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Investigation of Airy equations with random initial conditions

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Abstract

The paper investigates properties of mean-square solutions to the Airy equation with random initial data given by stationary processes. The result on the modulus of continuity of the solution is stated and properties of the covariance function are described. Bounds for the distributions of the suprema of solutions under φ-sub-Gaussian initial conditions are presented. Several examples are provided to illustrate the results. Possible extensions of the results are discussed.

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1 Introduction

Dispersive partial differential equations have been the topic of intensive studies for centuries, starting from the classical physics and going widely beyond. These equations serve to model various vibrating media such as waves on string, in liquids or in air. Probably the most familiar problems in which dispersive effects appear are the classical problems of water waves descriptions and one of the most famous equations is the Korteweg–de Vries (KdV) equation derived to model the propagation of low amplitude long water waves in a shallow canal. Nowadays many versions and higher order generalizations of the KdV equation are of use in different areas including hydrodynamics, plasma physics, electrodynamics, in studies of electromagnetic and acoustic waves, waves in elastic media, traffic flows, chemical processes and others.

In the present paper we consider the Airy equation or linear Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial^3 x},\ t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

subject to the random initial condition

$$u(0, x) = \eta(x),\ x \in \mathbb{R}, \tag{1.2}$$

with η being a stationary stochastic process.

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Initial value problem (1.1)-(1.2) was treated in [5], namely, the asymptotic behavior was analyzed for the rescaled solutions under weakly dependent stationary initial data. Note that in the recent literature such approach has been widely applied to study rescaled solutions to the heat, fractional heat, Burgers and other equations with Gaussian and non-Gaussian initial conditions possessing weak or strong dependence (see, for example, papers [3, 22] among many others).

The purpose of the present paper is to establish the upper bounds for the distribution of the supremum of solution to (1.1)-(1.2) under the assumption that the initial condition is given by a φ-sub-Gaussian process. These processes provide a natural generalization of Gaussian and sub-Gaussian ones and possess well described properties, which is important for applications. Theory developed for these processes allows to derive many useful bounds for the distribution of various functionals of these processes (see, [7]). The present paper is close to the papers [6, 15, 16, 10] where higher order dispersive equations and the heat equation were studied under φ-sub-Gaussian initial conditions.

Following [5], we consider the solution to the random initial value problem (1.1)-(1.2) in the mean square sense and write its representation in a form of stochastic integral, and corresponding representation for its covariance function. The solution field is a stationary (of the second order) both in time and space variables.

We state the result on the modulus of continuity of solution and reveal several interesting properties of the covariance function of solution. We show that the covariance function itself is a solution to a particular deterministic Airy equation and as such, it inherits well known properties of solutions to Airy equations. In particular, many useful bounds can be written for the covariance function.

It is important to note that from the statistical point of view, the covariance of the solution presents an example of non-separable space-time covariance function. So, we reveal a convenient way to construct a space-time stationary covariance with the use of another covariance model, which will possess a very transparent physical interpretation and a lot of well described properties due to its representation as an oscillatory integral.

From the probabilistic consideration, we also deduce such a general fact that if the Airy equation (1.1) is considered with a (nonrandom) initial condition $u(0, x) = u_0(x)$, $x \in \mathbb{R}$, given by a real positive-definite kernel, then the solution $u(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, is a positive definite kernel as well. To the best of our knowledge, this interesting fact was not reflected before in the literature on the Airy equations.

Next, using the methods developed for φ-sub-Gaussian processes, the bounds are obtained for the tails of the distribution of supremum of solution to (1.1)-(1.2). Bounds are presented in such a closed form which is convenient for possible applications.

The main condition for the bounds to hold is stated in terms of the spectral measure of the initial data η in (1.2). It is shown that this condition is satisfied for several models where the initial data process η is itself a solution to a stochastic differential equation. In particular, Matérn model, Ornstein-Uhlenbeck and fractional Ornstein-Uhlenbeck processes can be considered to model random initial data.

The paper is organized as follows. In Section 2, following [5], we give the expression in the form of a stochastic integral for the mean square solution to the initial value problem (1.1)–(1.2). In Section 3 the result on the modulus of continuity of solution is stated. Section 4 describes properties of the covariance function of the solution. Section 5 collects definitions and properties of φ-sub-Gaussian processes as a preparation for the study of the distribution of suprema of solutions to (1.1)–(1.2) under φ-sub-Gaussian initial conditions, which is done in Section 6. To illustrate the results stated, in Section 7 several examples of random initial data are given, for which assumptions for the stated results hold and all the constants involved can be calculated explicitly. Possible extensions of the results are discussed, in particular, to higher order and fractional Airy equations.

2 Solution to the Airy equation with initial condition given by a stationary stochastic process

Consider the Airy equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial^3 x},\ t > 0,\ x \in \mathbb{R}, \tag{2.1}$$

subject to the random initial condition

$$u(0, x) = \eta(x),\ x \in \mathbb{R}, \tag{2.2}$$

where η is a stochastic process satisfying the condition below.

A. $\eta(x), x \in \mathbb{R}$, is a real, measurable, mean-square continuous stationary (of the second order, that is, in a weak sense) centered stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Let $B_\eta(x), x \in \mathbb{R}$, be a covariance function of the process $\eta(x), x \in \mathbb{R}$, with the spectral representation

$$B_\eta(x) = \int_{\mathbb{R}} e^{i\lambda x} dF(\lambda), \tag{2.3}$$

where $F(\lambda), \lambda \in \mathbb{R}$, is a spectral measure, and for the process itself we can write the spectral representation

$$\eta(x) = \int_{\mathbb{R}} e^{i\lambda x} Z(d\lambda). \tag{2.4}$$

The stochastic integral (2.4) is considered as $L_2(\Omega)$ integral. The orthogonal complex-valued random measure Z is such that $\mathsf{E}|Z(d\lambda)|^2 = F(d\lambda)$.

