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OPTIMAL CONSUMPTION AND PORTFOLIO POLICIES WITH AN INFINITE HORIZON: EXISTENCE AND CONVERGENCE

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We provide sufficient conditions for the existence of a solution to a consumption and portfolio problem in continuous time under uncertainty with an infinite horizon. When the price processes for securities are diffusion processes, optimal policies can be computed by solving a linear partial differential equation. We also provide conditions under which the solution to an infinite horizon problem is the limit of the solutions to finite horizon problems when the horizon increases to infinity.

1. Introduction and summary. Recent advances in the understanding of dynamic asset markets have made available a set of new tools to analyze the optimal consumption and portfolio decision of an individual in continuous time under uncertainty; see Cox and Huang (1989, 1990), Karatzas, Lehoczky and Shreve (1987), He and Pearson (1989), Pagès (1989) and Pliska (1986). There are several attractive features of these new tools. First, the existence problem of an optimal consumption and portfolio policy for an individual can be analyzed with much ease while the admissible policies do not take their values in a compact set and the consumption must obey a positive constraint. Second, when securities prices follow a diffusion process, the optimal policies can be computed by solving a linear partial differential equation in contrast to a highly nonlinear Bellman equation in dynamic programming. Third, in some situations, optimal policies can even be computed directly by evaluating some integrals.

The aforementioned papers, however, address the *optimal consumption and portfolio problem* in economies with a finite horizon. The purpose of this paper is to show how this set of new tools can be brought to bear on the same problem in economies with an infinite horizon. Our conclusions are that, except for some important technical departures, the existence and computation of an optimal policy can be analyzed similarly with the same attractive features as in finite horizon economies.

We also study the convergence of optimal policies in finite horizon economies to those in infinite horizon economies. We show that pointwise convergence always occurs for optimal consumption policies. But we do not know whether this occurs for optimal portfolio policies. However, optimal portfolio policies converge in a certain norm involving taking expectations. Thus, if the optimal

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portfolio policies for finite horizon economies do converge at all pointwise, they must converge to the optimal policy in the infinite horizon economy. Note that this type of convergence result is very difficult to get by using dynamic programming.

The analysis in this paper utilizes Cox and Huang (1989, 1990) extensively, which the reader may want to consult. Two other papers complement our study. First, Merton (1989) uses the Cox–Huang technique to solve, in infinite horizon, the optimal consumption and portfolio policy in closed form for a class of utility functions when asset prices follow a geometric Brownian motion. Second, Foldes (1989), using a different but related technique, also analyzes the optimal consumption and portfolio problem with an infinite horizon in a stochastic environment more general than ours. He, however, does not give explicit conditions for existence nor does he study convergence properties of optimal policies.

The rest of this paper is organized as follows. Section 2 formulates the model. The existence and computation of optimal policies are analyzed in Section 3 and Section 4, respectively. Section 5 gives closed-form solutions for some commonly used utility functions. Section 6 gives results on convergence and Section 7 contains concluding remarks.

2. Formulation. We consider a securities market under uncertainty in continuous time with an infinite horizon. We will use an N-dimensional Brownian motion to model the evolution of exogenous uncertainty. Thus we take the state space Ω to be the space of continuous functions from $[0,\infty)$ to $\Re^N$ equipped with the topology of uniform convergence on compact subintervals of $[0,\infty)$. The collection of distinguishable events is the Borel sigma field of Ω denoted by $\mathscr{F}$ and the probability belief of the agent to be considered is the *Wiener measure* on $(\Omega, \mathscr{F})$ denoted by P. It is well known that under P, the *coordinate process*

$$w(\omega, t) \equiv \omega(t), \qquad \forall\, \omega \in \Omega,$$

is an N-dimensional standard Brownian motion (a standard Brownian motion is a Brownian motion that starts at zero with probability 1), where $\omega(t)$ is the value at time t of the $\Re^N$-valued continuous function $\omega \in \Omega$. Since a state of nature is a complete realization of w on the time interval $[0,\infty)$ and one learns the true state of nature by observing w over time, we model the intertemporal resolution of uncertainty by an increasing and right-continuous family of subsigma fields of $\mathscr{F}$ or a *filtration* $\mathbf{F} = \{\mathscr{F}_t;\ t \in \Re_+\}$, where $\mathscr{F}_t = \bigcap_{s>t}\mathscr{F}_s^{\,0}$ and $\mathscr{F}_t^{\,0}$ is the smallest sigma field generated by $\{w(s);\ 0 \le s \le t\}$. (Throughout this paper we will use weak relations. For example, positive means nonnegative, increasing means nondecreasing, and so on.) One can verify that $\mathscr{F} = \bigvee_{t\ge 0}\mathscr{F}_t$, that is, all the distinguishable events are generated by sample paths of w. As a standard Brownian motion starts from zero at time $t = 0$ with probability 1, $\mathscr{F}_0$ contains only subsets of $\mathscr{F}$ that are probability 1 or 0.

All the processes to appear will be adapted to $\mathbf{F}$. In our setup, however, adapted processes are progressively measurable; see Stroock and Varadhan

[(1979), Exercises 1.5.6 and 1.5.11]. Since progressively measurable processes are naturally adapted, the set of adapted processes is equivalent to the set of progressively measurable processes. A martingale X under P here is a (progressive) process that has right-continuous paths and has continuous paths with P probability 1 so that $E[X(s)|\mathscr{F}_t] = X(t)$ for all $s \geq t$, where $E[\cdot|\mathscr{F}_t]$ is the expectation under P conditional on $\mathscr{F}_t$. For any integrable random variable Y on $(\Omega, \mathscr{F}, P)$, there exists a (P)-martingale X so that $X(t) = E[Y|\mathscr{F}_t]$, P-a.s.; see Jacod and Shiryaev [(1987), Remark 1.37]. All the conditional expectations to appear will be martingales.

REMARK 1. In the above setup, the probability space $(\Omega, \mathscr{F}, P)$ is not complete [a probability space $(\Omega, \mathscr{F}, P)$ is complete if any subset of a probability zero set is an element of $\mathscr{F}$] and the filtration $\mathbf{F}$ does not satisfy the usual conditions. [A filtration $\mathbf{F}$ satisfies the *usual conditions* if: (i) it is right-continuous: $\mathscr{F}_t = \bigwedge_{s>t}\mathscr{F}_s$ for all $t \in \Re_+$; (ii) it is *complete*: the probability space $(\Omega, \mathscr{F}, P)$ is complete and $\mathscr{F}_0$ contains all the P-measure zero sets.] This is a departure from the earlier literature such as Cox and Huang (1989, 1990). This departure is important for our purpose; otherwise, Proposition 1 to follow is not valid.

We will use the following notation: If x is a matrix, then $|x| = (\mathrm{trace}(xx^{\mathsf{T}}))^{1/2}$, where T denotes transpose.

There are $N + 1$ securities traded continuously on the infinite time horizon $[0, \infty)$ indexed by $n = 0, 1, 2, \ldots, N$. Security $n \neq 0$ is risky and is represented by a process of right-continuous and bounded variation sample paths D_n with $D_n(t)$ representing cumulative dividends paid by security n from time 0 to time t. Denote the ex-dividend price of security $n \neq 0$ at time t by $S_n(t)$ and let $S(t) = (S_1(t), \ldots, S_N(t))^{\mathsf{T}}$ and $D(t) = (D_1(t), \ldots, D_N(t))^{\mathsf{T}}$. As these securities are traded ex-dividends, assume without loss of generality that $D_n(0) = 0$ for all $n = 1, 2, \ldots, N$. We assume also that $S + D$ is an N-vector process:

$$(1) \qquad S(t) + D(t) = S(0) + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dw(s), \qquad t \in \Re_+,$$

where b and σ, respectively, are N-vector and $N \times N$-matrix predictable processes (a process is predictable if it is measurable with respect to the predictable sigma field, which is the sigma field generated by all adapted processes with continuous paths) satisfying, for all n,

$$(2) \qquad \int_0^{T_n} |b(s)|\,ds < \infty, \qquad P\text{-a.s.}, \forall\, t \geq 0,$$

$$(3) \qquad \int_0^{T_n} |\sigma(s)|^2\,ds < \infty, \qquad P\text{-a.s.}, \forall\, t \geq 0$$

for some sequence of optional times (T_n) with $T_n \uparrow \infty$ P-a.s., and the second integral of (1) is a stochastic integral. [The usual definition of a stochastic integral depends on the completion of a probability space and a filtration satisfying the *usual conditions*; see, for example, Lipster and Shiryayev [(1977), Chapter 4.] The definition of a stochastic integral here is based on Jacod and Shiryayev [(1987), Chapter 1] and has all the usual properties. In particular, the stochastic integral of a continuous local martingale is a continuous local martingale and Itô's formula is valid, where we recall that a process X is a local martingale under P if there exists a sequence of optional times $T_n \uparrow \infty$, P-a.s., so that the process $\{X(T_n \wedge t); t \in \Re_+\}$ is a (uniformly integral) martingale for all T_n.]

