ON CONFIDENCE BOUNDS ASSOCIATED WITH MULTIVARIATE ANALYSIS OF VARIANCE AND NON-INDEPENDENCE BETWEEN TWO SETS OF VARIATES[1]

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1. Summary. A theory of simultaneous confidence bounds on certain parametric functions, and their 'partials,' associated with some problems in multivariate normal statistical analysis has been developed by S. N. Roy and his associates in a series of publications over the past decade (e.g., [6], [7], [8], also see the references in [8]). In this paper we have obtained simultaneous confidence bounds on the members of a class of parametric functions, together with their 'partials,' associated with the two problems mentioned in the title. Our results, not only contain Roy's results as particular cases but, also throw a new light on the parametric functions used by Roy. Furthermore, confidence bounds, which can be considered as being associated with Hotelling's trace criterian for the MANOVA problem can be obtained as an example from our results.

2. Introduction and preliminaries. In this section we shall present the models for the two problems, and the simultaneous confidence bounds by S. N. Roy for these two problems; and discuss, in brief, his method of constructing these bounds.

2.1. *The* MANOVA *model.* In the statistical literature several MANOVA models have been discussed (e.g., [1], [2], [7]). However under certain invariance restriction all these models can be reduced to the following canonical form:

Let $[\mathbf{X}(p \times s), \mathbf{Y}(p \times (n-r), \mathbf{Z}(p \times (r-s))]$, $s \leqq r \leqq n-p$, be a $(p \times n)$ random matrix of observations whose columns are independently, normally distributed with a common covariance matrix $\mathbf{\Sigma}$ and expectations given by

$$E\mathbf{X} = \mathbf{\Delta}(p \times s), \qquad E\mathbf{Y} = \mathbf{0}(p \times (n-r)), \qquad E\mathbf{Z} = \mathbf{\Gamma}(p \times (r-s)).$$

The MANOVA problem is that of testing the MANOVA hypothesis $\mathcal{H}_0 : \mathbf{\Delta} = \mathbf{0}(p \times s)$ against the alternative $\mathcal{H}: \mathbf{\Delta} \neq \mathbf{0}(p \times s)$, and that of estimating the expectation matrix $\mathbf{\Delta}(p \times s)$ on the basis of the observations. The invariance restriction mentioned above dictates that an invariant procedure should involve the observations only through the characteristic roots of $\mathbf{S}_h\mathbf{S}_e^{-1}$, where $\mathbf{S}_h$ and $\mathbf{S}_e$ are the sum-of-products matrices due to the hypothesis and due to error, and are given by

$$\mathbf{S}_h = \mathbf{X}\mathbf{X}'(p \times p), \qquad \mathbf{S}_e = \mathbf{Y}\mathbf{Y}'(p \times p).$$

Thus we are interested in the simultaneous confidence bounds estimates of $\mathbf{\Delta}$ depending only on the characteristic roots of $\mathbf{S}_h\mathbf{S}_e^{-1}$.

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2.2. *The union-intersection principle and Roy's confidence bounds for* MANOVA. Roy's method and philosophy of obtaining the simultaneous confidence bounds is very closely related (see [8]) to a heuristic method of test construction, also due to him [5], and sometimes referred to as the union-intersection principle by his associates. In this method, starting from a (usually multivariate) composite hypothesis H_0 and a (usually multivariate) composite hypothesis H_1, Roy expresses them as, respectively, an intersection and a union of more primitive (usually univariate) composite hypotheses

$$H_0 = \bigcap_{\gamma \varepsilon \Gamma} H_{0\gamma}, \qquad H_1 = \bigcup_{\gamma \varepsilon \Gamma} H_{1\gamma}, \tag{2.2.1}$$

where Γ is an index set, and each pair $(H_{0\gamma}, H_{1\gamma})$ is such that on certain prior optimality grounds one can accept the (univariate) hypothesis $H_{0\gamma}$ against the (univariate) alternative $H_{1\gamma}$ over, say, a_γ, $\gamma \varepsilon \Gamma$. The union-intersection principle, then, accepts the (multivariate) hypothesis H_0 against the (multivariate) alternative H_1 over $\mathcal{A} = \bigcap_{\gamma \varepsilon \Gamma} a_\gamma$, and rejects it otherwise. In the multivariate examples considered by Roy, it usually happens that, associated with each pair $(H_{0\gamma}, H_{1\gamma})$, $\gamma \varepsilon \Gamma$, there is a confidence interval based on $\mathcal{A} = \cap a_\gamma$, and the information in (2.2.1), then enables him to make simultaneous confidence statements about the parameters, other than the nuisance parameters, occurring in H_0 and H_1. Roy then uses various variational representations of the characteristic roots of matrices to compress these simultaneous confidence statements into a confidence bound on a parametric function, usually a function of the largest characteristic root of some matrix of parameters, which can be considered as a measure of the departure of the *nature* from H_0, in the direction of H_1.

In the MANOVA problem Roy expresses $\mathcal{H}_0 : \mathbf{\Delta} = \mathbf{0}(p \times s)$, as an intersection $\mathcal{H}_0 = \cap \mathcal{H}_{0\gamma}$, $\mathcal{H}_{0\gamma} : \boldsymbol{\gamma}'\mathbf{\Delta} = \mathbf{0}(p \times 1)$, where $\boldsymbol{\gamma}$ is a p-dimensional real vector, and the intersection is over R^p, the space of all p-dimensional real vectors. For each $\mathcal{H}_{0\gamma}$ there is Snedecor's F-test which accepts over

$$\boldsymbol{\gamma}'\mathbf{XX}'\boldsymbol{\gamma}/\boldsymbol{\gamma}'\mathbf{YY}'\boldsymbol{\gamma} \leqq \text{a constant} = \tau^2 \quad (\text{say}), \qquad \boldsymbol{\gamma} \varepsilon R^p, \tag{2.2.2}$$

and rejects it otherwise, and also yields Scheffé type confidence bounds [10]

$$\boldsymbol{\gamma}'\mathbf{X}\boldsymbol{\delta} - \tau(\boldsymbol{\gamma}'\mathbf{YY}'\boldsymbol{\gamma})^{\frac{1}{2}} \leqq \boldsymbol{\gamma}'\mathbf{\Delta}\boldsymbol{\delta} \leqq \boldsymbol{\gamma}'\mathbf{X}\boldsymbol{\delta} + \tau(\boldsymbol{\gamma}'\mathbf{YY}'\boldsymbol{\gamma})^{\frac{1}{2}}, \qquad \boldsymbol{\gamma}\varepsilon\mathbf{R}^p,\ \boldsymbol{\delta}\varepsilon\mathbf{R}^s. \tag{2.2.3}$$

