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ON THE RECURSIVE SEQUENCE $x_{n+1} = \frac{A}{\prod_{i=0}^{k}} + \frac{1}{\prod_{j=k+2}^{2(k+1)} x_{n-j}}$

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Abstract. In [6] the authors proposed two open problems concerning the boundedness and the periodic nature of positive solutions of the nonlinear difference equation in the title. In this paper we prove a global covergence result and solve the open problems in the case $A > 1$.

1. Introduction and Basic Observations

Recently there has been a lot of interest in studying the global attractivity, the boundedness character and the periodic nature of nonlinear difference equations. For some recent results see, for example, [1-5], [7-21].

In [5] the authors established that every positive solution of the difference equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n = 0, 1, \dots, \tag{1}$$

where $A \in (0, \infty)$, converges to a periodic two solution.

For closely related results concerning, among other problems, the periodic nature of scalar nonlinear difference equation see, for example, [1], [2], [7-12], [15], [18-20] and the references cited therein. In [7] and [16] two closely related global convergence results were established which can be applied to nonlinear difference equations in proving that every solution of these difference equations converges to a period-two solution (which is not the same for all solutions).

We believe that nonlinear rational difference equations are of great importance in their own right and furthermore results about such equations offer prototypes

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towards the development of the basic theory of the global behavior of nonlinear difference equations.

In [6] the authors proposed to investigate the behaviour of the solutions of the following, closely related to Eq. (1), difference equation:

$$x_{n+1} = \frac{A}{\prod_{i=0}^{k} x_{n-i}} + \frac{1}{\prod_{j=k+2}^{2(k+1)} x_{n-j}}, \quad n = 0, 1, \dots, \tag{2}$$

where $k = 0, 1, \dots$, and $x_{-2(k+1)}, \dots, x_{-1}, x_0, A \in (0, \infty)$.

It can be see that Eq. (2) has infinitely many solutions of period $k+2$ of the form

$$p_1,\ p_2,\ p_3,\ \dots,\ p_{k+2},\ p_1,\ p_2, \dots$$

where $p_1\ p_2\ \dots\ p_{k+2} = A + 1$.

Following [6], we can apply the change $y_n = x_n x_{n-1} x_{n-2} \dots x_{n-k-1}$ in Eq. (2) and obtain the equation

$$y_{n+1} = A + \frac{y_n}{y_{n-(k+1)}}, \quad n = 0, 1, \dots, \tag{3}$$

where $k = 0, 1, \dots$, and $y_{-(k+1)}, \dots, y_{-1}, y_0, A \in (0, \infty)$.

Note that Eq. (3) has a unique positive equilibrium, namely $\bar{y} = A + 1$. It is easy to see that the following equation represents the linearized equation of Eq. (3) near the equilibrium

$$z_{n+1} - \frac{1}{A+1} z_n + \frac{1}{A+1} z_{n-k-1} = 0.$$

In the case $A > 1$, all the zeros of the characteristic polynomial lies in the unit disc. Indeed, we have

$$\left| \frac{t^{k+1} - 1}{A+1} \right| < |t|^{k+2}, \quad \text{for} \quad t \in \{z \mid |z| = 1\},$$

from which the result follows by Rouchet's theorem.

Thus by the linearized stability result (see [11]), the positive equilibrium of Eq. (3) is a locally asymptotically stable (attractor) in this case.

The following conjectures and open problems were posed in [6].

Conjecture 1. *Show that every positive solution of Eq.* (2) *is bounded and persists.*

Open problem 1. *Obtain necessary and sufficient conditions for the global asymptotic stability of the positive equilibrium of Eq.* (3).

Open problem 2. *Obtain necessary and sufficient conditions under which every positive solution of Eq.* (2) *converges to a period* $k+2$ *solution.*

In this paper we confirm the conjecture and solve the open problems in the case $A > 1$.