We assume that $\sigma(t)$ is P-a.s. of full rank for all t. The 0th security, called the bond, is locally riskless. Its price at time t, $B(t)$, is $\exp\{\int_0^t r(s)\,ds\}$. One can view this security as a bank account that pays an instantaneous interest rate $r(t)$ at time t. So \$1 invested at time 0 grows to $B(t)$ at time t. For B to be well defined, we assume that the interest rate process r satisfies

$$(4) \qquad \int_0^t |r(s)|\,ds < \infty, \qquad P\text{-a.s.}, \forall\, t \geq 0.$$

Now consider an agent with a time-additive utility function for consumption, $u(c,t)$ and an initial wealth $W_0 > 0$. Assume throughout that $u(c,t)$ is continuous in (c,t), concave and increasing in c and is possibly unbounded from below at $c = 0$. This agent wants to manage a portfolio of the risky securities and the bond and withdraw funds out of the portfolio to maximize his expected utility of consumption over time. Our task here is to find conditions on the utility function and on the price processes to guarantee the existence of a solution to the agent's problem.

We refer to the *price system* as the N-vector of normalized prices defined by $S^*(t) = S(t)/B(t)$. Itô's formula implies that

$$S^*(t) + \int_0^t \frac{1}{B(s)}\,dD(s) = S(0) + \int_0^t \frac{b(s) - r(s)S(s)}{B(s)}\,ds + \int_0^t \frac{\sigma(s)}{B(s)}\,dw(s), \qquad P\text{-a.s.}, t \in \Re_+.$$

The process on the left-hand side is called the *gains process* and here it is expressed in units of the bond. Putting

$$G(t) = S^*(t) + \int_0^t \frac{1}{B(s)}\,dD(s),$$

one sees that the difference $G(t) - S(0)$ represents the accumulated capital gains and accumulated dividends on risky assets in units of the bond, where we note that since $B(0) = 1$, $S^*(0) = S(0)$.

A trading strategy is a $N + 1$-vector predictable process $(\alpha, \theta^{\mathrm{T}}) \equiv (\alpha, (\theta^1, \ldots, \theta^N))$, where $\alpha(t)$ and $\theta^n(t)$ are the number of shares of the bond and of risky asset n, respectively, owned by the investor at time t before trading. The investor's wealth in units of the bond at time t after the receipt of dividends is

$$W(t) = \alpha(t) + \theta(t)^{\mathrm{T}}(S^*(t) + \Delta D(t)/B(t)).$$

For now, a consumption plan c is a process that is positive (except on a set that is negligible with respect to the product measure generated by P and Lebesgue measure on $\mathfrak{R}_+$) with $c(t)$ denoting the consumption rate at time t. A trading strategy is said to *finance* the consumption plan c if the following intertemporal budget constraint holds:

$$(5)\quad \int_0^t \frac{c(s)}{B(s)}\,ds + W(t) = W(0) + \int_0^t \theta(t)^{\mathsf{T}}\,dG(t), \qquad P\text{-a.s.}, \forall\, t \in \mathfrak{R}_+,$$

provided that the stochastic integral on the right-hand side is well defined. This equation says that, in units of the bond, the value of the portfolio at time t is equal to its initial value plus accumulated capital gains or losses and minus accumulated withdrawals for consumption. For the stochastic integral of (5) to be well defined, we introduce the class $\mathscr{L}(G)$ of trading strategies (α, θ), the θ of which satisfies

$$\int_0^{T_n} \left| \frac{\theta(s)^{\mathsf{T}}\sigma(s)}{B(s)} \right|^2 ds < \infty, \qquad P\text{-a.s.},$$

for a sequence of optional times $T_n \uparrow \infty$, P-a.s.

For the agent's problem to be well-posed, however, it is necessary that something cannot be created from nothing through trading using reasonable trading strategies. This is the subject to which we now turn.

Up to now nothing was said about the existence of arbitrage opportunities. In fact with the strategies we have defined, it is well known that such arbitrage opportunities do exist, even in finite time; see the doubling strategy of Harrison and Kreps (1979). Were this the case, the consumption and portfolio choice problem would not be well posed. To rule out arbitrage opportunities, we will impose a regularity condition on the parameters of the price system and a natural institutional constraint that wealth cannot be strictly negative. This constraint has been analyzed by Dybvig and Huang (1989) and Harrison and Pliska (1981).

We make the following assumption throughout our analysis.

ASSUMPTION 1. Let $\kappa(t) = -\sigma(t)^{-1}(b(t) - r(t)S(t))$. There exists a positive constant $\overline{K} < \infty$ such that $|\kappa(\omega, t)| \le \overline{K}$ for all t, P-a.s.

This assumption is in particular satisfied in the models originally considered by Samuelson (1965) and Merton (1971). Now define a martingale under P that is almost surely strictly positive:

$$\xi(t) = \exp\left\{ \int_0^t \kappa(s)^{\mathsf{T}}\,dw(s) - \tfrac{1}{2}\int_0^t |\kappa(s)|^2\,ds \right\}, \qquad t \in \mathfrak{R}_+.$$

By Assumption 1, it is easily verified that $E[\xi(t)] = 1$. Thus

$$Q_t(A) = \int_A \xi(\omega, t) P(d\omega), \qquad \forall\, A \in \mathscr{F}_t,$$

is a probability measure equivalent to P. The family of probability measures $\{Q_t;\ t \in [0, \infty)\}$ is consistent in that Q_s equals Q_t on $\mathscr{F}_t$, where $\infty > s \geq t \geq 0$. We have the following result.

PROPOSITION 1. *There exists a probability Q on $(\Omega, \mathscr{F})$ such that its restriction on $\mathscr{F}_t$ is equivalent to the restriction of P on $\mathscr{F}_t$ for all $t \in \Re_+$. Moreover, under Q,*

$$G(t) = S(0) + \int_0^t \frac{\sigma(s)}{B(s)}\, dw^*(s), \qquad t \in [0, \infty),$$

and thus is a local martingale (see page 39 for a definition) under Q, where $w^*(t) \equiv w(t) - \int_0^t \kappa(s)\, ds$ is an N-dimensional standard Brownian motion under Q.

PROOF. Define a consistent family Q_t on $(\Omega, \mathscr{F}_t)$ as above. Stroock [(1987), Lemma 4.2] shows that there exists a unique Q on $(\Omega, \mathscr{F})$ so that $Q|\mathscr{F}_t = Q_t$ for all $t \in \Re_+$. Since Q is equivalent to P on $\mathscr{F}_t$, we have the first assertion. The second assertion follows from the definition of κ and Girsanov's theorem [see, e.g., Stroock (1987), Lemma 4.3]. □

Since P and Q are equivalent on $(\Omega, \mathscr{F}_t)$ for any finite t, they are said to be *locally* equivalent. On the other hand, P and Q may be mutually singular on the σ-field $\mathscr{F}$, as shown in the following lemma.

LEMMA 1. *The measures Q and P are mutually singular if and only if $\int_0^\infty |\kappa(t)|^2\, dt = \infty$, Q-a.s.*

PROOF. By the Radon–Nikodym theorem, there exists an extended-valued random variable ξ_∞ such that $P(\xi_\infty = \infty) = 0$ and

$$Q(A) = E[\xi_\infty I_A] + Q(A \cap \{\xi = \infty\}), \qquad \forall A \in \mathscr{F};$$

see, for example, Jacod [(1979), Theorem 7.1]. For P and Q to be mutually singular, it is necessary and sufficient that $Q(\xi_\infty < \infty) = 0$. But Theorem 8.19 of Jacod (1979) shows that in fact

$$\{\xi_\infty < \infty\} = \left\{\int_0^\infty |\kappa(t)|^2\, dt < \infty\right\}, \qquad Q\text{-a.s.},$$

and so P and Q are mutually singular if and only if $\int_0^\infty |\kappa(t)|^2\, dt = \infty$, Q-a.s., as desired. □

In the models of Samuelson or Merton, the process κ is constant, so by the lemma above P and Q are mutually singular. The *almost surely* statements on $\mathscr{F}$ can no longer be applied indifferently with respect to either probability, as they can be in the finite horizon case. However, one can still use almost surely with respect to both P and Q in restriction to $\mathscr{F}_t$ for all $t \in \Re_+$.

LEMMA 2. *Suppose $(\alpha, \theta) \in \mathscr{L}(G)$. Then the integral $\int_0^t \theta(s)^{\mathsf{T}}\, dG(s)$ is an Itô process under both P and Q and is the same process whether it is computed relative to $(\mathbf{F}, P)$ or to $(\mathbf{F}, Q)$.*

PROOF. This assertion follows from Stroock [(1987), III.4.3]. □

We are ready to show that there are no arbitrage oppportunities when the positive wealth constraint is in force. Our proof is a direct generalization of Dybvig and Huang [(1989), Theorem 2]. We first recall the following usual definition of an arbitrage opportunity.

DEFINITION 1. An arbitrage opportunity is a strategy $(\alpha, \theta) \in \mathscr{L}(G)$ with $W(0) = 0$ that finances a consumption plan c that is positive and nonzero.

We need a technical lemma to proceed.

LEMMA 3. *Let c be a consumption plan. Then*

$$E^*\left[\int_0^\infty \frac{c(t)}{B(t)}\, dt\right] = E\left[\int_0^\infty \frac{c(t)\xi(t)}{B(t)}\, dt\right],$$

where E^* denotes expectation under Q.

