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SHIFT PRESERVING OPERATORS ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract. We investigate shift preserving operators on locally compact abelian groups. We show that there is a one-to-one correspondence between shift preserving operators and range operators on $L^2(G)$ where G is a locally compact abelian group. We conclude that a shift preserving operator has several properties in common with its associated range operator, especially compactness of one implies compactness of the other. Moreover, we obtain a necessary condition for a shift preserving operator to be Hilbert Schmidt or of finite trace in terms of its range function.

1. Introduction and Preliminaries

A bounded linear operator $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is called shift preserving (which will be abbreviated to "SP") if $UT_k = T_k U$ for all $k \in \mathbb{Z}^n$, where T_k is the shift operator. As a special case of a shift operator is the time delay operator $T_k : l^2 \to l^2$ defined by $T_k u(n) = u(n-k),\ u \in l^2,\ k,\ n \in \mathbb{Z}$ where the action is to delay the signal u by k units. A digital filter U is a SP operator on l^2. In other words a filter is a time invariant operator in which delaying the input by k units of time is just to delay the output by k units. These operators play an important role in signal processing, such as to analyse, code, reconstruct signals and so on. They are often used to extract required frequency components from signals. For example, high frequency components of a signal usually contain the noise and fluctuations, which often have to be removed from the signal using different kinds of filters. For more details and examples of filters cf. [10, 4].

SP operators on $\mathbb{R}^n$ have been studied by Bownik in [3]. He gave a characterization of these operators in terms of range operators. Our goal in this paper is to investigate SP operators on locally compact abelian (which will be abbreviated

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to "LCA") groups. The major result in this paper is a novel characterization of SP operators on $L^2(G)$, where G is a LCA group. This allows us to handel SP operators (specially filters) on $L^2(G)$ in a unified manner. As an application of this approach, one is able to extend several results from the theory of filters on $\mathbb{R}^n$ to a general LCA group.

In what follows G is a LCA group with the dual group $\hat{G}$. We denote the Haar measure of G and $\hat{G}$ by dx and $d\xi$, respectively. As general references to the theory of LCA groups we mention [6, 9]. The Fourier transform $\hat{f}$ of any $f \in L^1(G)$ is defined by $\hat{f}(\xi) = \int_G f(x)\overline{\xi}(x)dx$, where ξ is an element in $\hat{G}$. The transformation $f \rightarrow \hat{f},\ L^1(G) \cap L^2(G) \rightarrow C_0(\hat{G})$ extends uniquely to a Hilbert space isomorphism from $L^2(G)$ onto $L^2(\hat{G})$, the so called Plancherel's transform. In the sequal, the Palncherel transform of a function $f \in L^2(G)$ will also be denoted by $\hat{f}$.

A subgroup L of G is called a uniform lattice if it is discrete and co-compact (i.e G/L is compact). A fundamental domain for L is a measurable subset $S_L \subseteq G$ such that every $x \in G$ can be uniquely written in the form $x = sd$ for some $s \in S_L$ and $d \in L$. It is shown in [13] that such a fundamental domain always exists in a second countable locally compact abelian group.

Let L be a uniform lattice in G. A bounded linear operator $U : L^2(G) \rightarrow L^2(G)$ is called SP (with respect to L) if $UT_k = T_kU$, for all $k \in L$, where T_k is the general shift operator $T_kf(x) = f(k^{-1}x)$. Let $L^\perp$ denote the annihilator of L in $\hat{G}$, i.e. the subgroup $\{\xi \in \hat{G};\ \xi(L) = \{1\}\}$, which is a uniform lattice in $\hat{G}$. Suppose $S_{L^\perp}$ is a fundamental domain for $L^\perp$. Let $L^2(S_{L^\perp}, l^2(L^\perp))$ denote the Hilbert space of square integrable functions from $S_{L^\perp}$ to $l^2(L^\perp)$ with inner product $< f, g >= \int_{S_{L^\perp}} < f(\xi), g(\xi) >_{l^2(L^\perp)} d\xi$. It is readily verified that this Hilbert space is isometrically isomorphic to $L^2(G)$. In fact the mapping $\mathcal{T} : L^2(G) \longrightarrow L^2(S_{L^\perp}, l^2(L^\perp))$, defined by $\mathcal{T}f(\xi) = (\hat{f}(\xi\eta))_{\eta \in L^\perp}$ is an isometric isomorphism between these two Hilbert spaces; [11] (see also [18, Proposition 1.3.2] for the case where $G = \mathbb{R}^n$).

A closed subspace $V \subseteq L^2(G)$ is called shift invariant (with respect to L) if $f \in V$ implies $T_kf \in V$, for any $k \in L$. For any subset $\phi \subseteq L^2(G)$, let $S(\phi) = \overline{span}\{T_k\varphi;\ \varphi \in \phi,\ k \in L\}$ be the shift invariant space generated by ϕ. For $\varphi \in L^2(G)$, $S(\{\varphi\})$ is the principle shift invariant space generated by φ and will be denoted by $S(\varphi)$. For a general orientation concerning shift invariant spaces on LCA groups the reader may consult [11, 12].

A range function is a mapping

$$J : S_{L^\perp} \rightarrow \{\text{closed subspaces of } l^2(L^\perp)\},$$

where $S_{L^\perp}$ is the fundamental domain for $L^\perp$ in $\hat{G}$. J is called measurable if the associated orthogonal projections $P(\xi) : l^2(L^\perp) \rightarrow J(\xi)$ are measurable in the sense that $\xi \mapsto < P(\xi)a, b >$ is measurable for all $a, b \in l^2(L^\perp)$. The following theorem which is [11, Theorem 3.1] asserts that there is a one to one correspondence between shift invariant subspaces of $L^2(G)$ and range functions.