The union-intersection test associated with the above decomposition of $\mathcal{H}_0$ accepts the MANOVA hypothesis over the intersection of all the regions in (2.2.2), that is over $\mathrm{Ch}_1(\mathbf{S}_h\mathbf{S}_e^{-1}) \leqq \mu_\alpha$, where μ_α, a function of τ, is a constant determined by α, the level of significance and $\mathrm{Ch}(\cdot)$ is defined by the following:

DEFINITION 2.1. $\mathrm{Ch}_i(\mathbf{A})$, $i = 1, 2, \cdots, p$, denotes the ith largest characteristic root of the real symmetric matrix $\mathbf{A}(p \times p)$. We shall also write $\mathrm{Ch}_{\min}(\mathbf{A})$ for $\mathrm{Ch}_p(\mathbf{A})$.

The simultaneous confidence statements (2.2.3) imply the confidence statement

$$\begin{aligned} \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_h) - \mu_\alpha^{\frac{1}{2}}\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e) &\leqq \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{\Delta\Delta}') \\ &\leqq \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_h) + \mu_\alpha^{\frac{1}{2}}\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e). \end{aligned} \tag{2.2.4}$$

Next suppose that I and J are any two subsets of distinct positive integers not greater than, respectively, the number of rows and the number of columns of a matrix $\mathbf{A}$. We shall refer to such sets I and J as 'valid sets.' For any valid sets I and J let us define (i) $\mathbf{A}^{(I,0)}$, (ii) $\mathbf{A}^{(0,J)}$ and (iii) $\mathbf{A}^{(I,J)}$ as the matrices obtained by replacing in $\mathbf{A}$, respectively

(i) rows with numbers in I,

(ii) columns with numbers in J, and

(iii) rows with numbers in I and columns with numbers in J,

all by $\mathbf{0}$-vectors. It is easy to see that the non-zero characteristic roots of $\mathbf{A}^{(I,J)}\mathbf{A}^{(I,J)\prime}$ are the same as those of the matrix obtained from $\mathbf{AA}'$ by *deleting* or, as Roy puts it, by '*cutting out*' the rows and columns of $\mathbf{A}$ with numbers in I and J respectively. Now let

$$\mathbf{S}_h(I, J) = \mathbf{X}^{(I,J)}\mathbf{X}^{(I,J)\prime}$$

and

$$\mathbf{S}_e(I, 0) = \mathbf{Y}^{(I,0)}\mathbf{Y}^{(I,0)\prime}.$$

Then the confidence statements (2.2.3) also imply,

$$\begin{aligned} \operatorname{Ch}_1^{\frac{1}{2}}(\mathbf{S}_h(I, J)) - \mu_\alpha^{\frac{1}{2}} \operatorname{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e(I, 0)) \\ \leqq \operatorname{Ch}_1^{\frac{1}{2}}(\mathbf{\Delta}^{(I,J)}\mathbf{\Delta}^{(I,J)\prime}) \\ \leqq \operatorname{Ch}_1^{\frac{1}{2}}(\mathbf{S}_h(I, J)) + \mu_\alpha^{\frac{1}{2}} \operatorname{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e(I, 0)), \end{aligned} \tag{2.2.5}$$

for all valid sets I and J. The parametric functions $\operatorname{Ch}_1^{\frac{1}{2}}(\mathbf{\Delta}^{(I,J)}\mathbf{\Delta}^{(I,J)\prime})$ have been termed by Roy as the '*partials*' of $\operatorname{Ch}_1^{\frac{1}{2}}(\mathbf{\Delta\Delta}')$. Thus for any α, $0 < \alpha < 1$, the simultaneous confidence bounds (2.2.4) and (2.2.5) hold with confidence coefficient not less than $(1 - \alpha)$. It may be noted that the intervals (2.2.5) on the partials are, in general, narrower than the interval (2.2.4) on the parametric function.

2.3. *The model and Roy's confidence bounds for the non-independence problem.* In the model of the problem concerning the independence between two sets of variates, we have a $(p + q)$-variate normal population, $p \leqq q$, with the covariance matrix

$$\mathbf{\Sigma}((p + q) \times (p + q)) = \begin{bmatrix} \mathbf{\Sigma}_{11}(p \times p) & \mathbf{\Sigma}_{12}(p \times q) \\ \mathbf{\Sigma}_{12}'(q \times p) & \mathbf{\Sigma}_{22}(q \times q) \end{bmatrix}.$$

From this population we have a random sample of size $(n + 1)$ with the sample dispersion matrix

$$\mathbf{S}((p + q) \times (p + q)) = \begin{bmatrix} \mathbf{S}_{11}(p \times p) & \mathbf{S}_{12}(p \times q) \\ \mathbf{S}_{12}'(q \times p) & \mathbf{S}_{22}(q \times q) \end{bmatrix}.$$

Under this model the hypothesis of independence the two sets of variates (the p-set and the q-set) is $\mathcal{H}_0 : \mathbf{\Sigma}_{12} = \mathbf{0}(p \times q)$ against the alternative $\mathcal{H}$: $\mathbf{\Sigma}_{12} \neq \mathbf{0}(p \times q)$. The invariant [2] tests of $\mathcal{H}_0$ depend only on the sample canonical correlation coefficients, that is, the characteristic roots of $\mathbf{S}_{11}^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{12}'$, and the population canonical correlation coefficients, that is, the characteristic roots of $\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{12}'$ can be considered as valid measures of non-independence between the p-set and the q-set of variates. However, Roy has obtained simultaneous confidence bounds on certain functions of some regression-like parameters associated with this problem. To be specific let us set $\boldsymbol{\beta} = \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}$, $\mathbf{B} = \mathbf{S}_{12}\mathbf{S}_{22}^{-1}$, and $\mathbf{S}_{1.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{12}'$. It is then well known (e.g., page 110 [7]) that

$$e_i = c_i/(1 - c_i), \qquad i = 1, 2, \cdots, p, \tag{2.3.1}$$

where

$$e_i = \mathrm{Ch}_i[\mathbf{S}_{1.2}^{-1}(\mathbf{B} - \boldsymbol{\beta})\mathbf{S}_{22}(\mathbf{B} - \boldsymbol{\beta})'],$$

and

$$c_i = \mathrm{Ch}_i[(\mathbf{S}_{11} - \mathbf{S}_{12}\boldsymbol{\beta}' - \boldsymbol{\beta}\mathbf{S}_{12}' + \boldsymbol{\beta}\mathbf{S}_{22}\boldsymbol{\beta}')^{-1}(\mathbf{S}_{12} - \boldsymbol{\beta}\mathbf{S}_{22})\mathbf{S}_{22}^{-1}(\mathbf{S}_{12}' - \mathbf{S}_{22}\boldsymbol{\beta}')],$$