PROOF. For any finite t,

$$E^*\left[\int_0^t \frac{c(s)}{B(s)}\, ds\right] = E\left[\int_0^t \frac{c(s)\xi(s)}{B(s)}\, ds\right];$$

see Dellacherie and Meyer [(1982), VI.57]. Given that P and Q are locally equivalent, c is positive on any finite subinterval of $[0, \infty)$ under either P or Q. Thus the integrands on both sides of the above relation are positive and increase in t. Thus letting $t \to \infty$, by monotone convergence theorem, we have the assertion. □

PROPOSITION 2. *Let c be a consumption plan (which by definition is a positive process) financed by $(\alpha, \theta) \in \mathscr{L}(G)$ with $W(0) = 0$ and with $W(t) \geq 0$, P-a.s. for all $t \in \Re_+$. Then P-almost surely, c is identically zero.*

PROOF. From (5) we have

$$\int_0^t \frac{c(s)}{B(s)}\, ds + W(t) = W(0) + \int_0^t \theta(t)^{\mathsf{T}}\, dG(t), \qquad \forall\, t \in \Re_+.$$

By Proposition 1 and the fact that $(\alpha, \theta) \in \mathscr{L}(G)$, the right-hand side of the above relation is a positive local martingale under Q since by the positive wealth constraint, the left-hand side is positive. It is known that a positive local martingale is a supermartingale; see, for example, Lemma 1 of Dybvig and Huang (1989).

By the fact that c is a positive process under P and P and Q are locally equivalent, we know c is a positive process under Q on any finite subinterval $[0, t]$. This implies that

$$\begin{aligned} E^*\left[\int_0^\infty \frac{c(t)}{B(t)}\, dt\right] &= \lim_{t\to\infty} E^*\left[\int_0^t \frac{c(s)}{B(s)}\, ds\right] \\ &\leq \lim_{t\to\infty} E^*\left[\int_0^t \frac{c(s)}{B(s)}\, ds + W(t)\right] \\ &\leq W(0) = 0, \end{aligned}$$

where $E^*[\cdot]$ is the expectation under Q, the first equality follows from the monotone convergence theorem, the first inequality follows from the fact that $W(t) \geq 0$, P-a.s. and thus Q-a.s. since P and Q are locally equivalent, and the second inequality follows from the above mentioned supermartingale property. Lemma 3 then implies that

$$E\left[\int_0^\infty \frac{c(t)\xi(t)}{B(t)}\, dt\right] \leq 0.$$

Since ξ is a strictly positive process under P, it must be the case that c is identically zero P-almost surely. □

Thus trading strategies from $\mathscr{L}(G)$ that satisfy the positive wealth constraint cannot be arbitrage opportunities.

In models of a finite horizon T such as Cox and Huang (1989, 1990), a consumption plan c is admissible if

$$E\left[\int_0^T |c(t)|^p\, dt\right] < \infty$$

for some given $p \in [1, \infty)$. A direct generalization of this space to our setup by taking $T = \infty$ is unsatisfactory since it does not include consumption plans that do not go to zero as time approaches infinity, which we do not want to rule out a priori. We will let the space of admissible consumption plans depend upon the impatience of the agent.

We assume that the utility function of the agent satisfies the following additional condition.

Assumption 2. For all $a > 0$, $t < s$, then $u(a, t) > u(a, s)$,

$$\int_0^\infty |u(a, t)|\, dt < \infty, \tag{6}$$

and there exists $K_a > 0$ such that

$$u_{a+}(a, t) \leq K_a |u(a, t)|, \quad \text{a.e. } t \in \Re_+, \tag{7}$$

where $u_{a+}(a, t)$ denotes the right-hand partial derivative of $u(a, t)$ with respect to its first argument. (Right-hand derivatives of a concave function always exist.)

Note that the first hypothesis of Assumption 2 implies impatience, while relation (6) requires that impatience be sufficiently strong. Relation (7) will later be used to show that the space of admissible consumption plans to be specified is invariant with respect to the choices of a among those a's so that $u(a,t) > 0$ for all t.

REMARK 2. If $u(c,t) = v(c)e^{-\delta_t t}$ with $\delta_t > 0$ for all $t > 0$, then it satisfies Assumption 2.

Pick some $a > 0$ and define a finite measure by

$$\lambda_a(A) = \int_A |u(a,t)|\,dt, \qquad \forall A \in \mathscr{B}(\Re_+),$$

and denote the product measure generated by P and λ_a by ν_a. Note that λ_a is equivalent to Lebesgue measure since by impatience $|u(a,t)| > 0$ except possibly for one t. Fix $p \in [1,\infty)$. We will say that a consumption plan c is admissible if

$$E\left[\int_0^\infty |c(t)|^p |u(a,t)|\,dt\right] < \infty.$$

The space of admissible consumption plans is the positive orthant of the space $L^p(\nu_a) \equiv L^p(\Omega \times \Re_+, \mathscr{PM}, \nu_a)$, where $\mathscr{PM}$ denotes the progressive sigma field. (Note that a process is progressively measurable if and only if it is measurable with respect to the progressive sigma field.) We will denote the positive orthant of $L^p(\nu_a)$ by $L^p_+(\nu_a)$.

The following lemma shows that the space of admissible consumption plans is invariant with respect to choices of $a > 0$ with $u(a,t) > 0$ for all t.

LEMMA 4. *For any strictly positive scalars, a and a' such that $u(a,t) > 0$ and $u(a',t) > 0$ for all t, $L^p_+(\nu_a) = L^p_+(\nu_{a'})$.*

PROOF. Let $c \in L^p(\nu_a)$. We can write

$$E\left[\int_0^\infty |c(t)|^p |u(a',t)|\,dt\right] \le E\left[\int_0^\infty |c(t)|^p |u(a,t)|\,dt\right] + E\left[\int_0^\infty |c(t)|^p |u(a',t) - u(a,t)|\,dt\right].$$

The first term on the right-hand side of the inequality is finite by assumption. If we can show that the second term is finite, then $c \in L^p_+(\nu_{a'})$. Assume first that $a' > a$. We have

$$\begin{aligned} E\left[\int_0^\infty |c(t)|^p |u(a',t) - u(a,t)|\,dt\right] &\le E\left[\int_0^\infty |c(t)|^p u_{a+}(a,t)(a'-a)\,dt\right] \\ &\le (a'-a)E\left[\int_0^\infty |c(t)|^p K_a |u(a,t)|\,dt\right] < \infty, \end{aligned}$$

where the first inequality follows from the concavity of u and the second inequality follows from (7).

Next suppose that $a' < a$. Arguments identical to those above prove the assertion by noting that

$$|u(a',t) - u(a,t)| \leq u_+(a',t)(a-a') \leq K_a u(a',t)(a-a') \leq K_a u(a,t)(a-a').$$

Similar arguments prove that $c \in L^p(\nu_{a'})$ implies $c \in L^p(\nu_a)$. □

Now we are ready to specify completely the agent's problem. He wants to solve the following program:

$$(8) \qquad \sup_{\substack{(\alpha,\theta)\in\mathscr{L}(G)\\ W(t)\geq 0}} E\left[\int_0^\infty u(c(t),t)\,dt\right] \begin{array}{l}\text{s.t. } W(0) \equiv \alpha(0)B(0) + \theta(0)^{\mathsf{T}}S(0) \leq W_0,\\ c \text{ is financed by } (\alpha,\theta) \text{ and } c \in L^p_+(\nu_a).\end{array}$$

We will say that there exists a solution to the program (8) if the value of the program, val(W_0), is finite and is attained by a consumption plan financed by an admissible trading strategy that satisfies the positive wealth constraint.

3. Existence of an optimal policy. We provide in this section sufficient conditions for the existence of a solution to (8). Our technique follows Cox and Huang (1990). We first transform the dynamic maximization problem into a static variational problem whose solution is well understood. The solution of the static problem is then implemented with a dynamic trading strategy uncovered by a martingale representation theorem.

Consider the following static variational problem:

$$(9) \qquad \sup_{c\in L^p_+(\nu_a)} E\left[\int_0^\infty u(c(t),t)\,dt\right] \quad \text{s.t.} \quad E\left[\int_0^\infty \frac{c(t)\xi(t)}{B(t)}\,dt\right] \leq W_0.$$

We first show that the dynamic program (8) is equivalent to the static variational program of (9).

PROPOSITION 3. *c is a feasible consumption plan in* (8) *if and only if it is one in* (9).

PROOF. Let $c \in L^p_+(\nu_a)$ be financed by $(\alpha,\theta) \in \mathscr{L}(G)$ satisfying the positive wealth constraint and with $W(0) \leq W_0$. From the proof of Proposition 2 we know that

$$\int_0^t \frac{c(s)}{B(s)}\,ds + W(t) = W(0) + \int_0^t \theta(t)^{\mathsf{T}}\,dG(t), \qquad \forall\, t > 0,\ P\text{-a.s.},$$

is a supermartingale under Q. Thus

$$E^*\left[\int_0^t \frac{c(s)}{B(s)}\,ds + W(t)\right] \leq W(0), \qquad \forall\, t \in \Re_+.$$

Because $W(t) \geq 0$ and $c(t) \geq 0$ under P, by the local equivalence of P and Q, we know

$$E^*\left[\int_0^t \frac{c(s)}{B(s)}\, ds\right] \leq W(0), \qquad \forall\, t \in \Re_+,$$

and

$$E^*\left[\int_0^\infty \frac{c(t)}{B(t)}\, dt\right] \leq W(0),$$

where the monotone convergence theorem is used for the second inequality. It then follows from Lemma 3 that

$$E\left[\int_0^\infty \frac{c(t)\xi(t)}{B(t)}\, dt\right] \leq W(0) \leq W_0.$$

Thus c is feasible in (9).