We say that a solution (x_n) of a difference equation is *bounded and presists* if there exist positive constants P and Q such that

$$P \cdot \; x_n \cdot \; Q \quad \text{for} \quad n = -1, 0, \ldots \; .$$

A *positive semicycle* of a solution (x_n) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to $\bar{x}$, with $l \geq -1$ and $m \cdot \; \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} < \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}$$

A *negative semicycle* of a solution (x_n) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than to $\bar{x}$, with $l \geq -1$ and $m \cdot \; \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}$$

The first semicycle of a solution starts with the term x_{-1} and is positive if $x_{-1} \geq \bar{x}$ and negative if $x_{l-1} < \bar{x}$

Following [17], we say that the real number $\bar{x}$ is a geometrically global attractor for some difference dquation DE if for each solution (x_n) of the DE there exist $L \in \mathbf{R}_+$ and $\theta \in [0, 1)$ such that

$$|x_n - \bar{x}| \cdot \; L\theta^n \quad \text{for all} \quad n \in \mathbf{N}.$$

The following lemma was proved in [17]:

Lemma A. *Let* (a_n) *be a sequence of positive numbers which satisfies the inequality*

$$a_{n+k} \cdot \; A \max\{a_{n+k-1}, a_{n+k-2}, \ldots, a_n\} \;\; for \;\; n \in \mathbf{N},$$

where $A \in (0, 1)$ *and* $k \in \mathbf{N}$ *are fixed. Then there exist* $L \in \mathbf{R}_+$ *such that*

$$a_{km+r} \cdot \; L\,A^m \;\; for\; all \;\; m \in \mathbf{N} \cup \{0\} \;\; and \;\; 1 \cdot \; r \cdot \; k.$$

Corollary A. *Let (a_n) be the sequence of positive numbers in Lemma A. Then there exists $M > 0$ such that*

$$a_n \cdot \quad M\left(\sqrt[k]{A}\right)^n.$$

2. A Global Convergence Result

In this section we present a global convergence result. By this result we confirm the conjecture in the case $A > 1$.

Theorem 1. *Consider the difference equation*

$$x_{n+1} = \alpha + \frac{x_n}{g(x_{n-k})}, \quad n = 0, 1, 2, \dots \tag{4}$$

where $\alpha > 1, k \in \mathbf{N}$, the initial conditions $x_{-k}, x_{-k+1}, \dots, x_{-1}$ and x_0 of Eq. (4) are arbitrary positive numbers and where $g(x)$ is a positive continuous real function defined on the interval $(0, \infty)$ which satisfies the following conditions:

(a) $g(x) > 1$ *for* $x > 1$;

(b) $\alpha g(x) - x$ *is increasing*;

(c) $x/g(x)$ *is nondecreasing.*

Then every positive solution of Eq. (4) converges.

Proof. First we prove that Eq. (4) has the unique positive equilibrium. The equlibrium points $\bar{x}$ of Eq. (4) satisfy the equation

$$\bar{x} = \alpha + \frac{\bar{x}}{g(\bar{x})}.$$

Let $F(x) = x - \alpha - \frac{x}{g(x)}$. It is clear that F is a contimous function on $[0, \infty)$ such that $F(0) = -\alpha < 0$ and $\lim_{x \longrightarrow +\infty} F(x) = +\infty$, because from (b) it follows that $g(x)$ is increasing. Thus it follows that there is an $x^* \in (0, \infty)$ such that $F(x^*) = 0$. On the other hand

$$\begin{aligned} F(x) - F(y) &= x - y + \frac{y}{g(y)} - \frac{x}{g(x)} \\ &= \frac{(x-y)g(y)(g(x)-1) + y(g(x)-g(y))}{g(x)g(y)} > 0, \end{aligned}$$

if $x > y$. So $F(x)$ is an increasing function and consequently x^* is the unique positive equilibrium of Eq. (4).