$i = 1, 2, \cdots, p$. It may be noted that $\mathcal{H}_0 : \mathbf{\Sigma}_{12} = \mathbf{0}$ is equivalent to $\boldsymbol{\beta} = \mathbf{0}$. Thus $\mathrm{Ch}(\boldsymbol{\beta}\boldsymbol{\beta}')$ are valid measures of non-independence between the two sets of variates. Roy's set of simultaneous confidence bounds with confidence coefficient not less than $(1 - \alpha)$, $0 < \alpha < 1$, obtained by using the method discussed above is

$$\begin{aligned} \boldsymbol{\gamma}'\mathbf{B}\boldsymbol{\delta} - \mu_\alpha \mathrm{Ch}^{\frac{1}{2}}(\mathbf{S}_{1.2})\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1}) & \\ \leqq \boldsymbol{\gamma}'\boldsymbol{\beta}\boldsymbol{\delta} \leqq \boldsymbol{\gamma}'\mathbf{B}\boldsymbol{\delta} + \mu_\alpha \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2})\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1}), & \end{aligned} \tag{2.3.2}$$

$$\begin{aligned} \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{B}\mathbf{B}') - \mu_\alpha^{\frac{1}{2}} \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2})\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1}) & \\ \leqq \mathrm{Ch}_1^{\frac{1}{2}}(\boldsymbol{\beta}\boldsymbol{\beta}') \leqq \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{B}\mathbf{B}') + \mu_\alpha^{\frac{1}{2}} \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2})\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1}), & \end{aligned} \tag{2.3.3}$$

and

$$\begin{aligned} &\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{B}(I, J)\mathbf{B}'(I, J)) - \mu_\alpha^{\frac{1}{2}} \cdot \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2}(I, 0))\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1}) \\ &\quad \leqq \mathrm{Ch}_1^{\frac{1}{2}}(\boldsymbol{\beta}(I, J)\boldsymbol{\beta}'(I, J)) \\ &\quad \leqq \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{B}(I, J)\mathbf{B}'(I, J)) + \mu_\alpha^{\frac{1}{2}} \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2}(I, 0))\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1}), \end{aligned} \tag{2.3.4}$$

for all valid sets I and J, where μ_α is the $\alpha\%$ point of the distribution of e_1, and $\mathbf{B}(I, J)$ etc. are the matrices obtained by replacing the variates in the p-set with numbers in I and the variates in the q-set with numbers in J, by variates which are, identically, equal to zero. It may be observed that replacing the variates by zero-variates is equivalent to cutting out these variates.

In Section 4 we shall obtain sumultaneous-confidence bounds with conservative confidence coefficients not less than $(1 - \alpha)$, $0 < \alpha < 1$, on symmetric gauge functions of the parametric functions $\mathrm{Ch}_i^{\frac{1}{2}}(\boldsymbol{\Delta}\boldsymbol{\Delta}')$, $i = 1, 2, \cdots, p$, and of $\mathrm{Ch}_i^{\frac{1}{2}}(\boldsymbol{\beta}\boldsymbol{\beta}')$, $i = 1, 2, \cdots, p$, and their partials. We shall obtain the bounds by direct inversion without an appeal to Roy's heuristic principle. Roy has often hinted at the 'distance' property of the parametric functions used by him (e.g. [8]). Section 5 will make this aspect more explicit. One of the good points of

Roy's bounds is that they involve only central distributions. Our bounds also use only on the central distributions. However, we shall discuss only the 'inversion' aspect of the bounds in this paper and defer the distribution aspect to another communication.

3. Symmetric gauge functions and the matrix theory. In this section we shall present the needed properties of symmetric gauge functions and their role in the matrix theory.

DEFINITION 3.1. A real valued function $\varphi(\mathbf{a}) = \varphi(a_1, a_2, \cdots, a_p)$ on the space of p-tuples of the real numbers is said to be a symmetric gauge function if

$$
(3.1)\quad
\begin{aligned}
&\text{(i) } \varphi(\mathbf{a}) \geqq 0 \text{ with the equality if, and only if, } a_1 = a_2 = \cdots = a_p = 0;\\
&\text{(ii) } \varphi(c\mathbf{a}) = |c|\varphi(\mathbf{a}) \text{ for any real } c;\\
&\text{(iii) } \varphi(\mathbf{a}_1 + \mathbf{a}_2) \leqq \varphi(\mathbf{a}_1) + \varphi(\mathbf{a}_2);\\
&\text{(iv) } \varphi(\epsilon_1 a_{i_1}, \epsilon_2 a_{i_2}, \cdots, \epsilon_p a_{i_p}) = \varphi(a_1, a_2, \cdots, a_p), \text{ where } \epsilon_i = \pm 1,\\
&i = 1, 2, \cdots, p, \text{ and } a_{i_1}, a_{i_2}, \cdots, a_{i_p} \text{ is a permutation of } a_1, a_2, \cdots, a_p.
\end{aligned}
$$

Sometimes, as convenience, one requires a symmetric gauge function φ to satisfy

(v) $\varphi(1, 1, \cdots, 1) = 1$.