Conversely, let $c \in L_+^p(\nu_a)$ be feasible in (9). Lemma 3 implies that

$$E^*\left[\int_0^\infty \frac{c(t)}{B(t)}\, dt\right] \leq W_0.$$

Thus

$$\int_0^\infty \frac{c(t)}{B(t)}\, dt \in L^1(\Omega, \mathscr{F}, Q).$$

Since all P-local martingales have the representation property relative to w (see Jacod and Shiryayev, Theorem III.4.33) and since Q and P are locally equivalent, all Q-local martingales also have the representation property relative to w^* (Jacod and Shiryayev, Theorem III.5.24). Hence there exists an N-vector process ρ and a sequence of optional times (T_n) with $\lim T_n \uparrow \infty$ Q-a.s. so that for all T_n,

$$\int_0^{T_n} |\rho(s)|^2\, ds < \infty, \qquad Q\text{-a.s.},$$

and

$$E^*\left[\int_0^\infty \frac{c(s)}{B(s)}\, ds \,\middle|\, \mathscr{F}_t\right] = E^*\left[\int_0^\infty \frac{c(s)}{B(s)}\, ds\right] + \int_0^t \rho(s)^\top\, dw^*(s), \quad t \in \Re_+,\ Q\text{-a.s.}$$

Let $\theta(t)^\top \equiv B(t)\rho(t)^\top\sigma(t)^{-1}$,

$$W(t) = E^*\left[\int_t^\infty \frac{c(s)}{B(s)}\, ds \,\middle|\, \mathscr{F}_t\right] \geq 0, \qquad \forall\, t,\ P\text{-a.s.}, \tag{10}$$

and

$$\alpha(t) \equiv W(t) - \theta(t)^\top\left(S^*(t) + \Delta D(t)/B(t)\right).$$

Since $c \geq 0$ under P, $W(t) \geq 0$, Q and P-a.s. Since Q and P are locally equivalent, $T_n \uparrow \infty$, P-a.s. Hence $(\alpha, \theta) \in \mathscr{L}(G)$. Also by construction and the local equivalence of P and Q, we have

$$W(t) + \int_0^t \frac{c(s)}{B(s)}\,ds = \alpha(0) + \theta(0)^{\mathsf{T}} S(0) + \int_0^t \theta(s)^{\mathsf{T}}\,dG(s), \qquad P\text{-a.s., } t \in \Re_+.$$

That is, (α, θ) finances c. Finally, it is easily seen that $W(0) \le W_0$. Thus c is feasible in (8). □

We record an immediate corollary of Proposition 3.

COROLLARY 1. *c is a solution to* (8) *if and only if it is a solution to* (9).

Given this corollary we can then concentrate on (9). Note that since (8) and (9) are equivalent, we will also use $\mathrm{val}(W_0)$ to denote the value of (9).

We first provide conditions under which $\mathrm{val}(W_0)$ is finite.

PROPOSITION 4. *Suppose that*:

(i) *For almost all t, $u(c,t)$ is unbounded from above in c and there exists $\beta_1 \ge 0$, $\beta_2 > 0$ and $b \in (0,1)$ such that*

$$(11) \qquad u(c,t) \le |u(a,t)| \left(\beta_1 + \frac{\beta_2}{1-b} (c^{1-b} - 1) \right), \quad a.e.\ t;$$

and

$$(12) \qquad \left(\frac{\xi}{B} \right)^{-1} |u(a,t)| \in L^{p/b}(\nu_a).$$

(ii) *Also suppose that, if $u(c,t)$ is unbounded from below at the origin on a set A of t with strictly positive Lebesgue measure,*

$$(13) \qquad E\left[\int_A \frac{\xi(t)}{B(t)}\,dt \right] < \infty.$$

Then $\mathrm{val}(W_0)$ *is finite.*

PROOF. We first show that $\mathrm{val}(W_0) < \infty$. When the utility function is $|u(a,t)|(\beta_1 + (\beta_2/(1-b))(c^{1-b} - 1))$ with $b \in (0,1)$, there exists a solution c to (9) only if there exists $\gamma > 0$ so that

$$c(t) = \left(\frac{\gamma}{\beta_2} \right)^{-1/b} \left(\frac{\xi(t)}{B(t)} \right)^{-1/b} |u(a,t)|^{1/b}.$$

Relation (12) ensures that $c \in L^p_+(\nu_a)$. For c to actually be a solution, it must also satisfy the budget constraint and yield a finite expected utility. For the former it is necessary and sufficient that

$$E\left[\int_0^\infty \frac{c(t)\xi(t)}{B(t)}\,dt \right] < \infty,$$

for then γ can be chosen to exhaust the initial wealth. Putting $\hat{p} = p/(1-b) > p$ and $1/\hat{p} + 1/\hat{q} = 1$, Hölder's inequality implies

$$E\left[\int_0^\infty \frac{c(t)\xi(t)}{B(t)}\,dt\right]$$
$$= \left(\frac{\gamma}{\beta_2}\right)^{-1/b} E\left[\int_0^\infty \left(\frac{\xi(t)}{B(t)}\right)^{1-(1/b)} |u(a,t)|^{-1+(1/b)}\,d\lambda_a(t)\right]$$
$$\leq \left(\frac{\gamma}{\beta_2}\right)^{-1/b} \left(E\left[\int_0^\infty \left(\frac{\xi(t)}{B(t)}\right)^{-p/b} |u(a,t)|^{p/b}\,d\lambda_a(t)\right]\right)^{1/\hat{p}}$$
$$\times \left(\int_0^\infty |u(a,t)|\,dt\right)^{1/\hat{q}} < \infty,$$

where the last inequality follows (12) and Assumption 2. Finally, we have to verify that the expected utility is finite. For this we note that

$$E\left[\int_0^\infty c(t)^{1-b}|u(a,t)|\,dt\right] = \frac{\gamma}{\beta_2} E\left[\int_0^\infty \frac{c(t)\xi(t)}{B(t)}\,dt\right]$$
$$\leq \frac{\gamma}{\beta_2} W_0 < \infty.$$

Thus $\mathrm{val}(W_0) < \infty$ by (11).

Next we take two cases. Suppose first that $u(c,t)$ is bounded from below for a.e. t. Then $\mathrm{val}(W_0) > -\infty$ and thus $\mathrm{val}(W_0)$ is finite. Otherwise, suppose that u is unbounded from below at the origin on a set A of t. Let

$$k \equiv E\left[\int_A \frac{\xi(t)}{B(t)}\,dt\right].$$

By (13), k is finite. Thus

$$c(\omega,t) = \frac{W_0}{k} 1_{\Omega\times A}(\omega,t) > 0, \qquad \forall\, \omega, t$$

is a feasible consumption plan. Thus

$$\mathrm{val}(W_0) \geq \int_A u(W_0/k, t)\,dt > -\infty,$$

where the inequality follows from Assumption 2. □

Note that, in (11), if $b > 1$, then $u(c,t) \leq |u(a,t)|\beta_1$. Thus the expected utility is always strictly less than $+\infty$.

Now our remaining task is to give conditions under which $\mathrm{val}(W_0)$ is attained. We will utilize Cox and Huang (1990) by rewriting (9) as follows:

$$\begin{aligned}
&\sup_{c \in L^p_+(\nu_a)} E\left[\int_0^\infty \frac{u(c(t),t)}{|u(a,t)|}\, d\lambda_a(t)\right] \\
(14)\qquad &\text{s.t.}\quad E\left[\int_0^\infty \frac{c(t)\xi(t)}{B(t)|u(a,t)|}\, d\lambda_a(t)\right] \le W_0.
\end{aligned}$$

In (14), both the objective functional of the maximization program and the space of feasible objects are defined on a measurable space $(\Omega \times \Re_+, \mathscr{PM})$ with a finite measure ν_a. This fits into the framework of Cox and Huang (1990).

Here is our first existence theorem:

THEOREM 1. *Under the conditions of Proposition* 4, *there exists a solution to* (14).

PROOF. The assertion follows from Cox and Huang [(1990), Theorem 4.1] by noting that Q and P are locally equivalent. □

The next theorem is for the case where the utility function is bounded from above by a multiple of $|u(a,t)|$.

THEOREM 2. *Suppose that*:

(i) $u(c,t) \le K|u(a,t)|$ *for a.e.* t;

(ii)

$$(\xi/B)^{-1}|u(a,t)| \in L^p(\nu_a);\ \textit{and} \tag{15}$$

(iii) (13) *holds when* $u(c,t)$ *is unbounded from below at the origin on a set* A *of* t *with strictly positive Lebesgue measure.*

Then there exists a solution to (14).

PROOF. By the first hypothesis, we know that

$$\frac{u(c,t)}{|u(a,t)|} \le K, \qquad \forall\, c, \text{a.e. } t.$$

Given (13) and (15), the assertion follows from a (trivial) generalization of Cox and Huang [(1990), Theorem 4.2] by noting that P and Q are locally equivalent. □

Before leaving this section, we give in the following two corollaries sets of explicit conditions on the parameters of prices for (12), (13) and (15) to be valid.