Further, we prove that every possitive solution of Eq. (4) is bounded. From (4) we can see that $x_n > \alpha$ for $n \geq 1$. Thus we obtain

$$
\begin{aligned}
x_{n+1} &= \alpha + \frac{x_n}{g(x_{n-k})} < \alpha + \frac{x_n}{g(\alpha)} \\
&= \alpha + \frac{x_n}{\beta}, \qquad n = k+1, k+2, \dots ,
\end{aligned} \tag{5}
$$

where $\beta = g(\alpha) > 1$.

From (5) using induction we obtain

$$
\begin{aligned}
x_{n+1} &< \frac{x_{k+1}}{\beta^{n-k}} + \alpha \left(1 + \frac{1}{\beta} + \cdots + \frac{1}{\beta^{n-k-1}} \right) \\
&< \frac{x_{k+1}}{\beta^{n-k}} + \frac{\alpha\beta}{\beta - 1},
\end{aligned}
$$

from which the boundedness follows.

Thus $\liminf_{n\to\infty} x_n = l$ and $\limsup_{n\to\infty} x_n = L$ are finite, moreover $l > 1$. Letting $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$ in (4) we obtain

$$
l \geq \alpha + \frac{l}{g(L)} \quad \text{and} \quad L \cdot \;\; \alpha + \frac{L}{g(l)}.
$$

From this and by (c) we obtain

$$
\alpha g(l) + L \geq L g(e) \geq l g(L) \geq \alpha g(L) + l.
$$

Since $\alpha > 1$ and by (b) we obtain $l = L$, as desired.

From Theorem 1 we obtain the following corollary.

Corollary 1. *Assume that $A > 1$. Then the positive equilibrium $A + 1$ of Eq. (3) is a globally asymptotically attractoor.*

Proof. Setting $g(x) = x$ and replace k by $k + 1$ in Theorem 1.

Remark 1. Corollary 1 solves Open problem 1 in the case $A > 1$.

Similarly, we can prove the following theorem. The proof will be omitted.

Theorem 1 a). *Consider the difference equation*

$$
x_{n+1} = \alpha + \frac{x_{n-k}}{g(x_n)}, \quad n = 0, 1, 2, \dots \tag{6}
$$

where $\alpha > 1, k \in \mathbf{N}$, the initial conditions $x_{-k}, x_{-k+1}, \dots , x_{-1}$ and x_0 of Eq. (6) are arbitrary positive numbers and where $g(x)$ is a continuous positive real function defined on the interval $(0, \infty)$ which satisfies the following conditions :

(a) $g(x) > 1$ *for* $x > 1$;

(b) $\alpha g(x) - x$ *is increasing*;

(c) $x/g(x)$ *is nondecreasing.*

Then every positive solution of Eq. (6) *converges.*

3. Boundedness and Global Attractivity of Eq. (3)

In contrast to convergence result on Eq. (3) we prove that the all solutions of some generalized equation are bounded.

Theorem 2. *Let* (y_n) *be a nontrivial positive solution of the difference equation*

$$y_{n+1} = \alpha_n + \frac{y_n}{y_{n-(k+1)}}, \quad n = 0, 1, \dots, \tag{7}$$

where $y_{-(k+1)}, \dots, y_{-1}, y_0 \in (0, \infty)$ *and* α_n *is a sequence which satisfies the following condition*

$$m < \alpha_n < M, \quad n = 0, 1, \dots, \tag{8}$$

for some $m, M \in (0, \infty)$*. Then* (y_n) *is bounded and persists.*

Proof. From (7) and (8) it follows that $y_n > m$ for all $n = 1, 2, \dots.$ On the other hand, from (7) we obtain