Theorem 1.1. *Suppose G is a second countable LCA group, L is a uniform lattice in G, and $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\hat{G}$. A closed subspace $V \subseteq L^2(G)$ is shift invariant (with respect to the uniform lattice L) if and only if $V = \{f \in L^2(G),\ \mathcal{T}f(\xi) \in J(\xi)\ for\ a.e.\ \xi \in S_{L^\perp}\}$, where J is a measurable range function and $\mathcal{T}$ is the isometric isomorphism, between $L^2(G)$ and $L^2(S_{L^\perp}, l^2(L^\perp))$. The correspondence between V and J is one to one if the range functions are identified when they are equal a.e. Moreover, if $V = S(\phi)$, for some countable set $\phi \subseteq L^2(G)$ then*

$$J(\xi) = \overline{span}\{\mathcal{T}\varphi(\xi); \varphi \in \phi\}. \tag{1.1}$$

Suppose H is a Hilbert space. $X \subseteq H$ is called a *frame* (for $\overline{span}(X)$), if there exist two numbers A and B with $0 < A \leq B < \infty$ such that

$$A\|h\|^2 \leq \sum_{\eta \in X} |< h, \eta >|^2 \leq B\|h\|^2 \quad for \quad h \in span(X). \tag{1.2}$$

The numbers A and B are called the frame bounds. Those sequences which satisfy only the upper inequality in (1.2), are called Bessel sequences.

The following proposition which is [11, Theorem 4.1] shows the relation between $\{T_k\varphi;\ \varphi \in \phi,\ k \in L\}$ being a frame for $S(\phi)$ (Bessel sequence) and $\{\mathcal{T}\varphi(\xi);\ \varphi \in \phi\}$ being a frame for $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$ (Bessel sequence).

Proposition 1.2. *Suppose G is a second countable LCA group, L is a uniform lattice in G, $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\hat{G}$, $\phi \subseteq L^2(G)$ is a countable set. Then $\{T_k\varphi;\ \varphi \in \phi,\ k \in L\}$ is a frame (Bessel sequence with bound B) for $S(\phi)$ with bounds A and B if and only if $\{\mathcal{T}\varphi(\xi);\ \varphi \in \phi\}$ is a frame (Bessel sequence with bound B) for $J(\xi)$ with bounds A and B, for a.e. $\xi \in S_{L^\perp}$.*

We now would like to define the (generalized) Gramian and dual Gramian operators on LCA groups, which play important roles in the study of shift invariant spaces (see [16]).

Let ϕ be a countable subset of $L^2(G)$. Define the (generalized) Gramian and dual Gramian operators $G_\phi : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ and $\widetilde{G}_\phi : l^2(L^\perp) \to l^2(L^\perp)$ of the system $\{\mathcal{T}\varphi_n(\xi);\ n \in \mathbb{N}\}$ for some fixed $\xi \in S_{L^\perp}$ (or briefly "Gramian and dual Gramian operators associated with J") by $G_\phi := K^*K$ and $\widetilde{G}_\phi := KK^*$, where K denotes the operator defined as follows:

$$K : l^2(\mathbb{N}) \to l^2(L^\perp),\ K((c_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} c_n \mathcal{T}\varphi_n(\xi),$$

for $(c_n)_{n\in\mathbb{N}}$ with compact support, and K^* denotes the adjoint of K given by

$$K^* : l^2(L^\perp) \to l^2(\mathbb{N}),\ K^*((a_\eta)_{\eta\in L^\perp}) = (< (a_\eta)_{\eta\in L^\perp}, (\mathcal{T}\varphi_n(\xi)(\eta))_{\eta\in L^\perp} >)_{n\in\mathbb{N}}.$$

In fact we easily see

$$G_\phi = \left(\sum_{\eta\in L^\perp} \hat{\varphi}_n(\xi\eta)\overline{\hat{\varphi}_m}(\xi\eta)\right)_{m,n\in\mathbb{N}},$$

and

$$\widetilde{G}_\phi = \left(\sum_{n\in\mathbb{N}} \hat{\varphi_n}(\xi\eta)\overline{\hat{\varphi}_n}(\xi\gamma)\right)_{\eta,\gamma\in L^\perp}.$$

The following corollary is an immediate consequence of Proposition 1.2, which is also a generalization of [16, Theorem 2.2.7].

Corollary 1.3. *Retain the assumptions of Proposition 1.2. Suppose that G_ϕ and $\widetilde{G}_\phi$ are the (generalized) Gramian and dual Gramian operators associated with the system $\{\mathcal{T}\varphi_n(\xi);\ n\in\mathbb{N}\}$ for some fixed $\xi\in S_{L^\perp}$. Then*

(i) $\{T_k\varphi;\ \varphi\in\phi,\ k\in L\}$ *is a Bessel sequence with bound B if and only if*

$$\textit{esssup}_{\xi\in S_{L^\perp}} \|G_\phi(\xi)\| \le B.$$

(ii) $\{T_k\varphi;\ \varphi\in\phi,\ k\in L\}$ *is a frame for $S(\phi)$ with bounds A and B if and only if*

$$A\|a\|^2 \le < \widetilde{G}_\phi(\xi)a, a > \le B\|a\|^2,$$

for $a\in$ span$\{\mathcal{T}\varphi_n(\xi);\ n\in\mathbb{N}\}$, for a.e. $\xi\in S_{L^\perp}$.

For a comprehensive account of shift invariant spaces on $L^2(\mathbb{R}^n)$ see [2, 16, 17].

Now suppose that G is a second countable LCA group, L is a uniform lattice in G, $S_{L^\perp}$ is a fundamental domain for $L^\perp$, V is a shift invariant subspace of $L^2(G)$ with the associated range function J, and $P(\xi)$ is the projection onto $J(\xi)$, for $\xi\in S_{L^\perp}$. A range operator on J is a mapping R from the fundamental domain $S_{L^\perp}$ to the set of bounded linear operators on closed subspaces of $l^2(L^\perp)$, so that the domain of $R(\xi)$ equals $J(\xi)$ for a.e. $\xi\in S_{L^\perp}$. R is called measurable if $\xi\mapsto < R(\xi)P(\xi)a, b >$ is a measurable scalar function for all $a, b\in l^2(L^\perp)$.