COROLLARY 2. *Suppose that there exists* $0 < \bar{r} < \infty$ *so that* $0 < r(t) \le \bar{r}$, *P-a.s. for a.e. t and there exists* $\infty > T > 0$, $\varepsilon > 0$, *such that*

$$-\frac{\ln|u(a,t)|}{t} > \frac{p}{2b}\bar{K}^2 + \frac{p}{b+p}\bar{r} + \varepsilon, \qquad \forall\, t \ge T. \tag{16}$$

Then (12) *is valid. When* (16) *holds with* $b = 1$, (15) *is valid.*

PROOF. Note that

$$E\left[\left(\frac{\xi(t)}{B(t)}\right)^{-p/b} |u(a,t)|^{(p/b)+1}\right]$$

$$= E\left[\overbrace{\exp\left\{-\frac{p}{b}\int_0^t \kappa(s)^{\mathsf{T}}\,dw(s) - \frac{p^2}{2b^2}\int_0^t |\kappa(s)|^2\,ds\right\}}^{\text{a } P \text{ martingale}}\right.$$

$$\left.\times \exp\left\{\frac{p}{2b}\left(\frac{p}{b}+1\right)\bar{K}^2 t + \frac{p}{b}\int_0^t r(s)\,ds + \left(1+\frac{p}{b}\right)\ln|u(a,t)|\right\}\right]$$

$$\le \exp\left\{\left(\frac{p}{2b}\left(\frac{p}{b}+1\right)\bar{K}^2 + \frac{p}{b}\bar{r} + \left(1+\frac{p}{b}\right)\frac{\ln|u(a,t)|}{t}\right)t\right\}.$$

Given that

$$-\frac{\ln|u(a,t)|}{t} > \frac{p}{2b}\bar{K}^2 + \frac{p}{b+p}\bar{r} + \varepsilon, \qquad \forall\, t \ge T,$$

Fubini's theorem then implies that (12) is valid.

Identical arguments proves the second assertion when $b = 1$. □

COROLLARY 3. *Suppose that the set A of* (13) *is of finite Lebesgue measure. Then if* $r(t) \ge 0$, (13) *is always valid. Otherwise, if there exists* $0 < \underline{r}$ *so that* $\underline{r} \le r(t)$ *P-a.s. for a.e. t, then* (13) *is valid.*

PROOF. Suppose first that A is of finite Lebesgue measure. We note that ξ is a martingale under P with unity expectation. Since $B(t) \ge 1$,

$$E\left[\left(\frac{\xi(t)}{B(t)}\right)\right] \le 1.$$

Fubini's theorem implies that (13) is valid.

Next suppose that A is of infinite measure. The hypothesis of this corollary yields

$$E\left[\left(\frac{\xi(t)}{B(t)}\right)\right] \le \exp\{-\underline{r}t\}.$$

Fubini's theorem again implies that (13) is valid. □

When the utility function is not unbounded from below, for existence, it suffices that the interest rate does not become unbounded and the utility function significantly discounts the future asymptotically. Otherwise, the agent may find it advantageous to keep accumulating his wealth and postpone consumption until $t = \infty$. When the utility function is unbounded from below at the origin, we will further require that the interest rate does not become too low so that it may become infeasible to maintain a certain level of minimum consumption over time and thus the expected utility may become $-\infty$.

4. Computation of optimal policies. This section is devoted to the computation of optimal policies. The idea follows that of Cox and Huang (1989) and thus we will be brief. The main result reported here is a verification theorem of optimal policies. We will show that if there exists a solution to a second order linear parabolic partial differential equation and if the solution satisfies certain conditions, then the optimal trading strategy in the form of feedback controls can be computed by taking derivatives of the solution.

For the purpose of this section, we will assume that $u(c,t)$ is strictly concave in c. Then define the inverse of the marginal utility function $f(x^{-1},t) = \inf\{c \in \Re_+ \colon u_{c+}(c,t) \le x^{-1}\}$, where u_{c+} denotes the right-hand partial derivative of u with respect to c. By the strict concavity of $u(c,t)$ in c, it is easily seen that $f(x^{-1},t)$ is continuous in x.

We assume throughout this section that f satisfies a growth condition:

$f(x^{-1},t) \le K_f |u(a,t)|^{1/b} x^{1/b}$ for some strictly positive constants K_f and b,

and (12) holds if $b \in (0,1)$ and (15) holds if $b \ge 1$. Moreover, if u is unbounded from below at the origin, (13) holds. Under these conditions, it is easily verified that the conditions of either Theorem 1 or Theorem 2 hold and there exists a solution to (14).

The object of computation here is the value over time of future optimal consumption. From (10) we know this value in units of the bond is

$$W(t) = E^*\left[\int_t^\infty \frac{c(s)}{B(s)}\, ds \,\middle|\, \mathscr{F}_t\right],$$

if c is the optimal consumption. Thus $F(t) \equiv W(t)B(t)$ is the present value of future optimal consumption. Under some conditions, F can be computed by solving a partial differential equation and optimal trading strategies are related to the derivatives of F.

Before proceeding, we record in the following proposition the first order condition for optimality, which is the cornerstone for the construction of an optimal policy.

PROPOSITION 5. *Under conditions of this section, there exists a solution to* (14) *if and only if there exists a* $\lambda > 0$ *so that*

$$c(t) = f\left(\frac{\lambda \xi(t)}{B(t)}, t\right) \in L_+^p(\nu_a) \tag{17}$$

and

$$E\left[\int_0^\infty \frac{c(t)\xi(t)}{B(t)}\, dt\right] = W_0.$$

PROOF. The only if part follows from the saddle-point theorem and Rockafellar (1975). The if part follows from the definition of f and concavity of u in c. □

For the purpose of computation, we specialize our model of securities prices as follows. Assume that S satisfies the stochastic integral equation

$$\begin{aligned} S(t) = S(0) &+ \int_0^t (b(S(s), s) - d(S(s), s))\, ds \\ &+ \int_0^t \sigma(S(s), s)\, dw(s), \qquad t \geq 0, \end{aligned} \tag{18}$$

where $d(S(t), t)$ is the N-vector dividend rate at time t when the risky asset prices are $S(t)$. Assume further that $r(t)$ can be written as $r(S(t), t)$. Thus $\kappa(t)$ can also be written as $\kappa(S(t), t)$.

Next define a process

$$\begin{aligned} Z(t) = Z(0) &+ \int_0^t (r(S(s), s) + |\kappa(S(s), s)|^2) Z(s)\, ds \\ &- \int_0^t \kappa(S(s), s)^{\mathrm{T}} Z(s)\, dw(s) \end{aligned} \tag{19}$$

for some constant $Z(0) > 0$. Using Itô's lemma, it is easily verified that

$$Z(t) = \frac{Z(0) B(t)}{\xi(t) B(0)}. \tag{20}$$

As pointed out by Cox and Huang (1989), $(\log Z(T) - \log Z(0))/T$ is the realized continuously compounded growth rate from time 0 to time T of the growth-optimal portfolio—the portfolio that maximizes the expected continuously compounded growth rate.

Now write (18) and (19) compactly together under Q:

$$\begin{aligned} \begin{pmatrix} S(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} S(0) \\ Z(0) \end{pmatrix} &+ \int_0^t \hat{\zeta}(S(s), Z(s), s)\, ds \\ &+ \int_0^t \hat{\sigma}(S(s), Z(s), s)\, dw^*(s), \end{aligned} \tag{21}$$

where we note that

$$\hat{\zeta}(S(t), Z(t), t) = \begin{pmatrix} r(S(t), t)S(t) - d(S(t), t) \\ r(S(t), t)Z(t) \end{pmatrix},$$

$$\hat{\sigma}(S(t), Z(t), t) = \begin{pmatrix} \sigma(S(t), Z(t), t) \\ -Z(t)\kappa^{\mathrm{T}}(S(t), t) \end{pmatrix}.$$

Assume throughout this section that $\hat{\zeta}$ and $\hat{\sigma}$ satisfy a local Lipschitz and a uniform linear growth condition on any finite time interval $[0, T]$.

[The functions $\hat{\zeta}$ and $\hat{\sigma}$ satisfy a local Lipschitz condition on every $[0, T]$ with $T < \infty$ if for every $T > 0$ and $M > 0$, there is a strictly positive constant K_T^M such that for all $y, z \in \Re^N \times (0, \infty)$ with $|y| \le M$ and $|z| \le M$ and all $t \in [0, T]$,

$$|\hat{\zeta}(y, t) - \hat{\zeta}(z, t)| \le K_T^M|y - z|, \qquad |\hat{\sigma}(y, t) - \hat{\sigma}(z, t)| \le K_T^M|y - z|.$$

These functions satisfy a uniform growth condition on every $[0, T]$ with $T < \infty$ if for every $T > 0$, there exists a strictly positive constant K_T such that, for all $x \in \Re^N \times (0, \infty)$ and $t \in [0, T]$,

$$|\hat{\zeta}(x, t)| \le K_T(1 + |x|), \qquad |\hat{\sigma}(x, t)| \le K_T(1 + |x|).]$$

Thus there exists a unique solution to (21) and (S, Z) has the strong Markov property under Q and thus is a diffusion process under Q.

The following notation will be utilized:

$$D_y^m = \frac{\partial^m}{\partial y^m} = \frac{\partial^{m_1 + m_2 + \cdots m_N}}{\partial y_1^{m_1} \cdots \partial yN^{m_N}}; \qquad m = m_1 + \cdots + m_N,$$

for positive integers $m_1, m_2, \ldots, m_N$. If g: $\Re^N \times [0, T] \mapsto \Re$ has partial derivatives with respect to its first N arguments, the vector $(\partial g/\partial y_1, \ldots, \partial g/\partial y_N)^{\mathrm{T}}$ is denoted by $D_y g$ or g_y.

