta,\alpha) + K^{(2)}(\delta,\alpha)$ with

$$\begin{aligned}
K^{(1)}(\delta,\alpha) &= \frac{\int_0^\infty dh\, x_a(h) y_a^{(\delta)}(h) \widehat{\mathbb{P}}^a_h(\max_{[0,\delta]} X \geq \alpha\delta^{-1/2})}{\langle x_a, y_a^{(\delta)}\rangle},\\
K^{(2)}(\delta,\alpha) &= \frac{\langle x_a, y_a^{(\delta,\alpha)}\rangle}{\langle x_a, y_a^{(\delta)}\rangle}.
\end{aligned} \tag{5.11}$$

To prove (5.3), we need the following technical lemma, which gives us control over the two numerators in (5.11).

LEMMA 5.2. *Fix $a < a^{**}$ and $\alpha \in (0, \infty)$. Then*:

(i) *There exists $d = d(\alpha) > 0$ such that, for any $R > 0$ and any $\delta > 0$ sufficiently small,*

$$\sup_{h \in [0,R]} \widehat{\mathbb{P}}_h^a\Big(\max_{[0,\delta]} X \geq \alpha\delta^{-1/2}\Big) \leq c\exp\big\{-d\delta^{-1/4}\big\}e^{c\sqrt{R}}, \tag{5.12}$$

$$\sup_{h \in [0,R]} \frac{y_a^{(\delta,\alpha)}(h)}{y_a(h)} \leq c\exp\big\{-d\delta^{-1/4}\big\}e^{c\sqrt{R}}. \tag{5.13}$$

(ii) *For any $\delta > 0$ sufficiently small,*

$$\inf_{h \in [0,\delta]} y_a^{(\delta)}(h) \geq c. \tag{5.14}$$

The proof is deferred to Section 6.

We use Lemma 5.2 to show that

$$\lim_{\delta \downarrow 0} K^{(1)}(\delta,\alpha) = \lim_{\delta \downarrow 0} K^{(2)}(\delta,\alpha) = 0, \tag{5.15}$$

which yields (5.3).

First note that, with the help of (5.14), the common denominator in (5.11) may be estimated from below by

$$\langle x_a, y_a^{(\delta)}\rangle \geq \int_0^\delta dh\, x_a(h) y_a^{(\delta)}(h) \geq c\int_0^\delta dh\, x_a(h) \geq c\delta, \tag{5.16}$$

where we use that x_a is bounded away from zero on $[0, \delta]$.

To estimate the numerator of $K^{(1)}(\delta,\alpha)$ from above, we split the integral in the numerator into two parts: $h \leq R$ and $h > R$. In the integral over $h \leq R$, estimate $y_a^{(\delta)} \leq y_a$ and use (5.12) to get the upper bound $c\exp\{-d\delta^{-1/4}\}e^{c\sqrt{R}}$. In the integral over $h > R$, estimate $y_a^{(\delta)} \leq y_a$, estimate the probability against 1 and use (2.2) and (2.17) to get the upper bound $c\exp\{-cR^{3/2}\}$. Pick R such that $c\sqrt{R} = (d/2)\delta^{-1/4}$ to obtain that the numerator of $K^{(1)}(\delta,\alpha)$ is at most $c\exp\{-(d/2)\delta^{-1/4}\}$.

In the same way we show, with the help of (5.13), that the numerator of $K^{(2)}(\delta,\alpha)$ in (5.11) is at most $c\exp\{-(d/2)\delta^{-1/4}\}$. Now combine the two estimates with (5.16) to obtain (5.15). □

6. Proof of Lemma 5.2. In this section we prove the technical assertions in Lemma 5.2, which were used to prove Proposition 5.1. In step 1 we provide precise asymptotics for the two functions x_a and y_a at $+\infty$. In step 2 we introduce an auxiliary martingale, to which we apply Doob's martingale inequality in step 3 to prove (5.12). In an analogous way we prove (5.13) in step 4, while in step 5 we prove (5.14).