Following [5], we can write the representation of the mean square solution to the problem (2.1)–(2.2) and the expression for its covariance function.

Consider the field $u(t, x)$, $t > 0$, $x \in \mathbb{R}$, defined by

$$u(t, x) = \int_{\mathbb{R}} g(t, x - y)\eta(y)dy, \tag{2.5}$$

where the function g is the fundamental solution to equation (2.1):

$$g(t, x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\alpha x - i\alpha^3 t}\, d\alpha = \frac{1}{\pi}\int_0^\infty \cos(\alpha x + \alpha^3 t)\, d\alpha = \frac{1}{\sqrt[3]{3t}}\mathrm{Ai}\Big(\frac{x}{\sqrt[3]{3t}}\Big),\ t > 0,\ x \in \mathbb{R}, \tag{2.6}$$

and

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\alpha x + \frac{\alpha^3}{3}\right) d\alpha,\ x \in \mathbb{R},$$

is the Airy function of the first kind. Properties of the Airy function are well studied due to numerous applications in various areas. In particular, this function has the following asymptotic behavior:

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}} \exp\left\{-\frac{2}{3}x^{3/2}\right\},\ x \to +\infty;\ \mathrm{Ai}(x) \sim \frac{1}{\sqrt{\pi}|x|^{1/4}} \cos\left\{\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right\},\ x \to -\infty.$$

The asymptotic expansions can be obtained, in particular, as a reflection of the well-known ones for the Bessel functions, since the Airy function can be expressed in terms of the Bessel function of order $\frac{1}{3}$: $\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{1/3}\big(\frac{2}{3}x^{3/2}\big)$. We refer, for example, to [20], [24], [26], [27].

In view of (2.4), the field (2.5) can be written in the following form

$$u(t,x) = \int_{\mathbb{R}} \exp\Big\{i\lambda x + i\lambda^3 t\Big\} Z(d\lambda). \tag{2.7}$$

The random field (2.7) can be interpreted as the mean-square or $L_2(\Omega)$ solution to the Cauchy problem (2.1)–(2.2) (see [5]).

From the representation (2.7) the covariance of the field u can be calculated:

$$\begin{aligned} \mathsf{Cov}\big(u(t,x), u(s,y)\big) &= \int_{\mathbb{R}} \exp(i\lambda(x-y) + i\lambda^3(t-s))dF(\lambda) \\ &= \int_{\mathbb{R}} \cos(\lambda(x-y) + \lambda^3(t-s))dF(\lambda) \\ &:= B(t-s, x-y). \end{aligned} \tag{2.8}$$

From the expression (2.8) one can see that the random field u is stationary with respect to time and space variables. For a fixed t we have:

$$\mathsf{Cov}\big(u(t,x), u(t,y)\big) = \int_{\mathbb{R}} \exp(i\lambda(x-y))dF(\lambda) = B_\eta(x-y),$$

that is, the covariance of the solution $u(t,\cdot)$ considered at any fixed time t coincides with the covariance function of the initial process η.

Note that one of the commonly used approaches of studying PDEs with stationary initial conditions is via the second order analysis, that is, by considering their mean square solutions represented by stochastic integrals. Such approach takes its origins in the paper by Rosenblatt [21] where the heat equation with stationary initial condition was treated and the mean square representation for the solution was given. In the recent literature this approach is widely used for various classes of PDEs with random initial conditions, in particular, for studying their rescaled solutions, see, e.g., [3, 5, 10, 22], to mention only few, see also references therein.

3 Modulus of continuity of the solution

In this section we state the general result on the modulus of continuity in the mean square of the field (2.7). This result is of interest by itself, but will also be used further in Section 6 for evaluation of the distribution of suprema of solutions. Denote $K = [a,b] \times [c,d]$ for arbitrary $a \geq 0$, $b, c, d \in \mathbb{R}$.

Theorem 3.1. *Let $u(t,x), t > 0, x \in \mathbb{R}$, be the random field given by* (2.7) *and assumption A hold. Suppose that for some $\beta \in (0,1]$*

$$\int_{\mathbb{R}} \lambda^{6\beta} F(d\lambda) < \infty, \tag{3.1}$$

then

$$\sigma(h) := \sup_{\substack{(t,x),(s,y)\in K:\\ |t-s|\leq h, |x-y|\leq h}} \Big(\mathsf{E}(u(t,x) - u(s,y))^2\Big)^{1/2} \leq c(\beta) h^\beta, \tag{3.2}$$

where

$$c(\beta) = 2^{1-\beta}\Big(\int_{\mathbb{R}} (\lambda + \lambda^3)^{2\beta} F(d\lambda)\Big)^{1/2}. \tag{3.3}$$

If $\int_{\mathbb{R}} |\lambda|^3 F(d\lambda) < \infty$, then

$$\sigma(h_1, h_2) := \sup_{\substack{(t,x),(s,y)\in K:\\ |t-s|\leq h_1, |x-y|\leq h_2}} \Big(\mathsf{E}(u(t,x) - u(s,y))^2\Big)^{1/2} \leq (c_1 h_1 + c_2 h_2)^{1/2}, \tag{3.4}$$

where $c_1 = 2\int_{\mathbb{R}} |\lambda|^3 F(d\lambda)$ and $c_2 = 2\int_{\mathbb{R}} |\lambda| F(d\lambda)$.