EXAMPLES. Suppose that $a_{(1)} \geqq a_{(2)} \geqq \cdots \geqq a_{(p)}$ are the ordered values of real numbers $|a_1|, |a_2|, \cdots, |a_p|$. Then the following are some examples of the symmetric gauge functions:

(1) $\varphi(\mathbf{a}) = \sum_{i=1}^{q} a_{(i)}$, $q = 1, 2, \cdots, p$. Thus the largest value $a_{(1)}$ and the sum $\sum_{i=1}^{p} |a_i|$ are symmetric gauge functions

(2) $\varphi(\mathbf{a}) = (\sum_{i=1}^{p} |a_i|^l)^{1/l}$, $1 \leqq l \leqq \infty$, is another example of a symmetric gauge function. We note that the condition (iii), in this case, is the Minkowski inequality. We also note that $\lim_{l\to\infty} (\sum_{i=1}^{p} |a_i|^l)^{1/l} = a_{(1)}$.

DEFINITION 3.2. For any function φ satisfying the conditions (i) and (ii) of (3.1) let us define the conjugate of φ to be the function

$$\psi(a_1, a_2, \cdots, a_p) = \sup\,[\textstyle\sum_{i=1}^{p} a_i b_i/\varphi(b_1, b_2, \cdots, b_p)],$$

where the sup is over either of the sets (i) $\mathbf{b} = (b_1, b_2, \cdots, b_p) \neq \mathbf{0}$, (ii) $\sum_{i=1}^{p}|b_i| = 1$ or (iii) $\varphi(b_1, b_2, \cdots, b_p) = 1$.

Then it is well known [4] that ψ satisfies, in addition to (i) and (ii) of (3.1), the condition (iii) also. Furthermore if φ is a symmetric gauge function then ψ is also a symmetric gauge function.

EXAMPLES. (1) $\varphi(\mathbf{a}) = (\sum_{i=1}^{p} |a_i|^l)^{1/l}$ and $\psi(\mathbf{a}) = (\sum_{i=1}^{p} |a_i|^m)^{1/m}$, $1 \leqq l$, $m \leqq \infty$, $1/l + 1/m = 1$, are conjugate symmetric gauge functions.

(2) $\varphi(\mathbf{a}) = a_{(1)}$ and $\psi(\mathbf{a}) = \sum_{i=1}^{p} |a_i|$ are mutually conjugate.

Let $\mathbf{\Phi}_p$ denote the class of all symmetric gauge functions on the p-dimensional space of p-tuples of real numbers. Then it is easy to prove the following.

LEMMA 3.1. *For any $\varphi \in \mathbf{\Phi}_p$ and k_i, $0 \leqq k_i \leqq 1$, $i = 1, 2, \cdots, p$, we have*

$$\varphi(k_1 a_1, k_2 a_2, \cdots, k_p a_p) \leqq \varphi(a_1, a_2, \cdots, a_p).$$

DEFINITION 3.3. For any real matrix $\mathbf{A}(p \times n)$, $p \leqq n$, and any $\varphi \varepsilon \mathbf{\Phi}_p$ let

$$\|\mathbf{A}\|_\varphi = \varphi(c_1^{\frac{1}{2}}, c_2^{\frac{1}{2}}, \cdots, c_p^{\frac{1}{2}}),$$

where $c_i = \mathrm{Ch}_i(\mathbf{AA}')$, $i = 1, 2, \cdots, p$.

LEMMA 3.2. *For any real matrix* $\mathbf{A}(p \times n)$, *any orthogonal matrices* $\mathbf{L}(p \times p)$ *and* $\mathbf{R}(n \times n)$, *any* $\varphi = \mathbf{\Phi}_p$,

$$\|\mathbf{A}\|_\varphi = \|\mathbf{LAR}\|_\varphi .$$

LEMMA 3.3. *For any real matrix* $\mathbf{A}(p \times n)$, *any valid sets* I *and* J, *and any* $\varphi \varepsilon \mathbf{\Phi}_p$,

$$\|\mathbf{A}^{(I,J)}\|_\varphi \leqq (\|\mathbf{A}^{(I,0)}\|_\varphi, \|\mathbf{A}^{(0,J)}\|_\varphi) \leqq \|\mathbf{A}\|_\varphi,$$

where either term in the middle brackets satisfies both inequalities.

PROOF. Suppose $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n]$. Then $\mathbf{AA}' = \sum_{i=1}^n \mathbf{a}_i\mathbf{a}_i' = \mathbf{A}^{(I,0)}\mathbf{A}^{(I,0)\prime} + \mathbf{B}$, where $\mathbf{B}$ is at least positive semidefinite. Hence $\mathrm{Ch}_i(\mathbf{AA}') \geqq \mathrm{Ch}_i(\mathbf{A}^{(I,0)}\mathbf{A}^{(I,0)\prime})$, $i = 1, 2, \cdots, p$. Thus replacing any number of rows of $\mathbf{A}$ by $\mathbf{0}$-vectors only decreases its ordered characteristic roots. Now since the non-zero characteristic roots of $\mathbf{A}'\mathbf{A}$ are the same as those of $\mathbf{AA}'$, the above conclusion holds for replacement of the rows of $\mathbf{A}$ by $\mathbf{0}$-vectors also. Hence etc.

The following is the result (A.3.6) of [7], and may also be considered as a form of polar factorization (e.g. [3]).

LEMMA 3.4. *For any real matrix* $\mathbf{A}(p \times n)$, $p \leqq n$, *there exists a symmetric matrix to be denoted by* $|\mathbf{A}|$,*and an orthonormal matrix* $\mathbf{R}(p \times n)$, $\mathbf{RR}' = \mathbf{I}(p)$, *where* $\mathbf{I}(p)$ *denotes the p-dimensional identity matrix such that*

$$\mathbf{A}(p \times n) = |\mathbf{A}|\mathbf{R}.$$

It is easy to see that $\mathrm{Ch}_i(|\mathbf{A}|) = \mathrm{Ch}_i^{\frac{1}{2}}(\mathbf{AA}')$, $i = 1, 2, \cdots, p$. Thus we may, in the sequel, refer to $|\mathbf{A}|$ as $(\mathbf{AA}')^{\frac{1}{2}}$. Also for any symmetric matrix $\mathbf{S}(p \times p)$ we can define a symmetric matrix $\mathbf{S}^{\frac{1}{2}}(p \times p)$ such that $\mathrm{Ch}_i(\mathbf{S}^{\frac{1}{2}}) = \mathrm{Ch}_i^{\frac{1}{2}}(\mathbf{S})$, $i = 1, 2, \cdots, p$. The following result follows immediately.