The rest of this paper is organized as follows:

In Section 2 we show that there is a one to one correspondence between SP operators on a LCA group and range operators defined as above. In section 3 we obtain some consequences of this characterization theorem. We conclude that a shift preserving operator has several properties in common with its associated range operator. We show that a shift preserving operator is an isometry (self adjoint) if and only if its corresponding range operator is an isometry (self adjoint). Moreover, we obtain a necessary condition for a shift preserving operator to be compact, Hilbert Schmidt or of finite trace in terms of its corresponding range operator. Finally we determine the range operator associated with the frame operator generated by shifts.

2. A Characterization of Shift Preserving Operators

Throughout this paper we always assume that G is a second countable LCA group, L is a uniform lattice in G, $S_{L^\perp}$ is a fundamental domain for $L^\perp$ and U is a SP operator on $L^2(G)$. The notation will be as in the previous section.

In this section we generalize a characterization of SP operators in terms of range operators. The main result is the following theorem:

Theorem 2.1. (The Characterization Theorem). *Suppose $V \subseteq L^2(G)$ is a shift invariant space and J is its associated range function. For every SP operator $U : V \to L^2(G)$, there exists a measurable range operator R on J such that*

$$(2.1) \qquad (\mathcal{T} \circ U)f(\xi) = R(\xi)(\mathcal{T}f(\xi)) \ \ \textit{for } a.e.\ \xi \in S_{L^\perp}, \ \textit{for all } f \in V,$$

where $\mathcal{T}$ is the isometric isomorphism between $L^2(G)$ and $L^2(S_{L^\perp}, l^2(L^\perp))$. Conversely, given a measurable range operator R on J with ess sup$_{\xi \in S_{L^\perp}} \|R(\xi)\| < \infty$, there is a bounded SP operator $U : V \to L^2(G)$, such that (2.1) holds. The correspondence between U and R is one-to-one under the usual convention that the range operators are identified if they are equal a.e.

An immediate consequence of Theorem 2.1 is [3, Theorem 4.5] which is obtained by putting $G = \mathbb{R}^n,\ L = \mathbb{Z}^n,\ L^\perp = \mathbb{Z}^n,\ S_{L^\perp} = \mathbb{T}^n$ in Theorem 2.1.

Before proving Theorem 2.1, we need some preparations. Also in the proof of this theorem we use some results of [12] that for the readers' convenience we state them here.

Let $\varphi \in L^2(G)$. We say $\varphi_0 \in L^2(G)$ is a Parseval frame generator of $S(\varphi)$ if $\sum_{k \in L} | < T_k\varphi_0, f > |^2 = \|f\|^2$, for all $f \in S(\varphi)$.

Suppose $\mathcal{T}$ is the isometric isomorphism, between $L^2(G)$ and $L^2(S_{L^\perp}, l^2(L^\perp))$. In [12] we have characterized all Parseval frame generators of $S(\varphi)$ as follows.

Proposition 2.2. *Let $\varphi \in L^2(G)$. Then φ is a Parseval frame generator of $S(\varphi)$, if and only if $\|\mathcal{T}\varphi(\xi)\|^2_{l^2(L^\perp)} (= \sum_{\eta \in L^\perp} |\hat{\varphi}(\xi\eta)|^2) = \chi_{\Omega_\varphi}(\xi),\ for\ a.e.\ \xi \in S_{L^\perp}$, where $\Omega_\varphi = \{\xi \in S_{L^\perp};\ \mathcal{T}\varphi(\xi) \neq 0\}$.*

Also we have shown the existence of a decomposition of a shift invariant subspace of $L^2(G)$ into an orthogonal sum of spaces each of which is generated by a single function whose shifts form a Parseval frame.

Proposition 2.3. *Let G be a second countable locally compact abelian group. If V is a shift invariant subspace in $L^2(G)$, then there exists a family of functions $\{\varphi_n\} \subseteq L^2(G)$ such that $V = \bigoplus_{n=1}^{\infty} S(\varphi_n)$, where each φ_n is a Parseval frame generator of $S(\varphi_n)$, for every $n \in \mathbb{N}$.*

Moreover, in the proof of Theorem 2.1 we need to determine how the information about orthogonality of $S(\varphi_1)$ and $S(\varphi_2)$ can be transferred into some other information about the generators φ_1 and φ_2 in $L^2(G)$. We have done this in [12] via the following proposition.

Proposition 2.4. *The spaces $S(\varphi_1)$ and $S(\varphi_2)$ are orthogonal if and only if*

$$\sum_{\eta \in L^{\perp}} \hat{\varphi}_1(\xi\eta)\overline{\hat{\varphi}_2}(\xi\eta) = 0 \quad a.e.\ \xi \in \hat{G}.$$

Remark 2.5. If φ is a Parseval frame generator of $S(\varphi)$ and $m \in L^2(S_{L^\perp})$ then obviously $m\mathcal{T}\varphi \in L^2(S_{L^\perp}, l^2(L^\perp))$. Indeed, by Proposition 2.2 we have

$$\begin{aligned} \|m\mathcal{T}\varphi\|^2 &= \int_{S_{L^\perp}} \|m\mathcal{T}\varphi(\xi)\|^2 d\xi \\ &= \int_{S_{L^\perp}} |m(\xi)|^2 \|\mathcal{T}\varphi(\xi)\|^2 d\xi \\ &= \int_{S_{L^\perp}\cap\Omega_\varphi} |m(\xi)|^2 d\xi \\ &\le \|m\|^2 < \infty. \end{aligned}$$

The following lemma will be needed in the proof of Theorem 2.1.

Lemma 2.6. *Let $\varphi \in L^2(G)$ be a Parseval frame generator of $S(\varphi)$, and $U : S(\varphi) \to L^2(G)$ be a SP operator. Then for every $m \in L^2(S_{L^\perp})$,*

$$(\mathcal{T} \circ U \circ \mathcal{T}^{-1})(m\mathcal{T}\varphi)(\xi) = m(\xi)(\mathcal{T} \circ U)\varphi(\xi) \ for\ a.e.\ \xi \in S_{L^\perp} \tag{2.2}$$

(*Note that by Remark 2.5 the left hand side of* (2.2) *is well defined*).