Proof. We have

$$\mathsf{E}\Big(u(t,x)-u(s,y)\Big)^2=\int_{\mathbb{R}}|b(\lambda)|^2F(d\lambda), \tag{3.5}$$

where

$$b(\lambda)=e^{i(\lambda x+\lambda^3 t)}-e^{i(\lambda y+\lambda^3 s)}.$$

By the direct calculations we obtain:

$$|b(\lambda)|^2=4\sin^2\left(\frac{\lambda(x-y)+\lambda^3(t-s)}{2}\right).$$

For $|t-s|\le h$ and $|x-y|\le h$ we can write for any $\beta\in(0,1]$:

$$4\sin^2\left(\frac{1}{2}(\lambda(x-y)+\lambda^3(t-s))\right)\le 4\min\left(\frac{h}{2}|\lambda+\lambda^3|,1\right)^2\le 4\frac{(h(\lambda+\lambda^3))^{2\beta}}{2^{2\beta}} \tag{3.6}$$

which implies the estimate

$$\Big(\int_{\mathbb{R}}|b(\lambda)|^2F(d\lambda)\Big)^{1/2}\le 2^{1-\beta}h^\beta\Big(\int_{\mathbb{R}}(\lambda+\lambda^3)^{2\beta}F(d\lambda)\Big)^{1/2}. \tag{3.7}$$

Therefore, under condition (3.1) we can write the bound (3.2).

Taking $|t-s|\le h_1$ and $|x-y|\le h_2$, analogously to the above we can write the estimate $|b(\lambda)|^2\le 2(h_1|\lambda|^3+h_2|\lambda|)$ which implies (3.4). □

4 Closer look at the covariance function of the solution

In this section we discuss the covariance function of the solution and reveal several interesting facts and properties coming from its representation.

Firstly, we note that the covariance function $B(t,x)$ of the random solution field $u(t,x)$ to the initial value problem (2.1)–(2.2) is itself a solution to the deterministic initial value problem for the Airy equation and as such, it possesses all the properties pertaining to the solution.

Theorem 4.1. *Let $u(t,x), t>0, x\in\mathbb{R}$, be the random field* (2.7) *representing the mean square solution to the initial value problem* (2.1)–(2.2)*, assumption A hold, and the covariance function of the field η be such that $B_\eta\in L^1\cap L^2$.*

Then the covariance function $B(t,x)$ of the field $u(t,x)$ is a solution to the initial value problem

$$\frac{\partial B}{\partial t}=-\frac{\partial^3 B}{\partial^3 x},\ t>0,\ x\in\mathbb{R}, \tag{4.1}$$

$$B(0,x)=B_\eta(x),\ x\in\mathbb{R} \tag{4.2}$$

and, therefore, it can be represented as

$$B(t,x)=\int_{\mathbb{R}}B_\eta(z)\frac{1}{\sqrt[3]{3t}}\mathrm{Ai}\Big(\frac{x-z}{\sqrt[3]{3t}}\Big)\,dz, \tag{4.3}$$

and the following estimates hold:

$$\|B(t,x)\|_{L^\infty_x}\le c|t|^{-1/3}\|B_\eta\|_{L^1}, \tag{4.4}$$

$$\|B(t,x)\|_{L^8_{t,x}}\le c\|B_\eta\|_{L^2}, \tag{4.5}$$

$$\|\partial_x B(t,x)\|_{L^\infty_x L^2_t}\le c\|B_\eta\|_{L^2}. \tag{4.6}$$

Proof. Under the conditions of theorem, there exists the spectral density $f(\lambda)$, $\lambda \in \mathbb{R}$, $f \in L^1$, of the initial value process η, therefore, the covariance B_η can be written in the form

$$B_\eta(x) = \int_{\mathbb{R}} e^{i\lambda x} f(\lambda)\, d\lambda,$$

and f itself can be represented as $f(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda x} B_\eta(x)\, dx = \widetilde{B}_\eta(\lambda)$ (denoting with tilde the Fourier transform). Formula (2.8) gives in such case the expression for the covariance B in the form

$$B(t,x) = \int_{\mathbb{R}} e^{i\lambda x + i\lambda^3 t} f(\lambda)\, d\lambda. \tag{4.7}$$

Consider the problem (4.1)–(4.2) and apply the standard methods (see, e.g. [24]). Taking the space Fourier transform, we obtain

$$\widetilde{B}(t,\xi) = e^{it\xi^3}\widetilde{B}_\eta(\xi), \tag{4.8}$$

that is, $\widetilde{B}(t,\xi) = e^{it\xi^3} f(\xi)$, and by applying the inverse transform we conclude that the solution is represented by the right hand side of formula (4.7), and thus, coincides with the covariance. On the other hand, once we found the Fourier transform in the form (4.8), we can use the convolution theorem for Fourier transform to recover B as B_η convoluted with the Airy function, that is, we come to formula (4.3). Now, having stated that the covariance B is the solution to the Airy equation, we use the known results on the properties of the solution to write the bounds (4.4)–(4.6). In particular, the bound (4.4) can be deduced directly from the representation (4.3) and the fact that the Airy function is uniformly bounded (see, e.g., [24, Chapter 8], as well as for (4.5)). Bounds (4.5)–(4.6) follow as particular cases of the general result stated in [11, Theorem 2.1]. □

Note that the representation of the covariance in the form (4.3) was obtained in [5] by direct calculations based on the representation of the solution field (2.5). Here we show that this representation follows immediately since the covariance appears to be the solution to the corresponding initial value problem for the Airy equation.