LEMMA 3.5. *For any real matrix* $\mathbf{A}(p \times n)$, $p \leqq n$, *and any* $\varphi \varepsilon \mathbf{\Phi}_p$,

$$\|\mathbf{A}\|_\varphi = \|\,|\mathbf{A}|\,\|_\varphi .$$

LEMMA 3.6. *For any two real matrices* $\mathbf{A}(p \times n)$, *and* $\mathbf{B}(p \times n)$ *and any* $\varphi \varepsilon \mathbf{\Phi}_p$,

$$\|\mathbf{A}\|_\varphi - \|\mathbf{B}\|_\varphi \leqq \|\mathbf{A} + \mathbf{B}\|_\varphi \leqq \|A\|_\varphi + \|B\|_\varphi .$$

PROOF. For the second inequality see [4], or for a direct matrix proof see [3]. The first inequality may then be obtained by observing that $\|-\mathbf{A}\|_\varphi = \|\mathbf{A}\|_\varphi$.

LEMMA 3.7. *For any two real matrices* $\mathbf{A}(p \times n)$ *and* $\mathbf{B}(n \times p)$ *and any* $\varphi \varepsilon \mathbf{\Phi}_p$,

$$\mathrm{Ch}_p^{\frac{1}{2}}(BB')\|A\|_\varphi \leqq \|\mathbf{AB}\|_\varphi \leqq \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{BB}')\|\mathbf{A}\|_\varphi .$$

PROOF. We know that

$$\mathrm{Ch}_p(\mathbf{BB}')\mathrm{Ch}_i(\mathbf{AA}') \leqq \mathrm{Ch}_i(\mathbf{ABB}'\mathbf{A}') \leqq \mathrm{Ch}_1(\mathbf{BB}')\mathrm{Ch}_i(\mathbf{AA}'),$$

for all $i = 1, 2, \cdots, \min(p, n)$. Thus

$$\|\mathbf{AB}\|_\varphi \leqq \varphi(\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{BB}')\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{AA}'),\ \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{BB}')\mathrm{Ch}_2^{\frac{1}{2}}(\mathbf{AA}'),\ \cdots,\ \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{BB}') \cdot\mathrm{Ch}_p^{\frac{1}{2}}(\mathbf{AA}')) \leqq \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{BB}')\cdot\|\mathbf{A}\|_\varphi .$$

The other inequality of the lemma may be established similarly.

The following represention of $\|\mathbf{A}\|_\varphi$, for $\mathbf{A}(p \times n)$ real and $\varphi \,\varepsilon\, \mathbf{\Phi}_p$, due to von Neumann [4] will be very important for constructing the confidence bounds on partials.

LEMMA 3.8. *For any real matrix $\mathbf{A}(p \times n)$, $p \leqq n$, and any $\varphi \,\varepsilon\, \mathbf{\Phi}_p$,*

$$\|\mathbf{A}\|_\varphi = \sup_{\mathbf{M}\neq\mathbf{0}} [\mathrm{tr}\ |\mathbf{A}|\mathbf{M}/\|\mathbf{M}\|_\psi],$$

where $\mathbf{M}(p \times p)$ are real matrices and ψ is the conjugate of φ.

This result may be expressed in slightly different form as

LEMMA 3.9. *For any real matrix $\mathbf{A}(p \times n)$, and $\varphi \,\varepsilon\, \mathbf{\Phi}_p$ with conjugate $\psi \,\varepsilon\, \mathbf{\Phi}_p$,*

$$\|\mathbf{A}\|_\varphi = \sup_{\|\mathbf{N}\|_\psi=1} \mathrm{tr}\ \mathbf{AN}',$$

where the supremum is taken over all real matrices $\mathbf{N}(p \times n)$ such that $\|\mathbf{N}\|_\psi = 1$.

PROOF. Suppose that $\mathbf{A} = |\mathbf{A}|\mathbf{R}$, $\mathbf{RR}' = \mathbf{I}(p)$. Then

$$\begin{aligned}\|\mathbf{A}\|_\varphi &= \|\ |\mathbf{A}|\ \|_\varphi \\ &= \sup_{\|\mathbf{M}\|_\psi=1} \mathrm{tr}\ |\mathbf{A}|\mathbf{M} \\ &= \sup_{\|\mathbf{M}\|_\psi=1} \mathrm{tr}\ \mathbf{AN}',\end{aligned}$$

where $\mathbf{N}(p \times n) = \mathbf{M}'\mathbf{R}$. The proof is complete since $\|\mathbf{N}\|_\psi = \|\mathbf{M}\|_\psi$.

One of the consequences of the Lemma 3.8 is the following:

$$\mathrm{tr}\ \mathbf{AN}' \leqq \|\mathbf{A}\|_\varphi \cdot \|\mathbf{N}\|_\psi . \tag{3.2}$$

Other results concerning symmetric gauge functions and of relevance to this work are contained in the following:

LEMMA 3.10. *If $\varphi_r(\mathbf{a}) = (\sum_{i=1}^p |a_i|^r)^{1/r}$, $r \geqq 1$, then we have*

$$\varphi_r(\mathbf{a}) \leqq \varphi_{r'}(\mathbf{a}), \qquad 1 \leqq r' \leqq r \leqq \infty.$$

If φ_1, φ_2 are two symmetric gauge functions of p variables with conjugates ψ_1, ψ_2 respectively, and if $\varphi_1 \leqq \varphi_2$ then $\psi_1 \geqq \psi_2$. Furthermore for all $\varphi \,\varepsilon\, \mathbf{\Phi}_p$ we have

$$\max(|a_1|, |a_2|, \cdots, |a_p|) \leqq \varphi(\mathbf{a}) \leqq \textstyle\sum_{i=1}^p |a_i|.$$

4. The simultaneous confidence bounds associated with the two problems. In this section we shall present the confidence bounds for the two problems. We shall discuss the bounds for the non-independence problem in some detail and derive from these the bounds for the MANOVA problem. The observation that this may be done is due to the referee, which I greatly appreciate. His suggestions have immensely influenced the form of this section.