Proof. First we show that (2.2) holds for polynomials. For $k \in L$, define $M_k \in L^2(S_{L^\perp})$ by $M_k(\xi) = \overline{\xi}(k)$, for $\xi \in S_{L^\perp}$ and note that $M_k\mathcal{T}\varphi = \mathcal{T}T_k\varphi$, a.e. for all $k \in L$. Then we have

$$\begin{aligned} (\mathcal{T} \circ U \circ \mathcal{T}^{-1})(M_k\mathcal{T}\varphi) &= (\mathcal{T} \circ U \circ \mathcal{T}^{-1})(\mathcal{T}T_k\varphi) \\ &= (\mathcal{T} \circ U \circ T_k)\varphi \\ &= (\mathcal{T} \circ T_k \circ U)\varphi \\ &= M_k(\mathcal{T} \circ U)\varphi. \end{aligned}$$

So, by linearity (2.2) holds for all polynomials $p(\xi) = \sum_{k\in L} a_k \overline{\xi}(k) \in L^2(S_{L^\perp})$. Moreover,

$$
\begin{aligned}
\int_{S_{L^\perp}} |p(\xi)|^2 \|(\mathcal{T}\circ U)\varphi(\xi)\|^2 d\xi &= \int_{S_{L^\perp}} \|(\mathcal{T}\circ U\circ\mathcal{T}^{-1})(p\mathcal{T}\varphi)(\xi)\|^2 d\xi \\
&= \|(\mathcal{T}\circ U\circ\mathcal{T}^{-1})(p\mathcal{T}\varphi)\|^2 \\
&\leq \|(\mathcal{T}\circ U\circ\mathcal{T}^{-1})\|^2 \|p\mathcal{T}\varphi\|^2 \\
&= \|U\|^2 \int_{S_{L^\perp}} |p(\xi)|^2 \|\mathcal{T}\varphi(\xi)\|^2 d\xi.
\end{aligned} \tag{2.3}
$$

Let $r \in L^\infty(\hat{L}) \subseteq L^2(\hat{L})$. Lusin's Theorem, together with compactness of $\hat{L}$ [6, Proposition 4.4], imply that there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials such that $p_n(\xi) \longrightarrow r(\xi)$ as $n \longrightarrow \infty$, for a.e. $\xi \in \hat{L}$. By the Lebesgue Dominated Convergence Theorem, and the fact that $L^2(\hat{L})$ is isometrically isomorphic to $L^2(S_{L^\perp})$ [13], (2.3) implies that, $\int_{S_{L^\perp}} |r(\xi)|^2 \|(\mathcal{T}\circ U)\varphi(\xi)\|^2 d\xi \leq \|U\|^2 \int_{S_{L^\perp}} |r(\xi)|^2 \|(\mathcal{T}\varphi(\xi)\|^2 d\xi$. Since $r \in L^\infty(\hat{L})$ was arbitrary, we get

$$\|(\mathcal{T}\circ U)\varphi(\xi)\| \leq \|U\| \|\mathcal{T}\varphi(\xi)\|, \text{ for } a.e.\ \xi \in S_{L^\perp}. \tag{2.4}$$

Now, let $m \in L^2(S_{L^\perp})$. Then there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials such that $p_n \longrightarrow m$ in $L^2(S_{L^\perp})$. Also $p_n\mathcal{T}\varphi \longrightarrow m\mathcal{T}\varphi$, and since $\mathcal{T}\circ U\circ\mathcal{T}^{-1}$ is continuous, $(\mathcal{T}\circ U\circ\mathcal{T}^{-1})(p_n\mathcal{T}\varphi) \longrightarrow (\mathcal{T}\circ U\circ\mathcal{T}^{-1})(m\mathcal{T}\varphi)$. On the other hand, passing to a subsequence we have

$$p_n(\xi) \longrightarrow m(\xi) \text{ for } a.e.\ \xi \in S_{L^\perp}, \tag{2.5}$$

and

$$(\mathcal{T}\circ U\circ\mathcal{T}^{-1})p_n\mathcal{T}\varphi(\xi) \longrightarrow (\mathcal{T}\circ U\circ\mathcal{T}^{-1})m\mathcal{T}\varphi(\xi) \text{ for } a.e.\ \xi \in S_{L^\perp}. \tag{2.6}$$

Moreover, (2.4) and (2.5) imply that

$$p_n(\xi)(\mathcal{T}\circ U)\varphi(\xi) \longrightarrow m(\xi)(\mathcal{T}\circ U)\varphi(\xi) \text{ for } a.e.\ \xi \in S_{L^\perp}. \tag{2.7}$$

Since (2.2) holds for all polynomials, (2.6) and (2.7) imply that $(\mathcal{T}\circ U\circ\mathcal{T}^{-1})m\mathcal{T}\varphi(\xi) = m(\xi)(\mathcal{T}\circ U)\varphi(\xi)$ for a. e. $\xi \in S_{L^\perp}$. ∎

Proof of Theorem 2.1. By Proposition 2.3, there exists a family of functions $\{\varphi_n\} \subseteq L^2(G)$ such that $V = \bigoplus_{n=1}^{\infty} S(\varphi_n)$, and each φ_n is a Parseval frame generator of $S(\varphi_n)$, for each $n \in \mathbb{N}$. Consider $V_k = \bigoplus_{i=1}^{k} S(\varphi_i)$, with the associated range function J_k. Note that since $S(\varphi_i) \perp S(\varphi_j)$ for $i \neq j$, by Proposition 2.4 we have $< \mathcal{T}\varphi_i(\xi), \mathcal{T}\varphi_j(\xi) >= \sum_{\eta\in L^\perp} \hat{\varphi}_i(\xi\eta)\overline{\hat{\varphi}_j}(\xi\eta) = 0,$ for $i \neq j$. So by (1.1) and Proposition 2.2, $\{\mathcal{T}\varphi_i(\xi)\}_{i=1}^k - \{0\}$ is an orthonormal basis of $J_k(\xi)$, for a.e. $\xi \in S_{L^\perp}$. Define $R_k(\xi) : J_k(\xi) \longrightarrow l^2(L^\perp)$ by

$$R_k(\xi)(\sum_{i=1}^k \alpha_i \mathcal{T}\varphi_i(\xi)) = \sum_{i=1}^k \alpha_i (\mathcal{T} \circ U)\varphi_i(\xi). \tag{2.8}$$