Here is the main theorem of this section:

THEOREM 3. *Suppose that*:

(i) *There exists a function* F: $(0, \infty) \times \Re^N \times [0, \infty) \to \Re_+$ *such that* $DF_y^m(y, t)$ *and* $F_t(y, t)$ *are continuous for* $m \le 2$, F *is a solution to the partial differential equation*

$$\mathscr{L}F(Z, S, t) - r(S, t)F(Z, S, t) + F_t(Z, S, t) + f(Z^{-1}, t) = 0$$

with the boundary condition

$$\lim_{T \to \infty} E^*\left[\frac{F(Z(T), S(T), T)}{B(T)}\right] = 0,$$

where $\mathscr{L}$ *is the differential of* (Z, S) *under* Q $[\mathscr{L}F = \frac{1}{2}\mathrm{tr}(F_{SS}\sigma\sigma^{\mathrm{T}}) + \frac{1}{2}F_{ZZ}Z^2|\kappa|^2 + F_{SZ}\sigma\kappa + F_S(rS - f) + F_Z rZ]$; *and, for all* $T > 0$, F *satisfies a uniform growth condition on every* $[0, T]$ *with* $T < \infty$, *that is, there exist* *strictly positive constants* K_1^T *and* γ^T *so that, for all* $t \in [0, T]$,

$$|F(y,t)| \le K_1^T\left(1 + |y|^{\gamma^T}\right) \qquad \forall\, y \in (0,\infty) \times \Re^N.$$

(ii) *Also suppose that there exists* Z_0 *such that* $F(Z_0, S(0), 0) = W_0$.

Then the optimal consumption and portfolio policies are

$$
\begin{aligned}
&c(t) = f\left(Z(t)^{-1}, t\right), \qquad \nu_a\text{-a.e.},\\
&\theta(t) = \Big[F_S(Z(t), S(t), t) + \left[\sigma(S(t),t)\sigma(S(t),t)^{\mathsf T}\right]^{-1}\\
(22)\qquad &\qquad \times [b(S(t),t) - r(S(t),t)S(t)]Z(t)F_Z(Z(t), S(t), t)\Big], \quad \nu_a\text{-a.e.},\\
&\alpha(t) = \left[F(Z(t), S(t), t) - \theta(t)^{\mathsf T}S(t)\right]/B(t), \qquad \nu_a\text{-a.e.},
\end{aligned}
$$

where we have put $Z(0) = Z_0$.

PROOF. Using the growth condition on f and F and the fact that F satisfies the partial differential equation, Cox and Huang [(1989), Theorem 2.3] implies that, for every t,

$$\frac{F(Z(t), S(t), t)}{B(t)} = E^*\left[\int_t^T \frac{f\left(Z(s)^{-1}, s\right)}{B(s)}\, ds + \frac{F(\dot{Z}(T), S(T), T)}{B(T)}\,\middle|\, \mathcal{F}_t\right].$$

Let $T \to \infty$ and use the boundary condition to get

$$\frac{F(Z(t), S(t), t)}{B(t)} = E^*\left[\int_t^\infty \frac{f\left(Z(s)^{-1}, s\right)}{B(s)}\, ds \,\middle|\, \mathcal{F}_t\right],$$

where we have used the monotone convergence theorem. Thus F gives the value of $f(Z^{-1}, t)$ over time. By the hypothesis, $Z(0) = Z_0$, thus

$$W_0 = \frac{F(Z(0), S(0), 0)}{B(0)} = E^*\left[\int_0^\infty \frac{f\left(Z(t)^{-1}, t\right)}{B(t)}\, dt\right].$$

This shows that $c(t) = f(Z(t)^{-1}, t)$ exhausts the initial wealth. If we can show that $c \in L^p(\nu_a)$, then by Proposition 5, c is a solution to (14). By the growth condition on f, (12) and (15), one quickly verifies that $c \in L^p(\nu_a)$ and thus is a solution to (14). The fact that the trading strategy that finances c is as described in (22) follows from identical arguments on Cox and Huang [(1989), Theorem 2.2]. □

5. A special case. We now specialize the market model developed in the earlier sections and consider the model with constant coefficients examined by Merton (1971) and revisited recently by Karatzas, Lehoczky and Shreve (1987). In this case explicit formulas for the optimal consumption and portfolio policies can be computed just as in Cox and Huang (1990). The method does not use stochastic control and takes care in a natural way of the nonnegative constraint on consumption. We illustrate our results with two examples taken from the family of constant absolute risk aversion and HARA utility functions.

We shall use the following specialization. The risky securities follow a geometric Brownian motion,

$$S(t) = S(0) + \int_0^t \left(I_{S(s)}\mathbf{b} - d(S(s), s)\right) ds + \int_0^t I_{S(s)}\boldsymbol{\sigma}\, dw(s), \qquad t \geq 0,$$

where $\mathbf{b}$ is an n-vector of constants, $\boldsymbol{\sigma}$ an $n \times n$ nonsingular matrix of constants and $I_{S(t)}$ an $n \times n$ diagonal matrix having $S_i(t)$ in the (i, i)th position. We furthermore assume that r is constant and write $\mathbf{r}$ for an n-vector of r's and $\boldsymbol{\kappa}$ for the constant vector $-\boldsymbol{\sigma}^{-1}(\mathbf{b} - \mathbf{r})$.

Given some initial data z, define ϕ as

$$\phi(z) = E^*\left[\int_0^\infty e^{-rt} f\left(Z(t)^{-1}, t\right) dt\right]; \qquad Z^{-1}(0) = z.$$

Assuming $u(c, t) = v(c)e^{-\beta t}$ (cf. Remark 2), we can write alternatively

$$\phi(z) = E^*\left[\int_0^\infty e^{-rt} g\left(e^{\beta t} Z(t)^{-1}\right) dt\right]; \qquad Z^{-1}(0) = z,$$

where g is the inverse of the time-independent marginal utility function $v_{c+}(c)$.

Since it is easily verified that

$$e^{\beta t} Z(t)^{-1} = z \exp\left\{\boldsymbol{\kappa}^{\mathrm{T}} w^*(t) + \left(\beta - r + |\boldsymbol{\kappa}|^2/2\right)t\right\},$$

one sees that the function F of Section 4 can be identified as $F(z, t) = \phi(e^{\beta t} z^{-1})$. Under Q, $e^{\beta t} Z(t)^{-1}$ is lognormally distributed with mean $\log z + (\beta - r + \varrho^2/2)t$ and variance $\varrho^2 t$, where ϱ is the square root of $|\boldsymbol{\kappa}|^2$. It follows that

$$\phi(z) = \int_0^\infty \int_{-\infty}^{+\infty} e^{-rt} \frac{1}{\varrho\sqrt{t}} g(e^x) n\left(\frac{x - \log z - (\beta - r + \varrho^2/2)t}{\varrho\sqrt{t}}\right) dt\, dx,$$

where n stands for the standard normal density function. This yields

$$\begin{aligned}
\phi(z) &= \int_{-\infty}^{+\infty} \frac{g(e^x)}{\varrho} \int_0^\infty \frac{e^{-rt}}{\sqrt{2\pi t}} \exp\left(-\frac{\left(x - \log z - (\beta - r + \varrho^2/2)t\right)^2}{2\varrho^2 t}\right) dt\, dx \\
&= \int_{-\infty}^{+\infty} \frac{g(e^x)}{\varrho\sqrt{2\pi}} \exp\left[\frac{\tilde{\beta}(x - \log z)}{\varrho^2}\right] \\
&\quad \times \int_0^\infty t^{-1/2} \exp\left[-\left(\frac{x - \log z}{\varrho\sqrt{2}}\right)^2 \frac{1}{t} - \frac{\alpha^2 t}{2}\right] dt\, dx,
\end{aligned}$$

where we have put $\tilde{\beta} = \beta - r + \varrho^2/2$ and $\alpha^2 = 2r + \tilde{\beta}^2/\varrho^2$,

$$= \int_{\infty}^{+\infty} \frac{2g(e^x)}{\varrho\sqrt{2\pi}} \exp\left[\frac{\tilde{\beta}(x - \log z)}{\varrho^2}\right] \left|\frac{x - \log z}{\varrho\alpha}\right|^{1/2} K_{1/2}\left(\left|\frac{x - \log z}{\varrho\alpha^{-1}}\right|\right) dx,$$

where $K_{1/2}$ is associated with a Bessel function and takes here the particular form $K_{1/2}(x) = \sqrt{\pi/2x}\, e^{-x}$; see Gradshteyn and Ryzhik [(1979), pages 340 and 967]. We then evaluate the last integral to obtain the formula

$$\phi(z) = \frac{1}{\varrho\alpha}\left[\int_0^{\infty} g(ze^{-u})e^{-\mu u}\, du + \int_0^{\infty} g(ze^{u})e^{-\mu' u}\, du\right],$$

where

$$\mu = \frac{\tilde{\beta}}{\varrho^2} + \frac{\alpha}{\varrho},$$

$$\mu' = -\frac{\tilde{\beta}}{\varrho^2} + \frac{\alpha}{\varrho}$$

are both positive. Using the formula above, it is then easy to compute ϕ for a wide range of utility functions. Note that when utility has a finite marginal utility at zero, optimal consumption may involve zero consumption. Indeed, $c_t = 0$ if and only if $v_{c+}(0) \le e^{\beta t} Z(t)^{-1}$, that is, if and only if nominal wealth $e^{rt}W(t)$ is less than the deterministic time-independent boundary given by

$$\underline{W} = \phi(v_{c+}(0)) = \frac{1}{\varrho\alpha}\int_0^{\infty} \left[g(v_{c+}(0)e^{-u})e^{-\mu u} + g(v_{c+}(0)e^{u})e^{-\mu' u}\right] du.$$

We now give two examples.