4.1 *The non-independence between two sets of variates.* Let us work in the framework of the model for the non-independence problem discussed in the Section 2.3. For any $\varphi \varepsilon \mathbf{\Phi}_p$ with the conjugate $\psi \varepsilon \mathbf{\Phi}_p$, the statistic $\varphi(e_1^{\frac{1}{2}}, e_2^{\frac{1}{2}}, \cdots, e_p^{\frac{1}{2}})$, where $e_i = c_i(1 - c_i)^{-1}$ and $c_1, c_2, \cdots, c_p$ are the sample canonical correlation coefficients, may be expressed as $\|S_{1.2}^{-\frac{1}{2}} B S_{22}^{\frac{1}{2}}\|_\varphi$. Let $C = \mu_{\alpha,\varphi}$ be the $\alpha\%$ point of the central distribution of this statistic. That is, let

$$(4.1.1) \qquad \|\mathbf{S}_{1.2}^{-\frac{1}{2}}(\mathbf{B} - \boldsymbol{\beta})\mathbf{S}_{22}^{\frac{1}{2}}\|_\varphi \leqq C$$

be true with probability $(1 - \alpha)$, $0 \leqq \alpha \leqq 1$. Then because of (3.1.2) we have for all $(q \times p)$ matrices $\mathbf{M}$

$$(4.1.2) \qquad |\mathrm{tr}\, \mathbf{S}_{1.2}^{-\frac{1}{2}}(\mathbf{B} - \boldsymbol{\beta})\mathbf{S}_{22}^{\frac{1}{2}} M| \leqq C\, \|\mathbf{M}\|_\psi .$$

Taking $\mathbf{M} = \mathbf{S}_{22}^{-\frac{1}{2}}\mathbf{N}\mathbf{S}_{1.2}^{\frac{1}{2}}$ we get the basic set of simultaneous confidence bounds

$$(4.1.3) \qquad \mathrm{tr}\,\mathbf{BN} - C\, \|\mathbf{S}_{22}^{-\frac{1}{2}}\mathbf{N}\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi \leqq \mathrm{tr}\,\boldsymbol{\beta}\mathbf{N} \leqq \mathrm{tr}\,\mathbf{BN} + C\, \|\mathbf{S}_{22}^{-\frac{1}{2}}\mathbf{N}\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi ,$$

valid for all $(q \times p)$ matrices $\mathbf{N}$ with the simultaneous confidence coefficient $(1 - \alpha)$.

Some Particular Cases. (i) Let φ_1 be a symmetric gauge function of p variables with the conjugate ψ_1, and such that $\varphi_1 \leqq \varphi$. From the Lemma 3.6 we have

$$\|\mathbf{S}_{22}^{-\frac{1}{2}}\mathbf{N}\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi \leqq \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2})\|\mathbf{N}\|_\psi .$$

Hence from (4.1.2) we get

$$\begin{aligned} |\mathrm{tr}\,(\mathbf{B} - \boldsymbol{\beta})\mathbf{N}| &\leqq C\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2})\|\mathbf{N}\|_\psi \\ &\leqq C\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2})\|\mathbf{N}\|_{\psi_1} , \end{aligned}$$

since $\psi_1 \geqq \psi$. An application of the Lemma 3.8, then yields

$$\begin{aligned} (4.1.4) \qquad \|\mathbf{B}\|_{\varphi_1} - C\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2}) &\leqq \|\mathbf{B}\|_{\varphi_1} \\ &\leqq \|\mathbf{B}\|_{\varphi_1} + C\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2}) \end{aligned}$$

true for all $\varphi_1 \leqq \varphi$ with simultaneous confidence coefficient $(1 - \alpha)$.

Moreover, we get from $\mathrm{tr}\,\boldsymbol{\beta}\mathbf{N} \geqq \mathrm{tr}\,\mathbf{BN} - C\, \|\mathbf{S}_{22}^{-\frac{1}{2}}\mathbf{N}\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi$ using the Lemmas 3.6 and 3.8 the following:

$$\begin{aligned} \|\boldsymbol{\beta}\|_{\varphi_1} \geqq \max\{&\mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})(\|\mathbf{B}\mathbf{S}_{22}^{\frac{1}{2}}\|_{\varphi_1} - C\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2})), \\ (4.1.5) \qquad &\mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_{1.2})(\|\mathbf{S}_{1.2}^{-\frac{1}{2}}\mathbf{B}\|_{\varphi_1} - C\, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})), \\ &\mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_{1.2})\mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})(\|\mathbf{S}_{1.2}^{-\frac{1}{2}}\mathbf{B}\mathbf{S}_{22}^{\frac{1}{2}}\|_\varphi - C)\}, \end{aligned}$$

with confidence $\geqq (1 - \alpha)$.

(ii) Now let $\varphi_2 > \varphi$ be any symmetric gauge function of p variables with the conjugate $\psi_2 < \psi$. To obtain a development of confidence bounds as above we notice that $\|\mathbf{N}\|_{\psi_2} \geqq \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{NN}')$ and hence

$$\|\mathbf{S}_{22}^{-\frac{1}{2}}\,\mathbf{N}\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi \leqq \min\{\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\|\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi \cdot \|\mathbf{N}\|_{\psi_2}, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2}^{\frac{1}{2}})\|\mathbf{S}_{1.2}^{-\frac{1}{2}}\|_\psi \cdot \|\mathbf{N}\|_{\psi_2}\}.$$

This with (4.1.3) and the Lemma 3.8 yields

$$\begin{aligned}(4.1.6)\quad \|\mathbf{B}\|_{\varphi_2} - C\min(\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\|\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi, \mathrm{Ch}_{1q}^{\frac{1}{2}}(\mathbf{S}_{1.2})\|\mathbf{S}_{22}^{-\frac{1}{2}}\|_\psi)& \\ \leqq \|\boldsymbol{\beta}\|_{\varphi_2} \leqq \|\mathbf{B}\|_{\varphi_2} + C\min(\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\|\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2})\|\mathbf{S}_{22}^{-\frac{1}{2}}\|_\psi)&.\end{aligned}$$

Furthermore, we get from $\mathrm{tr}\,\boldsymbol{\beta}\mathbf{N} \geqq \mathrm{tr}\,\mathbf{BN} - \|\mathbf{S}_{22}^{-\frac{1}{2}}\mathbf{N}\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi$, the following:

$$\begin{aligned}\|\boldsymbol{\beta}\|_{\varphi_2} \geqq \max\{&\mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})(\|\mathbf{B}\mathbf{S}_{22}^{\frac{1}{2}}\|_{\varphi_2} - C\,\|\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi),\\ (4.1.7)\qquad &\mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_{1.2})(\|\mathbf{S}_{1.2}^{-\frac{1}{2}}\mathbf{B}\|_{\varphi_2} - C\,\|\mathbf{S}_{22}^{-\frac{1}{2}}\|_\psi),\\ &\mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\,\mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_{1.2})(\|\mathbf{S}_{1.2}^{-\frac{1}{2}}\mathbf{B}\mathbf{S}_{22}^{\frac{1}{2}}\|_{\varphi_2} - C)\}\end{aligned}$$

with confidence $\geqq (1 - \alpha)$.