Remark 4.2. Theory of dispersive equations and, in particular, KdV and linear KdV equations, is well developed and presents many interesting and important results on the solutions (see, e.g., [25]). Basing on results for solutions to deterministic Airy equations one can deduce properties of the covariance function. Some of them are collected in the above theorem. As the solution to the Airy equation, $B(t,x)$ “disperses” as $t \to \infty$ in the form (4.4). In the recent literature, a whole direction of research is devoted to the general Airy–Strichartz inequalities, that is, mixed Lebesgue norms $L_t^p L_x^q$ estimates, of which we present here the simplest dispersive estimates (4.5)–(4.6). In particular, (4.6) is called the Kato smoothing effect. Giving space-time integrability of solutions, the Strichartz estimates are the fundamental tools to obtain well-posedness for nonlinear dispersive equations, and as such they have been intensively studied. For more detail, more estimates of this kind and historical background see, for instance, the seminal papers [11], [12]. On the other hand, we note that along with the “dispersive” behavior, $B(t,x)$ possesses some conserved quantities, namely, $\int_{\mathbb{R}} B(t,x)\, dx$ is constant in time, or conserved: $\int_{\mathbb{R}} B(t,x)\, dx = \int_{\mathbb{R}} B_\eta(x)\, dx$, and $B(t,x)$ also preserves the L_2 norm, that is, $\int_{\mathbb{R}} B^2(t,x)\, dx$ is constant in time: $\|B(t,x)\|_{L_x^2} = \|B_\eta\|_{L^2}$.

Remark 4.3. Considering $B(t,x)$ from the statistical point of view, we see that it gives an example of nonseparable space-time covariance function, constructed with the use of another covariance model, and possessing a very transparent physical interpretation and a lot of well described properties due to its representation as an oscillatory integral.

We state in the next theorem one more interesting fact coming as a feedback from the probabilistic consideration of the Airy equation. To the best of our knowledge, this fact was not reflected before in the literature on the Airy equations.

Theorem 4.4. *Consider the Airy equation $\partial_t u(t,x) = -\partial_{xxx} u(t,x)$ with initial condition $u(0,x) = u_0(x)$, $x \in \mathbb{R}$, given by a real positive-definite kernel, that is, $\sum_{i=1}^n c_i c_j u_0(x_i - x_j) \geq 0$ for all $n \in N$, $c_i \in \mathbb{R}$, $x_i \in \mathbb{R}$, and suppose $u_0 \in L^1 \cap L^2$.*

Then the solution $u(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, is a positive definite kernel as well: $\sum_{i=1}^n c_i c_j u(t_i - t_j, x_i - x_j) \geq 0$ for all $n \in N$, $c_i \in \mathbb{R}$, $(t_i, x_i) \in \mathbb{R}_+ \times \mathbb{R}$.

Proof. We recall the result due to Loève that the class of covariance functions of second order stochastic processes coincides with the class of positive definite functions. In view of Theorem 4.1, the solution $u(t,x)$ can be seen as a covariance function of the stationary random field representing the solution to the random initial value problem for the Airy equation with the stationary random initial condition having the covariance function u_0. Therefore, as a covariance function, $u(t,x)$ is a positive definite kernel. □

5 φ-sub-Gaussian stochastic processes to be used as initial conditions

We further aim to state results on the solution to initial value problem (2.1)–(2.2) under a φ-sub-Gaussian initial condition. To make the paper self-contained, we present in this section definitions and facts needed in our study. The main theory for the spaces of φ-sub-Gaussian random variables and stochastic processes was presented in [7, 9, 17] and has been developed in numerous recent studies. Such spaces can be considered as exponential type Orlicz spaces of random variables and provide generalizations of Gaussian and sub-Gaussian random variables and processes (see, [7, Ch.2]).

Definition 5.1. *[9, 17] A continuous even convex function φ is called an Orlicz N-function if $\varphi(0) = 0$, $\varphi(x) > 0$ as $x \neq 0$ and $\lim\limits_{x\to 0} \frac{\varphi(x)}{x} = 0$, $\lim\limits_{x\to\infty} \frac{\varphi(x)}{x} = \infty$.*

Condition Q. Let φ be an N-function which satisfies $\lim\inf\limits_{x\to 0} \frac{\varphi(x)}{x^2} = c > 0$, where the case $c = \infty$ is possible.

Definition 5.2. *[9, 17] Let φ be an N-function satisfying condition Q and $\{\Omega, L, \mathsf{P}\}$ be a standard probability space. The random variable ζ is φ-sub-Gaussian, or belongs to the space $\mathrm{Sub}_\varphi(\Omega)$, if $\mathsf{E}\zeta = 0$, $\mathsf{E}\exp\{\lambda\zeta\}$ exists for all $\lambda \in \mathbb{R}$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in \mathbb{R}$*

$$\mathsf{E}\exp\{\lambda\zeta\} \leq \exp\{\varphi(\lambda a)\}.$$

The random process $X(t)$, $t \in T$, is called φ-sub-Gaussian if the random variables $\{X(t), t \in T\}$ are φ-sub-Gaussian.

The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm (see [9, 17]):

$$\tau_\varphi(\zeta) = \inf\{a > 0 : \mathsf{E}\exp\{\lambda\zeta\} \leq \exp\{\varphi(a\lambda)\}.$$

Definition 5.3. *[9, 17] The function φ^* defined by $\varphi^*(x) = \sup_{y\in\mathbb{R}}(xy - \varphi(y))$ is called the Young-Fenchel transform (or convex conjugate) of the function φ.*

The function φ^* (known also as the Legendre or Legendre-Fenchel transform) plays an important role in the theory of φ-sub-Gaussian random variables and processes and involved in estimates for 'tail' probabilities, distributions of suprema and other functionals of these processes. If ζ is a φ-sub-Gaussian random variable, then for all $u > 0$ we have

$$P\{|\zeta| > u\} \leq 2\exp\left\{-\varphi^*\left(\frac{u}{\tau_\varphi(\zeta)}\right)\right\}. \tag{5.1}$$

Moreover, it is stated in [7] (see, Corollary 4.1, p. 68) that a random variable ζ is a φ-sub-Gaussian if and only if $\mathsf{E}\zeta = 0$ and there exist constants $C > 0$, $D > 0$ such that

$$P\{|\zeta| > u\} \leq C \exp\left\{-\varphi^*\left(\frac{u}{D}\right)\right\}. \tag{5.2}$$

As one can see, the property of φ-sub-Gaussianity can be characterized in a double way: by introducing a bound on the exponential moment of a random variable as prescribed by Definition 5.2, or by the tail behavior of the form (5.1) or (5.2), which is even more essential from the practical point of view.