EXAMPLE 1. Utility functions of constant absolute risk aversion. Let the time-independent utility function be

$$v(x) = -\frac{1}{\theta}e^{-\theta x},$$

where $\theta > 0$ is the coefficient of absolute risk aversion. In this case, we find

$$g(x) = \left[-\frac{1}{\theta}\log x\right]^{+}$$

and so

$$g(ze^{-u}) = \begin{cases} -(1/\theta)(\log z - u), & \text{if } u \ge \log z, \\ 0, & \text{otherwise,} \end{cases}$$

and similarly for $g(ze^u)$. Straightforward computations show that if $z \ge 1$,

$$\phi(z) = \frac{\Gamma(2, \mu \log z)}{\varrho\alpha\theta\mu^2} - \log z \frac{z^{-\mu}}{\varrho\alpha\theta\mu},$$

and that if $z \leq 1$,

$$\phi(z) = \frac{1}{\varrho\alpha\theta\mu^2} - \frac{\log z}{\varrho\alpha\theta\mu} - \frac{\log z}{\varrho\alpha\theta\mu'}(1 - z^{\mu'}) - \frac{\gamma(2, \mu' \log z^{-1})}{\varrho\alpha\theta\mu'^2},$$

where γ and Γ are the incomplete gamma functions

$$\gamma(\alpha, x) = \int_0^x e^{-t}t^{\alpha-1}\,dt,$$

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t}t^{\alpha-1}\,dt.$$

It is easily checked that F is twice continuously differentiable and that $F \to 0$ as $z \to \infty$ and $F \to \infty$ as $z \to 0$. The vector of optimal dollar amounts invested in the stocks is given by

$$A_t = I_{S(t)}\theta(t) = \frac{e^{-\beta\mu t}}{\varrho\alpha\theta\mu}Z(t)^{\mu}(\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}})(\mathbf{b} - \mathbf{r})$$

if $Z(t) \leq e^{\beta t}$ and by

$$A_t = \left[\frac{1}{\varrho\alpha\theta\mu} + \frac{1}{\varrho\alpha\theta\mu'}\left(1 - e^{\beta\mu' t}Z(t)^{-\mu'}\right)\right](\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}})(\mathbf{b} - \mathbf{r})$$

if $Z(t) \geq e^{\beta t}$.

EXAMPLE 2. HARA utility functions. Let v be in turn

$$v(x) = \frac{1-\gamma}{\gamma}\left(\frac{\rho x}{1-\gamma} + \eta\right)^{\gamma}$$

with $\rho > 0$, $\eta > 0$, $\gamma < 1$ and $\gamma \neq 0$. In the first condition of Proposition 4, it suffices to take $b = 1 - \gamma$ when $\gamma > 0$. In this case one finds

$$g(x) = \frac{1-\gamma}{\rho}\left[\left(\frac{x}{\rho}\right)^{-1/1-\gamma} - \eta\right]^{+}.$$

Computations whose length is the sole difficulty give

$$\phi(z) = \frac{1}{\varrho\alpha\mu(\mu - 1)/(1-\gamma)}\,\frac{1}{\rho}\left(\frac{z}{\rho}\right)^{-\mu}\eta^{1-\mu(1-\gamma)}$$

when $z \geq \rho\eta^{-(1-\gamma)} = v_{c+}(0)$ and

$$\phi(z) = \frac{1-\gamma}{\rho\tilde{\delta}}\left(\frac{z}{\rho}\right)^{-1/1-\gamma} - \frac{1-\gamma}{\rho r}\eta + \frac{1}{\varrho\alpha\mu'(\mu' + 1)/(1-\gamma)}\,\frac{1}{\rho}\left(\frac{z}{\rho}\right)^{\mu'}\eta^{1+\mu'(1-\gamma)}$$

when $z \leq v_{c+}(0)$, where

$$\tilde{\delta} = \frac{1}{1-\gamma}\left(\beta - \gamma\left(r + \frac{\varrho^2}{2(1-\gamma)}\right)\right).$$

The inequalities $\tilde{\delta} > 0$, $\mu > 1/(1-\gamma)$ and $\mu' > -1/(1-\gamma)$ all result from $\beta > \gamma r + (\gamma/1-\gamma)\varrho^2/2$ which itself follows from condition (16) of Corollary 2.

The case $\eta = 0$ is the one that is usually taken in the approach of dynamic programming, for this is the one for which the Bellman equation can be solved quite explicitly. This yields the simpler expression

$$\phi(z) = \frac{1-\gamma}{\rho\tilde{\delta}}\left(\frac{z}{\rho}\right)^{-1/1-\gamma},$$

from which one derives $c_t = \tilde{\delta} W_t e^{rt}$. It is clear that except in the degenerate case $\eta = 0$, the optimal consumption and portfolio policies are not linear functions of wealth, because of the nonnegativity constraint on consumption. When $z \geq v_{c+}(0)$, we have $z\phi'(z) = -\mu\phi(z)$. Hence when nominal wealth $e^{rt}W_t$ falls below the nonstochastic boundary $\underline{W}$, consumption is zero and the optimal dollar investment in the stocks is proportional to wealth with

$$A_t = \mu e^{rt} W_t (\sigma\sigma^{\mathrm{T}})^{-1}(\mathbf{b} - \mathbf{r}).$$

When $z \leq v_{c+}(0)$, that is, when nominal wealth is above $\underline{W}$, optimal policy is no longer linear. As z approaches zero, however, which corresponds to large values of wealth, the optimal consumption and investment policies are almost the linear functions of wealth given in Merton (1971).

6. Convergence of finite horizon to infinite horizon solutions. We study two convergence problems in this section. First we show that the optimal *consumption* policy in an infinite horizon economy, if one indeed exists, is a pointwise limit of those in corresponding finite horizon economies. Second, under some regularity conditions, the optimal portfolio policy in an infinite horizon economy is the limit, with respect to a norm, of the optimal policies of the corresponding finite horizon economies. Thus, there exists a subsequence of the latter that converges pointwise to the former. It follows that if the latter converges pointwise at all, the pointwise limit must be equal to the former almost everywhere.

The convergence results reported here are useful in two respects. First, for the class of models where closed-form solutions exist for finite horizon economies, we know conditions under which the optimal policy in infinite horizon can be gotten by letting $T \to \infty$ in the finite horizon policies. Second, in the numerical computation of the optimal policy for the infinite horizon problem, the horizon will have to be truncated. It is therefore imperative to know conditions under which the solutions to finite horizon problems are approximations to that to an infinite horizon problem.

Assume until further notice that the conditions of Theorem 1 or Theorem 2 hold and thus there exists a solution to (14), denoted by c^*. Also assume that the utility function is strictly concave in consumption and thus the optimal consumption plan is unique. Consider a class of finite horizon problems:

$$\begin{aligned} &\sup_{c\in L^p_+(\nu_a;T)} E\left[\int_0^T \frac{u(c(t),t)}{|u(a,t)|}\, d\lambda_a(t)\right] \\ (23)\qquad &\text{s.t.}\quad E\left[\int_0^T \frac{c(t)\xi(t)}{B(t)|u(a,t)|}\, d\lambda_a(t)\right] \le W_0, \end{aligned}$$

where we have used $L^p_+(\nu_a;T)$ to denote the positive orthant of the space of processes c such that

$$E\left[\int_0^T |c(t)|^p\, d\lambda_a(t)\right] \le \infty.$$

It is clear that under the conditions of Theorem 1 or Theorem 2, there exists a unique optimal consumption policy of (23), denoted by c^T. The following theorem shows that $c^T \to c^*$, ν_a-a.e., as $T \to \infty$. Thus, if we know the functional expression of c^T, which naturally depends on T, we will get the optimal consumption policy for the infinite horizon case by simply letting $T \to \infty$.

We first record a lemma.

LEMMA 5. *Let λ_T be such that $c^T(t) = f((\lambda_T\xi(t)/B(t)), t)$ ν_a-a.e. Then $\lambda_{T'} \ge \lambda_T$ if $T' > T$.*

PROOF. Suppose otherwise that $\lambda_{T'} < \lambda_T$. By the strict concavity of u in c, we know that f is strictly decreasing when it is nonzero. This implies that

$$\begin{aligned} W_0 &= E\left[\int_0^T \frac{\xi(t)\, f((\lambda_T\xi(t)/B(t)),t)}{B(t)}\, dt\right] \\ &\le E\left[\int_0^T \frac{\xi(t)\, f((\lambda_{T'}\xi(t)/B(t)),t)}{B(t)}\, dt\right] \\ &\le E\left[\int_0^{T'} \frac{\xi(t)\, f((\lambda_{T'}\xi(t)/B(t)),t)}{B(t)}\, dt\right]. \end{aligned}$$

We claim that the first inequality in the above relation must be a strict inequality and hence $c^{T'}$ violates the budget constraint for the finite horizon problem with horizon $[0, T']$ and constitutes a violation. Now suppose that

$$E\left[\int_0^T \frac{\xi(t)\, f((\lambda_T\xi(t)/B(t)),t)}{B(t)}\, dt\right] = E\left[\int_0^T \frac{\xi(t)\, f((\lambda_{T'}\xi(t)/B(t)),t)}{B(t)}\, dt\right].$$

This necessitates that on $[0, T]$, $c^T = 0$, ν_a-a.e. This clearly contradicts the fact that c^T is an optimal solution to (23). □

Now let λ^* be such that $c^*(t) = f((\lambda^*\xi(t)/B(t)), t)$, ν_a-a.e.