By (2.4) $R_k(\xi)$ is well defined. Obviously R_k is measurable. Choose $f \in V_k$ and write it as $f = \sum_{i=1}^k f_i$, for some $f_i \in S(\varphi_i),\ i = 1, ..., k$. Then $\mathcal{T}f = \sum_{i=1}^k m_i \mathcal{T}\varphi_i$, where $m_i \in L^2(S_{L^\perp}),\ i = 1, ..., k$. So

$$(\mathcal{T} \circ U)f(\xi) = R_k(\xi)(\mathcal{T}f(\xi)),\ for\ a.e.\ \xi \in S_{L^\perp}. \tag{2.9}$$

Indeed by Lemma 2.6,

$$\begin{aligned}
(\mathcal{T} \circ U)f(\xi) &= (\mathcal{T} \circ U \circ \mathcal{T}^{-1})(\mathcal{T}f(\xi)) \\
&= (\mathcal{T} \circ U \circ \mathcal{T}^{-1})(\sum_{i=1}^k m_i \mathcal{T}\varphi_i)(\xi) \\
&= \sum_{i=1}^k m_i(\xi)(\mathcal{T} \circ U)\varphi_i(\xi) \\
&= \sum_{i=1}^k m_i(\xi) R_k(\xi)((\mathcal{T}\varphi_i)(\xi)) \\
&= R_k(\xi)(\sum_{i=1}^k m_i(\xi)(\mathcal{T}\varphi_i)(\xi)) \\
&= R_k(\xi)(\mathcal{T}f(\xi)).
\end{aligned}$$

We claim that $\|R_k(\xi)\| \leq \|U\|,\ for\ a.e.\ \xi \in S_{L^\perp}$. To prove this, define $\psi_s \in L^2(S_{L^\perp}, l^2(L^\perp))$, by $\psi_s(\xi) = \sum_{i=1}^k s_i \mathcal{T}\varphi_i(\xi)$, for any $s = (s_1, .., s_k) \in \mathbb{C}^k, \|s\| = 1$. Then $ess\ sup_{\xi \in S_{L^\perp}} \|R_k(\xi)(\psi_s(\xi))\| \leq \|U\|$. Indeed, if not then there would exist $\epsilon > 0$, a measurable set $D \subseteq S_{L^\perp}$ with positive measure, such that $\|R_k(\xi)(\psi_s(\xi))\| > \varepsilon + \|U\|$ for $\xi \in D$. So,

$$\begin{aligned}
\|\mathcal{T} \circ U(\mathcal{T}^{-1}\psi_s.\chi_D)\| &= \|U(\mathcal{T}^{-1}\psi_s.\chi_D)\| \\
&\leq \|U\|\|\mathcal{T}^{-1}\psi_s\chi_D\| \\
&= \|U\|\|\mathcal{T} \circ \mathcal{T}^{-1}\psi_s\chi_D\| \\
&\leq \|U\|\|\psi_s\chi_D\|.
\end{aligned} \tag{2.10}$$

On the other hand

$$
\begin{aligned}
\|\mathcal{T}\circ U(\mathcal{T}^{-1}\psi_s.\chi_D)\| &= \int_{S_{L^\perp}} \|(\mathcal{T}\circ U)(\mathcal{T}^{-1}\circ\psi_s\chi_D)(\xi)\|^2 d\xi \\
&= \int_D \|(\mathcal{T}\circ U)(\sum_{i=1}^{k} s_i\varphi_i)(\xi)\|^2 d\xi \\
&= \int_D \|R_k(\xi)(\psi_s(\xi))\|^2 d\xi \\
&\geq (\varepsilon+\|U\|)^2 \int_D d\xi \\
&\geq (\varepsilon+\|U\|)^2 \int_D \|\psi_s(\xi)\|^2 d\xi \\
&= (\varepsilon+\|U\|)^2 \|\psi_s\chi_D\|^2,
\end{aligned} \tag{2.11}
$$

which contradicts (2.10). This easily implies $\|R_k(\xi)\| \leq \|U\|,\ for\ a.e.\ \xi \in S_{L^\perp}$. For any $l \leq k \in \mathbb{N}$ we have $R_l(x) = R_k(x)|_{J_l(x)}$. So we can define $R(\xi) : \bigcup_{k\in\mathbb{N}} J_k(\xi) \to l^2(L^\perp)$ by $R(\xi)(a) = R_k(\xi)(a)$, if $a \in J_k(\xi)$, for some $k \in \mathbb{N}$. Since $\|R_l(\xi)\| \leq \|U\|$, we have $\|R(\xi)(a)\| \leq \|U\|\|a\|$ for $a \in \bigcup_{l\in\mathbb{N}} J_l(\xi)$. Moreover, since $\overline{\bigcup_{k\in\mathbb{N}} J_k(\xi)} = J(\xi)$ we can extend $R(\xi)$ uniquely to $J(\xi)$ with the desired properties.