The class of φ-sub-Gaussian random variables is rather wide and comprises, for example, centered compactly supported distributions, reflected Weibull distributions, centered bounded distributions, Gaussian, Poisson distributions. In the case when $\varphi = \frac{x^2}{2}$, the notion of φ-sub-Gaussianity reduces to the classical sub-Gaussianity. Various classes of φ-sub-Gaussian processes and fields were studied, in particular, in [10, 14, 15, 16] (see also references therein).

Let us consider the metric space $(\mathbf{T}, \rho)$, $\mathbf{T} = \{a_i \leq t_i \leq b_i, i = 1, 2\}$, $\rho(t, s) = \max_{i=1,2} |t_i - s_i|$. Denote $T_i = b_i - a_i, i = 1, 2$. For a φ-sub-Gaussian process $X(t)$, $t \in \mathbf{T}$, introduce the following conditions.

B.1. $\varepsilon_0 = \sup_{t\in\mathbf{T}} \tau_\varphi(X(t)) < \infty$.
B.2. The process X is separable on the space $(\mathbf{T}, \rho)$.
B.3. For $c > 0$ and $0 < \beta \leq 1$ the estimate holds: $\sup_{\rho(t,s)<h} \tau_\varphi(X(t) - X(s)) \leq ch^\beta$.

We state the result which will be used in the next section.

Theorem 5.4. *Let for a φ-sub-Gaussian process $X(t)$, $t \in \mathbf{T}$, conditions B.1–B.3 hold. Then for all $\theta \in (0, 1)$ such that $\theta\varepsilon_0 < c(\min(T_1, T_2)/2)^\beta$ and $u > 0$*

$$P\Big\{\sup_{t\in\mathbf{T}} |X(t)| \geq u\Big\} \leq 2 \exp\Big\{ - \varphi^*\Big(\frac{u(1-\theta)}{\varepsilon_0}\Big)\Big\} \varkappa (ce)^{2/\beta} (\theta\varepsilon_0)^{-2/\beta}, \tag{5.3}$$

where $\varkappa = \frac{1}{2} \min(T_1, T_2)(T_1 + T_2)$.

Theorem 5.4 gives the improved bound in comparison with the analogous results stated in [16] (Corollary 3.1) and [10] (Corollary 2).

Proof. Theorem is obtained as a corollary of the more general result from [16]. Namely, according to Theorem 3.1 [16] the following estimate holds:

$$P\Big\{\sup_{t\in\mathbf{T}} |X(t)| \geq u\Big\} \leq 2 \exp\Big\{ - \varphi^*\Big(\frac{u(1-\theta)}{\varepsilon_0}\Big)\Big\} r^{(-1)} \left(\frac{I_r(\min(\theta\varepsilon_0, \gamma_0))}{\theta\varepsilon_0}\right), \tag{5.4}$$

where it is supposed

$$I_r(\delta) := \int_0^\delta r\Big(\prod_{i=1,2}\Big(\frac{T_i}{2\sigma^{(-1)}(v)} + 1\Big)\Big) \, dv < \infty, \tag{5.5}$$

for $0 < \delta \leq \gamma_0 := \sigma(\max(T_1, T_2))$, $\sigma(h)$, $h > 0$, is a monotonically increasing continuous function such that $\sigma(h) \to 0$ as $h \to 0$ and $\sup_{\rho(t,s)<h} \tau_\varphi(X(t) - X(s)) \leq \sigma(h)$, and $r(x)$, $x \geq 1$, is a non-negative, monotone increasing function such that $r(e^x), x \geq 0$, is convex.

In our case $\sigma(u) = cu^\beta$, $\sigma^{(-1)}(u) = (u/c)^{1/\beta}$. Choose $r(v) = v^\alpha - 1$, $0 < \alpha < \beta/2$, then $r^{(-1)}(v) = (v + 1)^{1/\alpha}$. For such $r(v)$ we estimate $I_r(\delta)$. Consider $\delta \in (0, \theta\varepsilon_0)$, and choose $\theta \in \big(0, \frac{c}{\varepsilon_0}\big(\frac{\min(T_1,T_2}{2}\big)^\beta\big)$. Then we can write:

$$\begin{aligned} I_r(\delta) &= \int\limits_0^\delta \Big[\prod_{i=1,2}\Big(\frac{T_i c^{1/\beta}}{2u^{1/\beta}} + 1\Big)^\alpha - 1\Big] \, du \leq \int\limits_0^\delta \Big[\Big(\frac{\min(T_1, T_2)c^{1/\beta}}{u^{1/\beta}}\Big)^\alpha \Big(\frac{(T_1 + T_2)c^{1/\beta}}{2u^{1/\beta}}\Big)^\alpha - 1\Big] \, du \\ &= \varkappa^\alpha c^{2\alpha/\beta} \Big(1 - \frac{2\alpha}{\beta}\Big)^{-1} \delta^{1-2\alpha/\beta} - \delta, \end{aligned}$$

denoting $\varkappa = \frac{1}{2}\min(T_1, T_2)(T_1 + T_2)$, and

$$r^{(-1)}\left(\frac{I_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\right) \le \varkappa c^{2/\beta}\left(1 - \frac{2\alpha}{\beta}\right)^{-1/\alpha}(\theta\varepsilon_0)^{-2/\beta}. \tag{5.6}$$

Now letting $\alpha \to 0$ we have $\left(1-\frac{2\alpha}{\beta}\right)^{-1/\alpha} \to e^{2/\beta}$, and the right hand side of (5.6) becomes $\varkappa(ce)^{2/\beta}(\theta\varepsilon_0)^{-2/\beta}$. Inserting this expression to (5.4) we obtain (5.3). □

Remark 5.5. Analogues to Theorem 5.4, with different bounds on the increments in assumption B.3, were applied in the literature in various contexts, in particular, in [14] for developing uniform approximation schemes for φ-sub-Gaussian processes, in [10, 16] for studying partial differential equations with random initial data, in [23] for evaluation of suprema of spherical random fields. Such theorems allow to calculate bounds for the distribution of suprema of φ-sub-Gaussian processes in the closed form.