$$\begin{aligned}
y_{n+1} &= \alpha_n + \frac{y_n}{y_{n-(k+1)}} = \alpha_n + \frac{\alpha_{n-1} + \frac{y_{n-1}}{y_{n-(k+2)}}}{y_{n-(k+1)}} \\
&= \alpha_n + \frac{\alpha_{n-1}}{y_{n-(k+1)}} + \frac{y_{n-1}}{y_{n-(k+1)} y_{n-(k+2)}} \\
&= \alpha_n + \frac{\alpha_{n-1}}{y_{n-(k+1)}} + \frac{\alpha_{n-2} + \frac{y_{n-2}}{y_{n-(k+3)}}}{y_{n-(k+1)} y_{n-(k+2)}} \\
&= \cdots\cdots\cdots \\
&= \alpha_n + \frac{\alpha_{n-1}}{y_{n-(k+1)}} + \frac{\alpha_{n-2}}{y_{n-(k+1)} y_{n-(k+2)}} + \cdots \\
&\quad \cdots + \frac{\alpha_{n-(k+1)}}{y_{n-(k+1)} \cdots y_{n-(2k+1)}} + \frac{1}{y_{n-(k+2)} \cdots y_{n-(2k+2)}} \\
&< M + \frac{M}{m} + \frac{M}{m^2} + \cdots + \frac{M}{m^{k+1}} + \frac{1}{m^{k+1}},
\end{aligned}$$

as desired.

Then main result in this section is the following.

Theorem 3. *Assume that $A > 1$. Then the positive equilibrium $A+1$ is a geometrically global attractor of all positive solutions of Eq.* (3).

Proof. Setting $y_n = z_n + \bar{y}$ we obtain

$$z_{n+1} = \frac{z_n - z_{n-(k+1)}}{z_{n-(k+1)} + \bar{y}}.$$

By Theorem 1, $z_n \to 0$ as $n \to \infty$. Thus we have $|z_n| < \varepsilon$ for $n \geq n_0(\varepsilon)$. First, let us choose $\varepsilon > 0$ such that $A+1-\varepsilon > 2$. Hence for sufficiently large n, for example $n \geq n_1$, we have

$$|z_{n+1}| \cdot \frac{|z_n| + |z_{n-(k+1)}|}{A+1-\varepsilon} \cdot \frac{2}{A+1-\varepsilon} \max\{|z_n|, |z_{n-(k+1)}|\}$$

hence by Corollary A we obtain that there is an $L > 0$ such that

$$|z_n| \cdot L\left(\sqrt[k+2]{\frac{2}{A+1-\varepsilon}}\right)^n \quad \text{for all} \quad n = 0, 1, \ldots ,$$

from which the result follows.

4. Boundedness and Periodic Character of the Solutions of Eq. (2)

In this section we investigate the boundedness and the periodic character of the solutions of Eq. (2).

Theorem 4. *Assume that $A > 1$. Then every solution of Eq.* (2) *is bounded.*

Proof. By Corollary 1 we have that

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = 1.$$

By Theorem 3, for $\varepsilon > 0$ such that $A+1-\varepsilon > 2$ and sufficiently large n, we have

$$\begin{aligned} &\left|\frac{x_{n+1}}{x_{n-k-1}} - 1\right| = \left|\frac{y_{n+1}}{y_n} - 1\right| = \left|\frac{z_{n+1} + \bar{y}}{z_n + \bar{y}} - 1\right| = \left|\frac{z_{n+1} - z_n}{z_n + \bar{y}}\right| \\ &\qquad \cdot \frac{2\max\{|z_{n+1}|, |z_n|\}}{A+1-\varepsilon} \cdot L\left(\sqrt[k+2]{\frac{2}{A+1-\varepsilon}}\right)^n = L\theta^n. \end{aligned} \tag{9}$$

Thus
$$\left|\frac{x_{n+1}}{x_{n-k-1}}\right| = \left|1 + \frac{y_{n+1}}{y_n} - 1\right| \cdot \; 1 + L\theta^n,$$
i.e.
$$|x_{n+1}| \cdot \; |x_{n-k-1}|(1 + L\theta^n),$$
for sufficiently large n.

The infinite product $\prod_{i=1}^{\infty}(1 + L\theta^i)$ converges for $|\theta| < 1$, hence the result follows.

Remark 2. Theorem 4 confirms Conjecture 1 in the case $A > 1$.