First we provide the following refinements of (2.2) and (2.17) for the functions x_a and y_a, respectively.

STEP 1. For $a < a^{**}$,

$$\lim_{h\to\infty} \frac{1}{\sqrt{h}} \log\left[\exp\left\{\frac{\sqrt{2}}{3}h^{3/2}\right\}x_a(h)\right] = \lim_{h\to\infty} \frac{1}{\sqrt{h}} \log\left[\exp\left\{\frac{\sqrt{2}}{3}h^{3/2}\right\}y_a(h)\right] = \frac{a}{\sqrt{2}}. \tag{6.1}$$

PROOF. The statement for y_a is well known and follows from (2.16) together with the asymptotics of the Airy function given by (see [5], page 43)

$$\mathrm{Ai}(h) = \frac{1}{2\pi h^{1/4}} \exp\left\{-\frac{2}{3}h^{3/2}\right\}[1+o(1)], \qquad h \to \infty. \tag{6.2}$$

To prove the statement for x_a, use [3], Theorem 2.1, pages 143–144. To this end, define

$$\zeta_1(h) = x_a(h^2), \qquad \zeta_2(h) = h^{-2}\zeta_1'(h). \tag{6.3}$$

Then the eigenvalue equation $\mathcal{K}^a x_a = \rho(a)x_a$ [recall (2.1)] can be written as (see also [3], Equation (5.3))

$$\zeta'(h) = h^2 B(h)\zeta(h), \tag{6.4}$$

with

$$\zeta(h) = \begin{pmatrix} \zeta_1(h) \\ \zeta_2(h) \end{pmatrix}, \qquad B(h) = \begin{pmatrix} 0 & 1 \\ 2 - 2a/h^2 + 2\rho(a)/h^4 & -3/h^3 \end{pmatrix}. \tag{6.5}$$

Note that $B(h) = \sum_{n=0}^{\infty} h^{-n}B^{(n)}$ ($B^{(0)} \neq 0$) is a convergent power series in h^{-1}, with $B^{(0)}$ having eigenvalues $\lambda_{1,2} = \pm\sqrt{2}$. Therefore (6.4) has formal solutions of the form

$$Z(h) = P(h)h^R e^{Q(h)}, \tag{6.6}$$

where the columns of the matrix Z are the two linearly independent solutions to (6.4), $P(h) = \sum_{n=0}^{\infty} h^{-n}P^{(n)}$ [$\det(P^{(0)}) \neq 0$] is a formal power series in h^{-1}, R is a complex diagonal matrix and $Q(h) = \frac{1}{3}h^3 Q^{(0)} + \frac{1}{2}h^2 Q^{(1)} + hQ^{(2)}$ is a matrix polynomial with $Q^{(0)}$, $Q^{(1)}$ and $Q^{(2)}$ diagonal. In our case,

$$Q^{(0)} = \mathrm{diag}\{-\sqrt{2}, +\sqrt{2}\}, \qquad Q^{(1)} = 0, \qquad Q^{(2)} = \mathrm{diag}\left\{\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right\}. \tag{6.7}$$

From the proof of [3], Theorem 2.1, it follows that $P(h)$, R, $Q(h)$ can be chosen to be real, because $B(h)$, λ_1, λ_2 are real. A further remark on [3], page 151, is that for every formal solution there exists an actual solution with the same asymptotics.

We need the solution that is in $L^2[0,\infty)$. By construction, we compute, for $R = \mathrm{diag}\{r_1, r_2\}$ (with r_1, r_2 some functions of a),

$$\begin{aligned} &h^R \exp\{\tfrac{1}{3}h^3 Q^{(0)} + \tfrac{1}{2}h^2 Q^{(1)} + hQ^{(2)}\} \\ &\quad = \begin{pmatrix} h^{r_1} e^{-(\sqrt{2}/3)h^3 + (a/\sqrt{2})h} & 0 \\ 0 & h^{r_2} e^{(\sqrt{2}/3)h^3 - (a/\sqrt{2})h} \end{pmatrix}. \end{aligned} \tag{6.8}$$