We will need some further properties of φ-sub-Gaussian variables.

Definition 5.6. *[13] A family Δ of φ-sub-Gaussian random variables is called strictly φ-sub-Gaussian if there exists a constant C_Δ such that for all countable sets I of random variables $\zeta_i \in \Delta$, $i \in I$, the inequality holds: $\tau_\varphi\big(\sum_{i\in I}\lambda_i\zeta_i\big) \le C_\Delta\big(\mathsf{E}\left(\sum_{i\in I}\lambda_i\zeta_i\right)^2\big)^{1/2}$. Random process $\zeta(t)$, $t \in T$, is called strictly φ-sub-Gaussian if the family of random variables $\{\zeta(t), t \in T\}$ is strictly φ-sub-Gaussian.*

Example 5.7. [13] Let ξ_k, $k = \overline{1,\infty}$ be independent φ-sub-Gaussian random variables and φ be such that $\varphi(\sqrt{x})$ is convex. If there exists $C > 0$ such that $\tau_\varphi(\xi_n) \le C(\mathsf{E}\xi_k^2)^{1/2}$ for any $k \ge 1$, and for a sequence of nonrandom functions $f_k(t)$, $t \in T$, $k \ge 1$, the series $\sum\limits_{k=1}^{\infty}\mathsf{E}\xi_k^2 f_k^2(t)$ converges for all $t \in T$, then the series $\sum\limits_{k=1}^{\infty}\xi_k f_k(t)$, $t \in T$, is strictly φ-sub-Gaussian random process with determining constant C.

Example 5.8. [13] Let K be a deterministic kernel and $X(t) = \int_T K(t,s)\,d\xi(s)$, where $\xi(t)$, $t \in T$, is a strictly φ-sub-Gaussian process and the integral is defined in the mean-square sense. Then $X(t)$, $t \in T$, is strictly φ-sub-Gaussian process with the same determining constant.

6 Distribution of suprema of solutions under stationary φ-sub-Gaussian initial conditions

Consider the initial value problem (2.1)–(2.2), where the process η is strictly φ-sub-Gaussian and satisfies condition A. Suppose that the solution $u(t,x)$ is considered in the domain $K = \{(t,x) : a \le t \le b, c \le x \le d\}$.

Denote $\widetilde{\varepsilon}_0 = \sup_{(t,x)\in K}\tau_\varphi(u(t,x))$, $T_1 = b - a$, $T_2 = d - c$, $\varkappa = \frac{1}{2}\min(T_1, T_2)(T_1 + T_2)$.

Theorem 6.1. *Let $u(t,x), (t,x) \in K$, be a separable modification of the stochastic process given by* (2.7)*, the process η be strictly φ-sub-Gaussian with the determining constant c_η and assumption A hold. Suppose that for for some $\beta \in (0,1]$ condition* (3.1) *holds.*

Then:

1)

$$\sup_{|t-s|\le h, |x-y|\le h}\tau_\varphi(u(t,x) - u(s,y)) \le c_\eta c(\beta)h^\beta, \tag{6.1}$$

where $c(\beta)$ is given by formula (3.3)*; if $\int_{\mathbb{R}}|\lambda|^3 F(d\lambda) < \infty$, then*

$$\sup_{|t-s|\le h_1, |x-y|\le h_2}\tau_\varphi(u(t,x) - u(s,y)) \le (c_1h_1 + c_2h_2)^{1/2}, \tag{6.2}$$

where $c_1 = 2c_\eta \int_{\mathbb{R}} |\lambda|^3 F(d\lambda)$ *and* $c_2 = 2c_\eta \int_{\mathbb{R}} |\lambda| F(d\lambda)$*;*

2) for all $\theta \in (0,1)$ *such that* $\theta\varepsilon_0 < c_\eta c(\min(T_1,T_2)/2)^\beta$ *and* $v > 0$ *it holds:*

$$P\Big\{ \sup_{(t,x)\in K} |u(t,x)| > v \Big\} \le 2\exp\Big\{ -\varphi^*\Big(\frac{v(1-\theta)}{\widetilde{\varepsilon}_0}\Big)\Big\} \varkappa (c_\eta c(\beta) e)^{2/\beta} (\theta\widetilde{\varepsilon}_0)^{-2/\beta}; \tag{6.3}$$

3) for any $p \in (0,1)$*, any* $h > 0$ *and* $\widehat{\varkappa} = \max(T_1,T_2)$ *the following bond holds:*

$$P\Big\{ \sup_{\substack{|t-s|\le h,\\ |x-y|\le h}} |u(t,x)-u(s,y)| > v \Big\} \le 2^{4/\beta} \exp\Big\{ -\varphi^*\Big(\frac{v(1-p)^2}{c_\eta c(\beta) h^\beta (3-p)}\Big)\Big\} \Big[\frac{2^{4/\beta-2}\widehat{\varkappa}^2}{ph^2} + 1\Big]. \tag{6.4}$$