Theorem 5. *Assume that* $A > 1$. *Then every positive solution of Eq.* (2) *converges to a period* $k + 2$ *solution.*

Proof. Let (x_n) be a nontrivial positive solution of Eq. (2), then by Theorem 4 we have that (x_n) is bounded and persists. Thus there is a constant $M > 0$ such that

(10) $$|x_n| < M \quad \text{for} \quad n = -2(k+1), -(2k+1), \ldots \, .$$

From (9) and (10) we obtain

(11) $$|x_{n+1} - x_{n-(k+1)}| \cdot \; ML\theta^n,$$

for sufficiently large n.

From (11) using Cauchy's criterion we obtain that the sequences $(x_{(k+2)n})$, $(x_{(k+2)n+1}) \ldots (x_{(k+2)n+k+1})$ converge, as desired.

Remark 3. Theorem 5 solves Open problem 2 in the case $A > 1$.

5. Local Stability When $0 < A \cdot \; 1$

In this section we present some examples of Eq. (3) when the unique equilibrium is unstable in the case $0 < A \cdot \; 1$.

As we already noted the following equation represents the linearized equation of Eq. (3) near the equilibrium

(12) $$z_{n+1} - \frac{1}{A+1} z_n + \frac{1}{A+1} z_{n-(k+1)} = 0.$$

Let k be an odd number i.e. $k = 2l - 1$, for some $l \in \mathbf{N}$. We can see that the characteristic polynomial $P_{k+2}(t)$, of Eq. (12) has the only one real root which belongs to the interval $(-1,0)$. Let t_1 be the real root and let $t_{2j}, t_{2j+1} = \bar{t}_{2j}$, $j = 1,\ldots,l$ be the remaining roots of the characteristic polynomial. Clearly

$$t_1 t_2 \cdots t_{k+2} = t_1 |t_2|^2 \cdots |t_{2l}|^2 = -\frac{1}{A+1}.$$

Thus if $t_{2j},\ t_{2j+1} = \bar{t}_{2j},\ j = 1,\ldots l$ lie in the unit disk then

$$t_1 \in \left(-1, -\frac{1}{A+1}\right).$$

Since

$$P\left(-\frac{1}{A+1}\right) = 1 - \frac{2}{(A+1)^{2l}},$$

we obtain that

$$t_1 \in \left(-1, -\frac{1}{A+1}\right) \quad \text{if} \quad A > \sqrt[2l]{2} - 1.$$

Hence by the linearized stability result (see [11]), we obtain the following theorem.

Theorem 6. *Assume $k = 2l-1, l \in \mathbf{N}$ and $A \in (0, \sqrt[2l]{2} - 1)$. Then $\bar{y}$ is an unstable equilibrium solution of Eq.* (3).

When $l = 1$, from the above we obtain more.

Theorem 7. (a) *Assume $A > \sqrt{2} - 1$. Then $\bar{y}$ is a locally asymptotically stable equilibrium solution of Eq.* (3).

(b) *Assume $A \in (0, \sqrt{2} - 1)$. Then $\bar{y}$ is an unstable equilibrium solution of Eq.* (3).

Remark 4. The case $k = 1, A \in [\sqrt{2} - 1, 1]$ has not yet been understood. We believe that when $A \in (\sqrt{2} - 1, 1]$, the positive equilibrium of Eq. (3) is globally asymptotically stable.

References

1. A. M. Amleh, E. A. Grove, G. Ladas and D. A. Georgiou, On the recursive sequence $y_{n+1} = \alpha + \frac{y_{n-1}}{y_n}$, *J. Math. Anal. Appl.* **233** (1999), 790-798.
2. R. C. De Vault, G. Dial, V. L. Kocic and G. Ladas, Global behaviour of solution of $x_{n+1} = ax_n + f(x_n, x_{n-1})$, *J. Differ. Equations Appl.* **4** (1998), 311-330.
3. R. De Vault, G. Ladas and S. W. Schultz, On the recursive sequence $x_{n+1} = \frac{A}{x_n^p} + \frac{B}{x_{n-1}^q}$, *Proceedings of the Second International Conference on Difference Equations* (Veszprem, Hungary, 1995), Gordon and Breach Science Publishers (to appear).