Partials. Let us recall the notation defined for the Equation (2.3.3) and replace $\mathbf{N}$ in

$$\mathrm{tr}\,(\mathbf{B} - \boldsymbol{\beta})\mathbf{N} \leqq C\,\|\mathbf{S}_{22}^{-\frac{1}{2}}\mathbf{N}\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi$$

by $\mathbf{N}(I, J)$ and get

$$\begin{aligned}\mathrm{tr}\,(\mathbf{B}(I, J) - \boldsymbol{\beta}(I, J))N = \mathrm{tr}\,(\mathbf{B} - \boldsymbol{\beta})\mathbf{N}^{(I,J)} &\leqq C\,\|\mathbf{S}_{22}^{-\frac{1}{2}}\mathbf{N}^{(I,J)}\mathbf{S}_{1.2}^{\frac{1}{2}}\|_\psi\\ &\leqq C\,\|\mathbf{S}_{22}^{-\frac{1}{2}}\mathbf{N}\mathbf{S}_{1.2}^{\frac{1}{2}}(I, 0)\|_\psi.\end{aligned}$$

Then arguments similar to the development of previous bounds yield the following:

$$\begin{aligned}(4.1.8)\quad \mathrm{tr}\,\mathbf{B}(I, J)\mathbf{N} - C\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2}(I, 0))&\\ \leqq \mathrm{tr}\,\boldsymbol{\beta}(I, J)\mathbf{N} \leqq \mathrm{tr}\,\mathbf{B}(I, J)\mathbf{N} + C\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2}(I, 0))&;\\ (4.1.9)\quad \|\mathbf{B}(I, J)\|_{\varphi_1} - C\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2}(I, 0))&\\ \leqq \|\boldsymbol{\beta}(I, J)\|_{\varphi_1} \leqq \|\mathbf{B}(I, J)\| + C\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{22}^{-1})\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_{1.2}(I, 0))&,\end{aligned}$$

and analogues of other bounds obtained by replacing $\mathbf{B}$ by $\mathbf{B}(I, J)$, $\boldsymbol{\beta}$ by $\boldsymbol{\beta}(I, J)$, and $\mathbf{S}_{1.2}$ by $\mathbf{S}_{1.2}(I, 0)$, all simultaneously true with confidence not less than $(1 - \alpha)$.

We shall summerize the results of this section in the following:

THEOREM 4.1. *For the non-independence problem, inequalities* (4.1.3) *to* (4.1.9), *where* φ, φ_1, φ_2 ($\varphi_1 \leqq \varphi < \varphi_2$) *are any symmetric gauge functions of* p *variables and* C *is defined by* (4.1.1), *provide a set of simultaneous confidence bounds with the simultaneous confidence coefficient not less than* $(1 - \alpha)$.

COROLLARY 4.1. *If we take* $\varphi(a) = \max(|a_1|, |a_2|, \cdots, |a_p|)$ *and* $\mathbf{N} = \boldsymbol{\gamma}\boldsymbol{\delta}'$ *where* $\boldsymbol{\gamma}(q \times 1)$ *and* $\boldsymbol{\delta}(p \times 1)$ *are any two vectors with unit modulus then we get Roy-type bounds*, e.g., (4.1.3) *yields* (2.3.2), (4.1.4) *with* $\varphi_1 = \varphi$ *reduces to* (2.3.3) *and* (4.1.9) *gives* (2.3.4).

4.2. *MANOVA*. Let us recall the model and notation for the MANOVA problem discussed in the Section 2.2. Let us start with a statistic $\|\mathbf{S}_h^{\frac{1}{2}}\mathbf{S}_e^{-\frac{1}{2}}\|_\varphi = \varphi(c_1^{\frac{1}{2}}, c_2^{\frac{1}{2}}, \cdots, c_p^{\frac{1}{2}})$, where $c_1, c_2, \cdots, c_p$ are the characteristic roots of $\mathbf{S}_h\mathbf{S}_e^{-1}$ and φ is any symmetric gauge function of p variables. Let $\mu_{\alpha,\varphi} = C$ be the $\alpha\%$ point of the central distribution of this statistic. That is let

$$\|(\mathbf{X} - \mathbf{\Delta})\mathbf{S}_e^{-\frac{1}{2}}\|_\varphi \leqq C \tag{4.2.1}$$

be true with probability $(1 - \alpha)$, $0 < \alpha < 1$.

A comparison between (4.1.1) and (4.1.2) shows that the bounds for the MANOVA problem may be obtained from the bounds for the non-independence problem by replacing $\mathbf{B}$ by $\mathbf{X}$, $\boldsymbol{\beta}$ by $\mathbf{\Delta}$, $\mathbf{S}_{22}$ by $\mathbf{I}(q)$, where $\mathbf{I}(q)$ is the identity matrix, and $\mathbf{S}_{1.2}$ by $\mathbf{S}_e$, etc., and making relevant changes in the dimensions of the matrices. In this way we get, for example, the following:

$$\operatorname{tr}\mathbf{XN} - C\,\|\mathbf{S}_e\mathbf{N}\|_\psi \leqq \operatorname{tr}\mathbf{\Delta N} \leqq \operatorname{tr}\mathbf{XN} + C\,\|\mathbf{S}_e\mathbf{N}\|_\psi\,; \tag{4.2.2}$$

$$\|\mathbf{X}\|_{\varphi_1} - C\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e) \leqq \|\mathbf{\Delta}\|_{\varphi_1} \leqq \|\mathbf{X}\|_{\varphi_1} + C\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e); \tag{4.2.3}$$

$$\|\mathbf{\Delta}\|_{\varphi_1} \geqq \mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_e)(\|\mathbf{S}_e^{-\frac{1}{2}}\mathbf{X}\|_{\varphi_1} - C); \tag{4.2.4}$$