Therefore the solution in $L^2[0,\infty)$ must be

$$\zeta(h) = h^{r_1} \exp\left\{-\frac{\sqrt{2}}{3}h^3 + \frac{a}{\sqrt{2}}h\right\} \sum_{n=0}^{\infty} h^{-n} \begin{pmatrix} P_{11}^{(n)} \\ P_{21}^{(n)} \end{pmatrix}, \tag{6.9}$$

where $P_{ij}^{(n)}$ denotes the element in the ith row and the jth column of the matrix $P^{(n)}$. Now return to (6.3) to read off the claim. □

Pick a' such that $a < a' < a^{**}$ and define [recall (2.9)]

$$\begin{aligned} M_t &= \frac{D_t^{(a')}}{D_t^{(a)}} \\ &= \frac{x_a(X_0)}{x_{a'}(X_0)} \frac{x_{a'}(X_t)}{x_a(X_t)} e^{-t[\rho(a')-\rho(a)]} \exp\left\{(a'-a)\int_0^t X_v\,dv\right\}, \qquad t \ge 0. \end{aligned} \tag{6.10}$$

Step 2. For any $h \ge 0$, $(M_t)_{t\ge 0}$ is a martingale under $\widehat{\mathbb{P}}_h^a$.

PROOF. Fix $0 \le s < t$. If ϕ_s denotes the time shift by $s \ge 0$ [i.e., $\phi_s((X_t)_{t\ge 0}) = (X_{s+t})_{t\ge 0}$], then it is clear that $M_t = M_s(M_{t-s} \circ \phi_s)$. Hence, using the Markov property at time s, we see that, for any $h \ge 0$,

$$\widehat{\mathbb{E}}_h^a(M_t | M_s) = M_s \widehat{\mathbb{E}}_h^a(M_{t-s} \circ \phi_s | M_s) = M_s \widehat{\mathbb{E}}_h^a\big(\widehat{\mathbb{E}}_{X_s}(M_{t-s})\big). \tag{6.11}$$

Now use that, for any $x \ge 0$, according to the construction of the transformed process in (2.9) and (2.10),

$$\widehat{\mathbb{E}}_x^a(M_{t-s}) = \mathbb{E}_x\big(D_{t-s}^{(a)} M_{t-s}\big) = \mathbb{E}_x\big(D_{t-s}^{(a')}\big) = 1. \tag{6.12}$$ □

Step 3. Equation (5.12) is valid.

PROOF. Use step 2, Doob's martingale inequality and (6.12) to obtain

$$\widehat{\mathbb{P}}_h^a\Big(\max_{[0,\delta]} M \ge K\Big) \le \frac{1}{K} \max_{t\in[0,\delta]} \widehat{\mathbb{E}}_h^a(M_t) = \frac{1}{K}, \qquad h \ge 0,\ K > 0. \tag{6.13}$$

Next note that by step 1, for any $R > 0$,

$$\inf_{[0,R]} \frac{x_a}{x_{a'}} \geq e^{-c\sqrt{R}}. \tag{6.14}$$

Substitute this into (6.10) to get

$$M_t \geq c\frac{x_{a'}(X_t)}{x_a(X_t)}e^{-c\sqrt{R}}, \qquad \widehat{\mathbb{P}}_h^a\text{-a.s., } 0 \leq t \leq 1,\ 0 \leq h \leq R. \tag{6.15}$$

Pick $g_a : [0,\infty) \to (0,\infty)$ to be the largest increasing function not exceeding $x_{a'}/x_a$ anywhere on $[0,\infty)$. Then, by (6.15), $M_t \geq cg_a(X_t)e^{-c\sqrt{R}}$ $\widehat{\mathbb{P}}_h^a$-a.s., $0 \leq t \leq 1$, $0 \leq h \leq R$. Now use (6.13) to estimate, for $0 \leq h \leq R$,

$$\begin{aligned}
\widehat{\mathbb{P}}_h^a\Big(\max_{[0,\delta]} X \geq \alpha\delta^{-1/2}\Big) &= \widehat{\mathbb{P}}_h^a\Big(\max_{t\in[0,\delta]} cg_a(X_t) \geq cg_a(\alpha\delta^{-1/2})\Big)\\
&\leq \widehat{\mathbb{P}}_h^a\Big(\max_{t\in[0,\delta]} M_t \geq cg_a(\alpha\delta^{-1/2})e^{-c\sqrt{R}}\Big)\\
&\leq \frac{1}{cg_a(\alpha\delta^{-1/2})}e^{c\sqrt{R}}.
\end{aligned} \tag{6.16}$$