4. R. De Vault, G. Ladas and S. W. Schultz, Necessary and sufficient conditions for the boundedness of $x_{n+1} = \frac{A}{x_n^p} + \frac{B}{x_{n-1}^q}$, *J. Differ. Equations Appl.* **4**(3) (1998), 259-266.

5. R. De Vault, G. Ladas and S. W. Schultz, On the recursive sequence $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$, *Proc. Amer. Math. Soc.* **126**(11) (1998), 3257-61.

6. R. De Vault, G. Ladas and S. W. Schultz, Open problems and conjectures, On the recursive sequence $x_{n+1} = \frac{A}{x_n x_{n-1}} + \frac{1}{x_{n-3}x_{n-4}}$, *J. Differ. Equations Appl.* (2000) (to appear).

7. H. El-Metwally, E. A. Grove and G. Ladas, A global convergence result with applications to periodic solutions, *J. Math. Anal. Appl.* **245** (2000), 161-170.

8. C. H. Gibbons, M. R. S. Kulenović and G. Ladas, On the recursive sequence $x_{n+1} = \frac{\alpha+\beta x_{n-1}}{\gamma+x_n}$, *Math. Sci. Res. Hot-Line* **4**(2) (2000), 1-11.

9. C. H. Gibbons, M. R. S. Kulenović and G. Ladas, On the dynamics of $x_{n+1} = \frac{\alpha+\beta x_{n-1}+\gamma x_{n-1}}{A+Bx_n}$, *Proceedings of the Fifth International Conference on Difference Equations*, Temuco-Chile, Jan. 3-10, 2000, Gordon and Breach Science Publishers (to appear).

10. E. A. Grove, C. M. Kent, G. Ladas and S. Valicenti, Global stability in some population models, Proceedings of the Fourth International Conference on Difference Equations and Applications, August 27-31, 1999, Poznan, Poland, Gordon and Breach Science Publishers, (2000), (to appear).

11. V. L. Kocic and G. Ladas, *Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

12. M. R. S. Kulenović, G. Ladas and W. Sizer, On the recuursive sequence $y_{n+1} = \frac{\alpha y_n+\beta y_{n-1}}{\gamma y_n+C y_{n-1}}$, *Math. Sci. Res. Hot-Line*, **2**(5) (1998), 1-16.

13. Ch. G. Philos, I. K. Purnaras and Y. G. Sficas, Global attractivity in a nonlinear difference equation, *Appl. Math. Comput.* **62** (1994), 249-258.

14. S. Stević, A note on blunded sequences satisfying linear inequalities, *Indian J. Math.* **43**(2) (2001), 223-230.

15. S. Stević, On the recursive sequence $x_{n+1} = -\frac{1}{x_n} + \frac{A}{x_{n-1}}$, *Int. J. Mat. Mat. Sci.* **27**(1) (2001), 1-6.

16. S. Stević, A generalization of the Copson's theorem concerning sequences which satisfies linear inequality, *Indian J. Math.* **43**(3) (2001), 277-282.

17. S. Stević, Behaviour of the positive solutions of the generalized Beddington-Holt equation, *Panamer. Math. J.* **10** no. 4 (2000), 77-85.

18. S. Stević, A global convergence results with applications to periodic solutions, *Indian J. Pure Appl. Math.* **33**(1) (2002), 45-53.

19. S. Stević, On the recursive sequence $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$, *Appl. Math. Lett.* **15** (2002), 305-308.

20. S. Stević, On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$, *Taiwanese J. Math.* **6**(3) (2002), 405-414.

21. S. Stević, A note on the difference equation $x_{n+1} = \sum_{i=0}^{k} \frac{\alpha_i}{x_{n-i}^{p_i}}$, *J. Differ. Equations Appl.* **8**(7) (2002), 641-647.

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