Proof. The process $u(t,x)$ is strictly φ-sub-Gaussian with the determining constant c_η. Therefore, in view of Theorem 3.1 we can write

$$\sup_{|t-s|\le h, |x-y|\le h} \tau_\varphi(u(t,x)-u(s,y)) \le c_\eta \sup_{|t-s|\le h, |x-y|\le h} \Big(\mathsf{E}(u(t,x)-u(s,y))^2\Big)^{1/2} \le c_\eta c(\beta) h^\beta,$$

and, analogously, (6.2) follows from (3.4). The assertion 2) of the theorem follows from Theorem 5.4. Assertion 1) gives the validity of condition B.3. So, it is left to check that $\widetilde{\varepsilon}_0 = \sup_{(t,x)\in K} \tau_\varphi(u(t,x)) < \infty$. We have indeed

$$\widetilde{\varepsilon}_0 \le c_\eta \Big(\mathsf{E}|u(t,x)|^2\Big)^{1/2} = c_\eta \Big(\int_{\mathbb{R}} F(d\lambda)\Big)^{1/2} < \infty$$

Assertion 3) follows from Theorem 3 in [10]. □

Corollary 6.2. *Let the process* η *be Gaussian. Then* $\widetilde{\varepsilon}_0 = (B_\eta(0))^{1/2} = (\int_{\mathbb{R}} dF(\lambda))^{1/2}$ *and*

$$P\Big\{ \sup_{(t,x)\in K} |u(t,x)| > v \Big\} \le 2\exp\Big\{ -\frac{v^2(1-\theta)^2}{2\widetilde{\varepsilon}_0^2}\Big\} \varkappa (c(\beta) e)^{2/\beta} (\theta\widetilde{\varepsilon}_0)^{-2/\beta}. \tag{6.5}$$

Example 6.3. An important natural generalization of Gaussian processes is obtained with $\varphi(x) = \frac{|x|^\alpha}{\alpha}$, $1 < \alpha \le 2$. For this case $\varphi^*(x) = \frac{|x|^\gamma}{\gamma}$, where $\gamma \ge 2$, and $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$. For such φ-sub-Gaussian initial data the exponential term in (6.3) takes the form $\exp\big\{ -(u^\gamma(1-\mu)^\gamma)/(\gamma\tilde{\varepsilon}_0^\gamma)\big\}$. We can also conclude from assertion 3) of Theorem 6.1 that the solution u is sample continuous with probability 1. Indeed, in this case we have that the right hand side of the formula (6.4) tends to 0 for $h \to 0$, then $P\Big\{ \sup_{|t-s|\le h, |x-y|\le h} |u(t,x)-u(s,y)| > v\Big\} \to 0$. Therefore, as $h \to 0$, $\sup_{|t-s|\le h, |x-y|\le h} |u(t,x)-u(s,y)| \to 0$ in probability, but also (due to the monotonicity of the supremum) with probability 1.

Remark 6.4. In [16] similar results on the distribution of suprema were obtained for solutions to higher-order linear dispersive equations with harmonizable φ-sub-Gaussian initial data. Solutions therein were considered as classical solutions, that is, satisfying the corresponding equations with probability 1, under the appropriate set of conditions. Considering in the present paper the simpler case of the Airy equation with stationary initial conditions, we use the approach via second order analysis and treat the solutions in the mean square sense. The bounds and conditions are presented in such an explicit form, which is convenient for practical applications.

7 Examples and discussion

In this section we present several examples of processes which can be used as initial conditions. For these processes the condition (3.1) is satisfied, the constant $c(\beta)$ can be calculated in the closed form and the estimate (6.5) holds.

1. Consider the well-known and popular in various applications Matérn model.

Let $\eta(x)$, $x \in \mathbb{R}$ be a Gaussian stochastic process with the spectral density

$$f(\lambda) = \frac{\sigma^2}{(1+\lambda^2)^{2\alpha}},\ \lambda \in \mathbb{R}. \tag{7.1}$$

The corresponding covariance function is of the form:

$$B_\eta(x) = \frac{\sigma^2}{\sqrt{\pi}\Gamma(2\alpha)}\Big(\frac{|x|}{2}\Big)^{2\alpha-1/2} K_{2\alpha-1/2}(|x|),\ x \in \mathbb{R}, \tag{7.2}$$

where K_ν is the modified Bessel function of the second kind, in particular, $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$. Covariances (7.2) have a parameter $\nu = 2\alpha - 1/2 > 0$ that controls the level of smoothness of the stochastic process. Matérn class comprises a broad range of covariances, e.g., exponential covariance, which we consider in our next example.

The Gaussian stochastic process with the above covariance and spectral density can be obtained as solution to the following stochastic fractional differential equation:

$$\Big(1 - \frac{d^2}{dx^2}\Big)^\alpha \eta(x) = w(x),\ x \in \mathbb{R}, \tag{7.3}$$

with w being a white noise: $\mathsf{E}w(x) = 0$ and $\mathsf{E}w(x)w(y) = \sigma^2\delta(x-y)$ (see, for example, [8, Theorem 3.1]). Note that the relation between the Matérn covariance in $\mathbb{R}^d$, $d \geq 1$, and the corresponding stochastic fractional equation was established by Whittle in 1963.

Consider the initial value problem (2.1)–(2.2) with initial data η represented as solution to equation (7.3). Due to the form of the spectral density (7.1), we are able to calculate the constant $c(\beta)$ which is given by (3.3) and appears in the estimates (3.2), (6.1), (6.3). We have

$$\int_{\mathbb{R}} \lambda^{2\beta}(1+\lambda^2)^{2\beta-2\alpha} = \int_0^\infty \frac{t^{\beta+1/2-1}}{(1+t)^{\beta+1/2+2\alpha-3\beta-1/2}}\,dt = \mathcal{B}(\beta+1/2, 2\alpha-3\beta-1/2),$$

where $\mathcal{B}$ is the Beta-function, $\beta \in (0,1]$, $2\alpha - 3\beta - 1/2 > 0$ and we used the formula $\int_0^\infty \frac{t^{\mu-1}}{(1+t)^{\mu+\nu}}\,dt = \mathcal{B}(\mu,\nu)$.