By step 1, it is possible to pick g_a such that

$$g_a(h) \geq e^{-c\sqrt{h}}, \qquad h \to \infty. \tag{6.17}$$

This implies the bound in (5.12) with $d(\alpha) = \sqrt{\alpha}$. $\square$

Step 4. Equation (5.13) is valid.

PROOF. Fix $a < a^{**}$. Define

$$D_t^{(a,\star)} = \frac{y_a(X_t^\star)}{y_a(X_0^\star)} \exp\left\{\int_0^t [aX_v^\star - (X_v^\star)^2]\,dv\right\}, \qquad t \geq 0. \tag{6.18}$$

Then it is easy to check (see [16], Section VIII.3) that $(D_t^{(a,\star)})_{t\geq 0}$ is a martingale under $\mathbb{P}_h^\star$ for any $h \geq 0$ [y_a is a strictly positive solution to the differential equation $2y_a''(h) = (h-a)y_a(h)$ on $[0,\infty)$; recall (2.6) and (2.16)]. Hence, analogously to (2.10), we may construct a transformed process via a Girsanov transformation by taking $D_t^{(a,\star)}$ formally as a density with respect to BESQ^0. Denote by $\widehat{\mathbb{P}}_h^{a,\star}$ and $\widehat{\mathbb{E}}_h^{a,\star}$ probability and expectation with respect to this transformed process starting at $h \geq 0$.

Recall that $y_a(0) = 1$. We have the following representation for the function $y_a^{(\delta,\alpha)}$ [recall (5.10)]:

$$y_a^{(\delta,\alpha)}(h) = y_a(h)\widehat{\mathbb{P}}_h^{a,\star}(X_\delta^\star = 0, \max X^\star \geq \alpha\delta^{-1/2}). \tag{6.19}$$

The proof of (5.13) is now analogous to steps 2 and 3. Indeed, use (6.19), drop the restriction $X^\star_\delta = 0$ and proceed analogously. Step 1 provides the necessary asymptotic bounds for y_a and $y_{a'}$, provided that $a < a' < a^{**}$. □

Step 5. Equation (5.14) is valid.

PROOF. We return to the right-hand side of (4.11) and obtain a lower bound by inserting the indicator of the event $\{\max X^\star \le 2\delta\}$. On this event, we may estimate the exponential from below by c. Hence, for $0 \le h \le \delta$

$$\begin{aligned} y_a^{(\delta)}(h) &\ge c\mathbb{P}^\star_h(\max X^\star \le 2\delta, X^\star_\delta = 0) \\ &= c\big[\mathbb{P}^\star_h(X^\star_\delta = 0) - \mathbb{P}^\star_h(\max X^\star > 2\delta, X^\star_\delta = 0)\big]. \end{aligned} \tag{6.20}$$

Using the Markov property at the first time the BESQ^0 hits 2δ, we see that the latter probability is at most $\mathbb{P}^\star_{2\delta}(X^\star_\delta = 0)$. Since the first probability is decreasing in h, we therefore have the bound

$$y_a^{(\delta)}(h) \ge c\big(\mathbb{P}^\star_\delta(X^\star_\delta = 0) - \mathbb{P}^\star_{2\delta}(X^\star_\delta = 0)\big). \tag{6.21}$$

Now use that $\mathbb{P}^\star_h(X^\star_\delta = 0) = e^{-h/2\delta}$ for any $h, \delta \ge 0$ (see [16], Corollary XI(1.4)) to complete the proof. □

Acknowledgment. Most of the work of the first named author was carried out at Delft University of Tecnology.