Conversely, let R be a measurable range operator on $J(\xi)$ with $ess\ sup_{\xi\in S_{L^\perp}} \|R(\xi)\| < \infty$. Define, $U : V \to L^2(G)$ by $Uf = \mathcal{T}^{-1}F$, where $F : S_{L^\perp} \to l^2(L^\perp), F(\xi) = R(\xi)(\mathcal{T}f(\xi))$. Then U is linear and also bounded:

$$
\begin{aligned}
\|Uf\|^2 = \|\mathcal{T}\circ Uf\|^2 &= \|F\|^2 \\
&= \int_{S_{L^\perp}} \|F(\xi)\|^2 d\xi \\
&= \int_{S_{L^\perp}} \|R(\xi)(\mathcal{T}f(\xi))\|^2 d\xi \\
&\leq ess \sup_{\xi\in S_{L^\perp}} \|R(\xi)\|^2 \int_{S_{L^\perp}} \|\mathcal{T}f(\xi)\|^2 d\xi \\
&= ess \sup_{\xi\in S_{L^\perp}} \|R(\xi)\|^2 \|f\|^2.
\end{aligned}
$$

Moreover U is shift preserving, since

$$
\begin{aligned}
(\mathcal{T}\circ U)T_k f(\xi) &= R(\xi)(\mathcal{T}T_k f(\xi)) \\
&= R(\xi)(M_k(\xi)\mathcal{T}f(\xi)) \\
&= M_k(\xi)R(\xi)((\mathcal{T}f)(\xi)) \\
&= M_k(\xi)(\mathcal{T}\circ U)f(\xi) \\
&= \mathcal{T}(T_k\circ U)f(\xi),
\end{aligned}
$$

where $M_k(\xi) = \overline{\xi}(k)$. Since $\mathcal{T}$ is one-to-one, $T_k U = U T_k$. Moreover, by (2.1) the correspondence between U and R is unique. ■

3. Some Consequences of the Characterization Theorem

Our goal in this section is to investigate some properties of a SP operator in view of its corresponding range operator, as consequences of Theorem 2.1. In fact we propose some properties of a SP operator in view of its corresponding range operator. In other words, we derive some necessary, and some necessary and sufficient conditions for a shift preserving operator in terms of its corresponding range operator, as consequences of Theorem 2.1.

In the following two propositions we establish a necessary condition for a shift preserving operator to be a compact operator, a Hilbert Schmidt operator or of finite trace. Let H be a Hilbert space. A linear map $U : H \to H$ is compact if $U(B)$ is relatively compact in H, where B is the closed unit ball of H.
Note that if a bounded operator is of finite rank, then it is compact.

Let U be an operator on a Hilbert space H, and suppose that E is an orthonormal basis for H. The Hilbert-Schmidt norm of U, denoted by $\|U\|_{HS}$, is defined as $(\sum_{x \in E} \|U(x)\|^2)^{1/2}$. Note that the definition is independent of the choice of basis [14]. An operator U is called a Hilbert-Schmidt operator if $\|U\|_{HS} < \infty$. U is called of finite trace if $tr(U) < \infty$, where $tr(U) = \sum_{x \in E} < U(x), x >$ is the trace of U. The definition of trace is also independent of the chosen orthonormal basis [14, Lemma 2.4.12]. For a detailed exposition of these operators confer [14].

Theorem 3.1. *Suppose $V \subseteq L^2(G)$ is a shift invariant space, J is its associated range function and $U : V \to V$ is a SP operator with its corresponding range operator R. If U is compact then $R(\xi)$ is compact for a.e. $\xi \in S_{L^\perp}$.*

Proof. First reduce the case to self adjoint operators: If U is compact and SP, then so is $|U|$. In fact, one can achieve $U = |U|S$ with S a SP partial isometry (A general fact about polar decomposition is that the positive part $|U|$ is contained in the C^*-algebra generated by U, and S is contained in the von Neumann algebra generated by U, both are contained in the algebra of SP operators; (see [14])). Accordingly, we have $R(\xi) = |R(\xi)|W(\xi)$, with a partial isometry $W(\xi)$, $\xi \in S_{L^\perp}$. If we know that $|R(\xi)|$ is compact, then the same follows for $R(\xi)$, since the space of all compact operators is an ideal.
For positive compact SP operator U one has $U = \sum_n \lambda_n p_n$, with finite rank operators p_n and suitable λ_n (see also [19]). Each p_n is again SP, thus by Theorem 2.1, corresponding to a field $R_n(\xi)$ with rank less than or equal to rank(p_n), for a.e. $\xi \in S_{L^\perp}$. Therefore the decomposition of U provides $R(\xi) = \sum_n \lambda_n R_n(\xi)$. Moreover, as the decomposition of U converges in the operator norm, the same is

true for the decomposition of $R(\xi)$, for a.e. $\xi \in S_{L^\perp}$. For these ξ, $R(\xi)$ is the norm limit of a sequence of finite rank operators, thus compact. ■

Remark 3.2. (A Useful Orthonormal Basis). Let V be a shift invariant subspace of $L^2(G)$. Suppose $V = \bigoplus_{n=1}^{\infty} S(\varphi_n)$ with $(\varphi_n)_{n\in\mathbb{N}}$ as in Proposition 2.3. Then $\{T_k\varphi_n;\ k \in L,\ n \in \mathbb{N}\}$ is a Parseval frame for V [12, Remark 3.11]. Also obviously $\{\mathcal{T}\varphi_n(\xi);\ n \in \mathbb{N}\} - \{0\}$ is an orthonormal basis for $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$ (see the argument at the beginning of the proof of Theorem 2.1) and for $\varphi_n \neq 0,\ n \in \mathbb{N},\ k \in L$ we have $\|T_k\varphi_n\|_2^2 = \|\varphi_n\|_2^2 = \|\mathcal{T}\varphi_n\|^2_{L(S_{L^\perp}, l^2(L^\perp))} = \int_{S_{L^\perp}} \|\mathcal{T}\varphi_n(\xi)\|^2_{l^2(L^\perp)} d\xi = 1$ (note that $\|\mathcal{T}\varphi_n(\xi)\|_{l^2(L^\perp)} = 1$ and $|S_{L^\perp}| = 1$). So [5, Theorem 4.5.1] implies that $\{T_k\varphi_n;\ k \in L,\ n \in \mathbb{N}\}$ is an orthonormal basis for V.