Therefore, in this case we obtain $c(\beta) = 2^{1-\beta}(\mathcal{B}(\beta+1/2, 2\alpha-3\beta-1/2))^{1/2}$. In particular, having in (7.1) $\alpha > 1$ and choosing $\beta = 1/2$ we get $c(1/2) = 1/(\alpha-1)$.

2. Consider a stationary Gaussian Ornstein-Uhlenbeck process η defined by the equation

$$d\eta(x) = -\eta(x)dx + \gamma dW(x),\ x \in \mathbb{R}, \tag{7.4}$$

where W is a Brownian motion or Wiener process with $\mathsf{E}W(x) = 0$, $\mathrm{Var}W(x) = |x|$, $x \in \mathbb{R}$. Stationary Gaussian solution to (7.4) has the following covariance function and spectral density:

$$B_\eta(x) = \frac{\gamma^2}{2}e^{-|x|},\ x \in \mathbb{R},\ f(\lambda) = \frac{\gamma^2}{2\pi(1+\lambda^2)},\ \lambda \in \mathbb{R}.$$

These covariance and spectral density are particular cases of those considered in the previous example, and the calculations for $c(\beta)$ are valid as well.

3. Ornstein–Uhlenbeck equation driven by a fractional Brownian motion. Consider the linear stochastic differential equation

$$d\eta(x) = -\eta(x)dx + dW_H(x),\ x \in \mathbb{R}, \tag{7.5}$$

where W_H, $\frac{1}{2} < H < 1$, is a fractional Brownian motion, that is, a zero mean Gaussian process with $W_H(0) = 0$, stationary increments and covariance

$$\mathrm{Cov}(W_H(x), W_H(y)) = \frac{c}{2}\big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\big),\ x, y \in \mathbb{R},$$

where $c = \mathrm{Var} W_H(1)$. One can show that there exists a unique continuous solution of equation (7.5) in the form

$$\eta(x) = \int_{-\infty}^{x} e^{-(x-y)}\, dW_H(y), x \in \mathbb{R},$$

which is a stationary Gaussian process with the spectral density

$$f(\lambda) = \frac{\sigma^2}{1+\lambda^2}|\lambda|^{1-2H},\ \lambda \in \mathbb{R},$$

where $\sigma^2 = c\Gamma(2H+1)\sin(\pi H)/(2\pi)$ (see, e.g., [4]). Similar to example 1, we calculate

$$\int_{\mathbb{R}} \frac{\lambda^{2\beta}(1+\lambda^2)^{2\beta}}{1+\lambda^2}|\lambda|^{1-2H} = \mathcal{B}(\beta+1-H, H-3\beta),$$

where we should choose $\beta < H/3$. Therefore, $c(\beta) = 2^{1-\beta}(\sigma^2\mathcal{B}(\beta+1-H, H-3\beta))^{1/2}$.

We discuss some possible extensions of the obtained results for further research.

Remark 7.1. The results obtained can be extended to higher-order and fractional Airy equations. Namely, one can consider the equations

$$\frac{\partial u}{\partial t} = (-1)^n \frac{\partial^{2n+1} u}{\partial^{2n+1} x},\ t>0,\ x \in \mathbb{R}, n \in N, \quad \text{and} \quad \frac{\partial u}{\partial t} = \mathcal{D}_x^{\alpha} u,\ t>0,\ x \in \mathbb{R},\ \alpha > 1,$$

where $\mathcal{D}_x^{\alpha}$ represents the Riesz–Feller fractional derivative (see [19]). Formulas for the fundamental solutions are available in the form (see, e.g., [19] and references therein):

$$g_\alpha(t,x) = \frac{1}{\pi}\int_0^\infty \cos(\gamma x + \gamma^\alpha t)\, d\gamma = \frac{1}{(\alpha t)^{1/\alpha}}\mathrm{Ai}_\alpha\Big(\frac{x}{(\alpha t)^{1/\alpha}}\Big),\ t>0,\ x \in \mathbb{R},$$

with the generalized Airy function $\mathrm{Ai}_\alpha(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\gamma x + \frac{\gamma^\alpha}{\alpha}\right) d\gamma,\ x \in \mathbb{R},\ \alpha > 1.$ The derivations of previous sections can be adjusted for these equations.

Remark 7.2. One of the main classical tools in studying boundedness properties of Gaussian processes is metric entropy integral estimates due to Dudley, which were extended to wider classes of processes in the monograph [7] (see also references therein). Results on the bounds for distribution of supremum for random fields in Sections 5, 6 were derived basing on this approach. A number of approaches and techniques have been developed for deriving approximations for excursion probabilities (tail probabilities) $\mathsf{P}\big\{\sup_{t\in\mathbf{T}} X(t) \geq u\big\}$ for $u \to \infty$ (see, e.g., [1], [2] and references therein). An interesting question for future research would be to compare the obtained in the present paper bounds with corresponding asymptotic results, in particular, for the examples presented above and using numerical methods.

Remark 7.3. Another approach to the investigation of solutions of PDE subject to φ-sub-Gaussian random initial conditions was developed in [13, 18] and some others. For particular classes of equations, conditions were obtained under which solutions can be given by series representations and methods of approximations of solutions by means of partial sums of the corresponding series were presented, as well as conditions of convergence of the approximations in different functional spaces. It would be useful from the practical point of view to apply the mentioned methods for further research of considered in the present paper equations and, in particular, to use for initial condition a non-Gaussian process of the form presented in Example 5.7.

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