REFERENCES

[1] ABRAMOWITZ, M. and STEGUN, I. (1970). *Handbook of Mathematical Functions*, 9th ed. Dover, New York.

[2] BOLTHAUSEN, E. (1993). On the construction of a three-dimensional polymer measure. *Probab. Theory Related Fields* **97** 81–101.

[3] CODDINGTON, E. A. and LEVINSON, N. (1955). *Theory of Ordinary Differential Equations*. McGraw-Hill, New York.

[4] DEMBO, A. and ZEITOUNI, O. (1996). *Large Deviations Techniques and Applications*. Jones and Bartlett, Boston.

[5] ERDÉLYI, A. (1956). *Asymptotic Expansions*. Dover, Toronto.

[6] GRIFFEL, D. H. (1981). *Applied Functional Analysis*. Wiley, New York.

[7] VAN DER HOFSTAD, R. (1998). On the constants in the central limit theorem for the one-dimensional Edwards model. *J. Statist. Phys.* **90** 1295–1306.

[8] VAN DER HOFSTAD, R. and DEN HOLLANDER, F. (1995). Scaling for a random polymer. *Comm. Math. Phys.* **169** 397–440.

[9] VAN DER HOFSTAD, R. and KÖNIG, W. (2001). A survey of one-dimensional random polymers. *J. Statist. Phys.* **103** 915–944.

[10] VAN DER HOFSTAD, R., DEN HOLLANDER, F. and KÖNIG, W. (1997). Central limit theorem for the Edwards model. *Ann. Probab.* **25** 573–597.

[11] VAN DER HOFSTAD, R., DEN HOLLANDER, F. and KÖNIG, W. (2002). Weak interaction limits for one-dimensional random polymers. *Probab. Theory Related Fields* **125** 483–521.

[12] DEN HOLLANDER, F. (2000). *Large Deviations*. Amer. Math. Soc., Providence, RI.

[13] LAWLER, G., SCHRAMM, O. and WERNER, W. (2002). On the scaling limit of planar self-avoiding walk. Preprint.

[14] MADRAS, N. and SLADE, G. (1993). *The Self-Avoiding Walk*. Birkhäuser, Boston.

[15] MARCH, P. and SZNITMAN, A.-S. (1987). Some connections between excursion theory and the discrete Schrödinger equation with random potentials. *Probab. Theory Related Fields* **75** 11–53.

[16] REVUZ, D. and YOR, M. (1994). *Continuous Martingales and Brownian Motion*, 2nd ed. Springer, Berlin.

[17] VANDERZANDE, C. (1998). *Lattice Models of Polymers*. Cambridge Univ. Press.

[18] VARADHAN, S. R. S. (1969). Appendix to "Euclidean quantum field theory," by K. Symanzik. In *Local Quantum Field Theory* (R. Jost, ed.). Academic Press, New York.

[19] WESTWATER, J. (1981). On the Edwards model for long polymer chains. II. The self-consistent potential. *Comm. Math. Phys.* **79** 53–73.

[20] WESTWATER, J. (1984). On Edwards' model for polymer chains. In *Trends and Developments in the Eighties* (S. Albeverio and Ph. Blanchard, eds.) 384–404. World Scientific, Singapore.

R. VAN DER HOFSTAD
DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE
EINDHOVEN UNIVERSITY OF TECHNOLOGY
P.O. BOX 513
5600 MB EINDHOVEN
THE NETHERLANDS
E-MAIL: rhofstad@win.tue.nl

F. DEN HOLLANDER
EURANDOM
P.O. BOX 513
5600 MB EINDHOVEN
THE NETHERLANDS
E-MAIL: denhollander@eurandom.tue.nl

W. KÖNIG
INSTITUT FÜR MATHEMATIK
TU BERLIN
STRASSE DES 17. JUNI 136
D-10623 BERLIN
GERMANY
E-MAIL: koenig@math.tu-berlin.de