THEOREM 4. $\lim_{T\to\infty} \lambda_T = \lambda^*$ *and* $c^T \to c^*$, ν_a-*a.e.*

PROOF. Once we can show that $\lambda_T \to \lambda^*$ as $T \to \infty$, the second assertion follows from the fact that f is continuous in its first argument by the strict concavity of u in c.

By Lemma 5, there exists a limit

$$\bar{\lambda} \equiv \lim_{T\to\infty} \lambda_T.$$

We claim that $\bar{\lambda} = \lambda^*$.

We take two cases. Suppose first that $\bar{\lambda} > \lambda^*$. Then there exists $T < \infty$ such that $\lambda_T > \lambda^*$. Arguments identical to those used in the proof of Lemma 5 prove that

$$W_0 < E\left[\int_0^\infty \frac{\xi(t)\, f((\lambda^*\xi(t)/B(t)),t)}{B(t)}\, dt\right],$$

which is a contradiction.

Next suppose that $\bar{\lambda} < \lambda^*$. Pick $\hat{\lambda} \in (\bar{\lambda}, \lambda^*)$. Arguments identical to those used in the proof of Lemma 5 show that

$$\begin{aligned} W_0 &= E\left[\int_0^T \frac{\xi(t)\, f((\lambda_T\xi(t)/B(t)),t)}{B(t)}\, dt\right] \\ &> E\left[\int_0^T \frac{\xi(t)\, f\big((\hat{\lambda}\xi(t)/B(t)),t\big)}{B(t)}\, dt\right], \qquad \forall\, T > 0. \end{aligned}$$

Letting $T \to \infty$ on the right-hand side of the second inequality gives

$$\begin{aligned} W_0 &\geq E\left[\int_0^\infty \frac{\xi(t)\, f\big((\hat{\lambda}\xi(t)/B(t)),t\big)}{B(t)}\, dt\right] \\ &\geq E\left[\int_0^\infty \frac{\xi(t)\, f((\lambda^*\xi(t)/B(t)),t)}{B(t)}\, dt\right] = W_0. \end{aligned}$$

If the second inequality above is an equality, then it must be that $c^* = 0$, ν_a-a.e., which is clearly suboptimal. Thus the inequality must be a strict inequality and it leads to a contradiction. □

Next we turn our attention to the convergence of trading strategies. For this we restrict our attention to the case where $p \geq 2$ and make the following assumption.

ASSUMPTION 3. (i) If u satisfies the conditions of Theorem 1, then

$$E\left[\int_0^\infty \xi(t)\left(\frac{\xi(t)}{B(t)}\right)^{-2/b} |u(a,t)|^{2/b}\, d\lambda_a(t)\right] < \infty. \tag{24}$$

(ii) If u satisfies the conditions of Theorem 2, then

$$E\left[\int_0^\infty \xi(t)\left(\frac{\xi(t)}{B(t)}\right)^{-2}|u(a,t)|^2\,d\lambda_a(t)\right] < \infty. \tag{25}$$

We record below a useful technical lemma.

Lemma 6. *Let c be the solution to* (14) *financed by $(\alpha, \theta) \in \mathscr{L}(G)$. Then, under Assumption* 3,

$$E^*\left[\left(\int_0^\infty \frac{c(t)}{B(t)}\,dt\right)^2\right] < \infty$$

and

$$E^*\left[\int_0^\infty \left|\frac{\theta(t)^\top \sigma(t)}{B(t)}\right|^2 dt\right] < \infty.$$

Proof. Using the fact that P and Q are locally equivalent, the first assertion follows from similar arguments of Cox and Huang [(1990), Theorem 4.1].

From the proof of Proposition 3, we know that

$$E^*\left[\int_0^\infty \frac{c(s)}{B(s)}\,ds \,\middle|\, \mathscr{F}_t\right] = E^*\left[\int_0^\infty \frac{c(s)}{B(s)}\,ds\right] + \int_0^t \frac{\theta(s)^\top \sigma(s)}{B(s)}\,dw^*(s),$$

$$\forall\, t \geq 0,\ Q\text{-a.s.}$$

The left-hand side is a square-integrable martingale under Q. Thus Jacod (1979, 2.48) shows that the second assertion of this lemma is valid. □

Now we present our results of convergence of trading strategies. We will show the convergence of trading strategies in a metric involving taking expectation under Q.

Theorem 5. *Let (α, θ) and (α_T, θ_T) be the trading strategies that finance c^* and c^T, respectively. Then*

$$\lim_{T\to\infty} E^*\left[\int_0^\infty \left|\frac{(\theta_T(t) - \theta(t))^\top \sigma(t)}{B(t)}\right|^2 dt\right] = 0.$$

PROOF. Note first that $c^T(t) = f((\lambda_T \xi(t)/B(t)), t)1_{[0,T]}(t)$, $t \geq 0$, and $c^*(t) = f((\lambda^* \xi(\theta)/B(t)), t) \geq 0$. The triangle inequality implies

$$\left(E^*\left[\int_0^\infty |c^*(t) - c^T(t)|^2\, dt\right]\right)^{1/2}$$

$$\leq \left(E^*\left[\int_0^\infty \left| f\left(\frac{\lambda^* \xi(t)}{B(t)}, t\right) - f\left(\frac{\lambda_T \xi(t)}{B(t)}, t\right)\right|^2 dt\right]\right)^{1/2}$$

$$+ \left(E^*\left[\int_0^\infty \left| f\left(\frac{\lambda_T \xi(t)}{B(t)}, t\right) - f\left(\frac{\lambda_T \xi(t)}{B(t)}, t\right)1_{[0,T]}(t)\right|^2 dt\right]\right)^{1/2}.$$

Recall from Lemma 5 that λ_T is an increasing function of T and from Theorem 4 that $\lambda^* = \lim_{T\to\infty} \lambda_T$. Thus both

$$\left| f\left(\frac{\lambda^* \xi(t)}{B(t)}, t\right) - f\left(\frac{\lambda_T \xi(t)}{B(t)}, t\right)\right|^2$$

and

$$\left| f\left(\frac{\lambda_T \xi(t)}{B(t)}, t\right) - f\left(\frac{\lambda_T \xi(t)}{B(t)}, t\right)1_{\{t \leq T\}}\right|^2$$

decrease to zero ν_a-a.e. and the Lebesgue convergence theorem yields that

$$\lim_{T\to\infty} \left(E^*\left[\int_0^\infty \left|\frac{c^*(t) - c^T(t)}{B(t)}\right|^2 dt\right]\right)^{1/2} = 0.$$

The assertion then follows from the fact that

$$E^*\left[\int_0^\infty \left|\frac{c^*(t) - c^T(t)}{B(t)}\right|^2 dt\right] = E^*\left[\int_0^\infty \left|\frac{(\theta_T(t) - \theta(t))^\top \sigma(t)}{B(t)}\right|^2 dt\right]. \qquad \square$$

Theorem 5 is not enough for us to conclude that the optimal trading strategy for the infinite horizon case can be gotten by letting $T \to \infty$ in the optimal strategies for the finite horizon cases as we need almost everywhere convergence for this. The following theorem gives a sufficient condition for this operation to be valid.

THEOREM 6. *Let (α, θ) and (α_T, θ_T) be the trading strategies that finance c^* and c^T, respectively. Suppose that $\lim_{T\to\infty} \theta_T(t)$ exists ν_a-a.e.; then $\lim_{T\to\infty} \theta_T(t) = \theta(t)$, ν_a-a.e.*

PROOF. First recall that $\theta_T \to \theta$ in the sense of Theorem 5. Chung [(1974), Theorems 4.1.4 and 4.2.3] implies that there exists a subsequence $\{T_n\}$ with $T_n \uparrow \infty$ so that $\theta_{T_n} \to \theta$, ν_a-a.e. as $n \to \infty$. Given the hypothesis that $\lim_{T\to\infty} \theta_T(t)$ exists ν_a-a.e., it follows that $\lim_{T\to\infty} \theta_T(t) = \theta(t)$, ν_a-a.e. $\square$

In words, if the optimal portfolio policies for finite horizon economies do converge pointwise, they must then converge to the optimal policy for the infinite horizon economy.

7. Concluding remarks. We have assumed throughout this paper that there are as many linearly independent risky securities as the dimension of the underlying uncertainty, or, markets are dynamically complete. When markets are dynamically incomplete, it is straightforward to borrow from He and Pearson (1989) and show that when prices of securities together with some other processes follow a diffusion process, then the optimal policy can be computed by solving a quasilinear partial differential equation, and, in addition, a general existence theorem is available for the case where the coefficient of the Arrow–Pratt measure of the relative risk aversion [see Arrow (1970) and Pratt (1964)] of the individual's utility function is less than 1.

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