$$\|\mathbf{X}\|_{\varphi_2} - C\min(1, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e)) \leqq \|\mathbf{\Delta}\|_{\varphi_2} \leqq \|\mathbf{X}\|_{\varphi_2} + C\min(1, \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e)); \tag{4.2.5}$$

$$\|\mathbf{\Delta}\|_{\varphi_2} \geqq \max\{(\|\mathbf{X}\|_{\varphi_2} - C\,\|\mathbf{S}_e^{\frac{1}{2}}\|_\psi),\ \mathrm{Ch}_{\min}^{\frac{1}{2}}(\mathbf{S}_e)(\|\mathbf{S}_e^{-\frac{1}{2}}\mathbf{X}\|_{\varphi_2} - C)\}; \tag{4.2.6}$$

$$\begin{aligned}\|\mathbf{S}_h(I, J)\|_{\varphi_1} - C\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e(I, 0)) &\leqq \|\mathbf{\Delta}(I, J)\|_{\varphi_1} \\ &\leqq \|\mathbf{S}_h(I, J)\|_{\varphi_1} + C\,\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{S}_e(I, 0));\end{aligned} \tag{4.2.7}$$

and other analogues of the bounds for the non-independence problem. We summarize these results in the following:

THEOREM 4.2. *The inequalities* (4.2.2) *to* (4.2.7), *where* C *is given by* (4.2.1), *and* $\varphi, \varphi_1, \varphi_2$ ($\varphi_1 \leqq \varphi \leqq \varphi_2$) *are many symmetric gauge functions of* p *variables provide a set of simultaneous confidence bounds for the* MANOVA *problem with confidence coefficient not less than* $(1 - \alpha)$.

COROLLARY 4.2. *Theorem* 4.2 *with* $\varphi(\mathbf{a}) = \max(|a_1|, |a_2|, \cdots, |a_n|)$ *and* $\mathbf{N} = \boldsymbol{\gamma}\boldsymbol{\delta}'$, *where* $\boldsymbol{\gamma}(s \times 1)$ *and* $\boldsymbol{\delta}(p \times 1)$ *are any two vectors with unit modulus, provide Roy-type confidence bounds. In particular,* (4.2.2) *yields* (2.2.3), (4.2.3) *with* $\varphi_1 = \varphi$ *reduces to* (2.2.4) *and* (4.2.7) *gives* (2.2.5).

COROLLARY 4.3. *Theorem* 4.2 *with* $\varphi(\mathbf{a}) = (\sum_{i=1}^p a_i^2)^{\frac{1}{2}}$ *provides simultaneous confidence bounds which use the distribution of Hotelling's trace criterion.*

5. Remarks. (1). For any $\varphi \in \mathbf{\Phi}_p$, and real $\mathbf{A}(p \times n)$, $\|\mathbf{A}\|_\varphi$ is a unitarily invariant norm, that is, it satisfies

(i) $\|\mathbf{A}\|_\varphi \geqq 0$, with equality if, and only if, $\mathbf{A} = \mathbf{0}$,

(ii) $\|c\mathbf{A}\|_\varphi = |c| \cdot \|\mathbf{A}\|_\varphi$ for any real c,

(iii) $\|\mathbf{A}_1 + \mathbf{A}_2\|_\varphi \leqq \|\mathbf{A}_1\|_\varphi + \|\mathbf{A}_2\|_\varphi$,

(iv) $\|\mathbf{LAR}\|_\varphi = \|\mathbf{A}\|_\varphi$, where $\mathbf{L}$ and $\mathbf{R}$ are orthogonal.

Furthermore, it is known [9] that $\|\mathbf{A}\|_\varphi$, $\varphi \in \mathbf{\Phi}_p$, are the only unitarily invariant norms. It is, therefore, clear that the parametric functions introduced by Roy, namely, the largest characteristic roots of the parametric matrices, indeed have the "distance properties" which he vaguely claimed. As a matter of fact, because of the Lemma 3.9, his parametric functions happen to be the smallest unitarily invariant norms of the parametric matrices. We have obtained simultaneous confidence bounds on all unitarily invariant norms of the matrices of the noncentrality parameters.

(2) Let us consider the confidence bounds of the type (4.1.4) for the nonindependence problem or (4.2.3) for the MANOVA. The geometrical width of these confidence bounds is proportional to the constant $C = \mu_{\alpha,\varphi}$, which in turn depends upon the confidence coefficient $(1 - \alpha)$ and the symmetric gauge function φ, with which we start. Now by the Lemma 3.9 we have, for any $\varphi \varepsilon \mathbf{\Phi}_p$,

$$\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{AA}') \leqq \|\mathbf{A}\|_\varphi \leqq \textstyle\sum_{i=1}^p \mathrm{Ch}_i^{\frac{1}{2}}(\mathbf{AA}').$$

Therefore for any $\varphi \varepsilon \mathbf{\Phi}_p$, $\|\mathbf{A}\|_\varphi \leqq C$ implies that $\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{AA}') \leqq C$. Thus if $\mathbf{AA}' = \mathbf{S}_h\mathbf{S}_e^{-1}$ or $\mathbf{AA}' = \mathbf{S}_{1.2}^{-1}\mathbf{BS}_{22}\mathbf{B}'$, then

$$\mathrm{Prob}\,\{\|\mathbf{A}\|_\varphi \leqq \mathbf{C}\} \leqq \mathrm{Prob}\,\{\mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{AA}') \leqq C\}.$$

That is, the constant C is the least when $\|\mathbf{A}\|_\varphi = \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{AA}')$. This shows that among all the confidence bounds of the type (4.1.4) or (4.2.3) those obtained by starting from the maximum root tests are the shortest. It may, however, be noted that the only φ_1 satisfying $\varphi_1 \leqq \varphi$, when $\varphi = \mathrm{Ch}_1^{\frac{1}{2}}(\mathbf{AA}')$, is φ itself. Thus larger the symmetric gauge function φ we start with, it will provide more simultaneous confidence bounds of type (4.1.4) or (4.2.3), but they will be wider.

(3) It may be noted that Hotelling's trace criterion for the MANOVA problem is based on the self conjugate symmetric gauge function $\varphi(\mathbf{a}) = (\sum a_i^2)^{\frac{1}{2}}$. Statistical implication of this fact deserves further investigation.

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