With this in mind we are ready to prove that for a.e. $\xi \in S_{L^\perp},\ R(\xi)$ is a Hilbert Schmidt operator or of finite trace whenever U has the same property:

Theorem 3.3. *Suppose $V \subseteq L^2(G)$ is a shift invariant space, J is its associated range function and $U : V \to V$ is a SP operator with its corresponding range operator R.*

(1) *If U is a Hilbert Schmidt operator then so is $R(\xi)$ for a.e. $\xi \in S_{L^\perp}$.*

(2) *If U is positive and of finite trace then so is $R(\xi)$ for a.e. $\xi \in S_{L^\perp}$.*

Proof. Suppose U is Hilbert Schmidt. Denote by $\|U\|_{HS}$ the Hilbert Schmidt norm of U. Then

$$\begin{aligned}\infty > \|U\|^2_{HS} &= \sum_{k\in L, n\in\mathbb{N}} \|U(T_k\varphi_n)\|_2^2 \\ &= \sum_{k\in L, n\in\mathbb{N}} \|T_kU\varphi_n\|_2^2 \\ &= \sum_{k\in L, n\in\mathbb{N}} \|U\varphi_n\|_2^2.\end{aligned}$$

Thus $\sum_{n\in\mathbb{N}} \|U\varphi_n\|_2^2 < \infty$. So we have

$$\begin{aligned}\int_{S_{L^\perp}} \sum_{n\in\mathbb{N}} \|R(\xi)(\mathcal{T}\varphi_n(\xi))\|^2 d\xi &= \int_{S_{L^\perp}} \sum_{n\in\mathbb{N}} \|\mathcal{T} \circ U\varphi_n(\xi)\|^2 d\xi \\ &= \sum_{n\in\mathbb{N}} \|\mathcal{T} \circ U\varphi_n\|^2 \\ &= \sum_{n\in\mathbb{N}} \|U\varphi_n\|^2 < \infty.\end{aligned}$$

that is $\|R(\xi)\|_{HS} = \sum_{n\in\mathbb{N}} \|R(\xi)(\mathcal{T}\varphi_n(\xi))\|^2 < \infty$, for a.e. $\xi \in S_{L^\perp}$. This completes the proof of (3.3).

For (3.3) Suppose U is of finite trace. Denote by $tr(U)$, the trace of U. Then

$$\begin{aligned}\infty > tr(U) &= \sum_{k\in L, n\in\mathbb{N}} < UT_k\varphi_n, T_k\varphi_n > \\ &= \sum_{k\in L, n\in\mathbb{N}} < \mathcal{T}\circ UT_k\varphi_n, \mathcal{T}T_k\varphi_n > \\ &= \sum_{k\in L, n\in\mathbb{N}} \int_{S_{L^\perp}} < R(\xi)(\mathcal{T}\circ T_k\varphi_n(\xi)), \mathcal{T}T_k\varphi_n(\xi) > d\xi \\ &= \sum_{k\in L, n\in\mathbb{N}} \int_{S_{L^\perp}} < R(\xi)(M_k\mathcal{T}\varphi_n(\xi)), M_k\mathcal{T}\varphi_n(\xi) > d\xi \\ &= \sum_{k\in L, n\in\mathbb{N}} \int_{S_{L^\perp}} < R(\xi)(\mathcal{T}\varphi_n(\xi)), \mathcal{T}\varphi_n(\xi) > d\xi.\end{aligned}$$

So $tr(R(\xi)) = \sum_{n\in\mathbb{N}} < R(\xi)(\mathcal{T}\varphi_n(\xi)), \mathcal{T}\varphi_n(\xi) >< \infty$ a.e. That is $R(\xi)$ is of finite trace for a.e. $\xi \in S_{L^\perp}$. ∎

In the following proposition we show that a necessary and sufficient condition for a SP operator to be an isometry is that its corresponding range operator is an isometry.

Proposition 3.4. *Suppose $V \subseteq L^2(G)$ is a shift invariant space, J is its associated range function and $U : V \to V$ is a SP operator with its corresponding range operator R. Then U is an isometry if and only if $R(\xi)$ is an isometry for a.e. $\xi \in S_{L^\perp}$.*

Proof. By Theorem 2.1, it is enough to show that

$$\|Uf\| \geq \|f\| \text{ for all } f \in V, \tag{3.1}$$

if and only if for a.e. $\xi \in S_{L^\perp}$,

$$\|R(\xi)a\| \geq \|a\|, \text{ for all } a \in J(\xi). \tag{3.2}$$

To prove this, first assume (3.2). Then $\|Uf\|^2 = \|\mathcal{T}\circ Uf\|^2 = \int_{S_{L^\perp}} \|\mathcal{T}\circ Uf(\xi)\|^2 d\xi = \int_{S_{L^\perp}} \|R(\xi)(\mathcal{T}f(\xi))\|^2 d\xi \geq \int_{S_{L^\perp}} \|\mathcal{T}f(\xi)\|^2 d\xi = \|f\|^2$.

Conversely, assume (3.1). Let $\{d_i\}_{i=1}^\infty$ be a dense subset of $l^2(L^\perp)$ (note that since G is second countable, $l^2(L^\perp)$ is separable). Then

$$\|R(\xi)P(\xi)d_i\| \geq \|P(\xi)d_i\| \text{ for } a.e.\ \xi \in S_{L^\perp},\ i \in \mathbb{N}, \tag{3.3}$$

where $P(\xi)$ is the projection onto $J(\xi)$, a.e. $\xi \in S_{L^\perp}$. By the contrary, if (3.3) fails then there exists a measurable set $D \subseteq S_{L^\perp}$ with positive measure, $\varepsilon > 0,\ j \in \mathbb{N}$ such that $\|R(\xi)P(\xi)d_j\| \leq \|P(\xi)d_j\|(1-\varepsilon),$ for $\xi \in D$. Let $f \in V$ be given by $\mathcal{T}f(\xi) = \chi_D(\xi)P(\xi)d_j$. Then

$$\begin{aligned}
\|Uf\| = \|\mathcal{T}Uf\| &= \int_{S_{L^\perp}} \|\mathcal{T}\circ Uf(\xi)\|^2 d\xi \\
&= \int_{S_{L^\perp}} \|R(\xi)(\mathcal{T}f(\xi))\|^2 d\xi \\
&= \int_{D} \|R(\xi)(P(\xi)d_j)\|^2 d\xi \\
&\leq (1-\varepsilon)^2 \int_{D} \|P(\xi)d_j\|^2 d\xi \\
&= (1-\varepsilon)^2 \int_{S_{L^\perp}} \|P(\xi)d_j\chi_D(\xi)\|^2 d\xi \\
&= (1-\varepsilon)^2 \int_{S_{L^\perp}} \|\mathcal{T}f(\xi)\|^2 d\xi \\
&= (1-\varepsilon)^2 \|f\|^2,
\end{aligned}$$

which is a contradiction. So (3.3) holds. ∎

The adjoint of a shift preserving operator is again a shift preserving operator. More precisely we have the following proposition.

Proposition 3.5. *Suppose $V \subseteq L^2(G)$ is a shift invariant space and $U : V \to V$ is a SP operator with its corresponding range operator R. Then the adjoint operator $U^* : V \to V$ is shift preserving and its corresponding range operator R^* is given by $R^*(\xi) = R(\xi)^*$, for a.e. $\xi \in S_{L^\perp}$. In Particular, U is self adjoint if and only if $R(\xi)$ is self adjoint.*

Proof. Obviously U^* is SP. Indeed, $< U^*T_k f, g >=< T_k f, Ug >=< f, T_{k^{-1}} Ug >=< f, UT_{k^{-1}} g >=< U^* f, T_{k^{-1}} g >=< T_k U^* f, g >$, for any $f, g \in L^2(G)$ and $k \in L$. Note that the operator R^* given by $R^*(\xi) = R(\xi)^*$, for a.e. $\xi \in S_{L^\perp}$ is measurable and $ess\ sup_{\xi \in S_{L^\perp}} \|R^*(\xi)\| < \infty$. By Theorem 2.1, there exists a SP operator W so that $(\mathcal{T}\circ W)f(\xi) = R(\xi)^*(\mathcal{T}f(\xi))$, for a.e. $\xi \in S_{L^\perp}$, for all $f \in V$. We have $U^* = W$. Indeed, for $f, g \in L^2(G)$

$$\begin{aligned}
< U^* f, g >=< f, Ug > &= < \mathcal{T}f, \mathcal{T}\circ Ug > \\
&= \int_{S_{L^\perp}} < \mathcal{T}f(\xi), \mathcal{T}\circ Ug(\xi) > d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \int_{S_{L^\perp}} < \mathcal{T}f(\xi), R(\xi)(\mathcal{T}g(\xi)) > d\xi \\
&= \int_{S_{L^\perp}} < R(\xi)^*(\mathcal{T}f(\xi)), (\mathcal{T}g(\xi)) > d\xi \\
&= \int_{S_{L^\perp}} < \mathcal{T} \circ Wf(\xi), \mathcal{T}g(\xi) > d\xi \\
&= < \mathcal{T}Wf, \mathcal{T}g > \\
&= < Wf, g > . \qquad \blacksquare
\end{aligned}
$$

As an application of Theorems 3.1, 3.3 and Propositions 3.4 and 3.5 for $G = \mathbb{R}^n$, we have the following corollary.

Corollary 3.6. *Suppose $V \subseteq L^2(\mathbb{R}^n)$ is a shift invariant space, J is its associated range function and $U : V \to V$ is a SP operator with its corresponding range operator R.*

(1) *U is an isometry if and only if $R(\xi)$ is an isometry for a.e. $\xi \in \mathbb{T}^n$.*

(2) *The adjoint operator $U^* : V \to V$ is SP and its corresponding range operator R^* is given by $R^*(\xi) = R(\xi)^*$, for a.e. $\xi \in \mathbb{T}^n$. In Particular, U is self adjoint if and only if $R(\xi)$, for a.e. $\xi \in \mathbb{T}^n$ is self adjoint.*

(3) *If U is compact then $R(\xi)$ is compact for a.e. $\xi \in \mathbb{T}^n$.*

(4) *If U is a Hilbert Schmidt operator then so is $R(\xi)$ for a.e. $\xi \in \mathbb{T}^n$.*

(5) *If U is positive and of finite trace then so is $R(\xi)$ for a.e. $\xi \in \mathbb{T}^n$.*

Let $\{T_k\varphi;\ k \in L,\ \varphi \in \phi\}$ be a Bessel sequence. Consider the frame operator $S : S(\phi) \to S(\phi)$, corresponding to $\{T_k\varphi;\ \varphi \in \phi,\ k \in L\}$, defined by

$$
Sf = \sum_{\varphi \in \phi, k \in L} < f, T_k\varphi > T_k\varphi. \tag{3.4}
$$

It is easily seen that S is SP. We are interested in finding the range operator associated with S. We show, as a consequence of Theorem 2.1, that its range operator is nothing but the dual Gramian operator associated with J.

Corollary 3.7. *Let $\{T_k\varphi;\ k \in L,\ \varphi \in \phi\}$ be a Bessel sequence and S be the frame operator corresponding to $\{T_k\varphi;\ \varphi \in \phi,\ k \in L\}$, as in (3.4). The range operator associated with S is given by $R(\xi) := \widetilde{G}_\phi(\xi)$, where $\widetilde{G}_\phi$ is the dual Gramian operator associated with the system $\{\mathcal{T}\varphi_n(\xi);\ n \in \mathbb{N}\}$, for a.e. $\xi \in S_{L^\perp}$.*

Proof. Let R be the range operator associated with S. Using (2.1), we have

$$
\begin{aligned}
< Sf, f > &= < \mathcal{T}Sf, \mathcal{T}f > \\
&= \int_{S_{L^\perp}} < \mathcal{T}Sf(\xi), \mathcal{T}f(\xi) > d\xi \\
&= \int_{S_{L^\perp}} < R(\xi)\mathcal{T}f(\xi), \mathcal{T}f(\xi) > d\xi
\end{aligned} \tag{3.5}
$$

On the other hand by a simple calculation we have (one may see also [11, Lemma 4.2]),

$$
\begin{aligned}
< Sf, f > &= \sum_{\varphi\in\phi}\sum_{k\in L} | < T_k\varphi, f > |^2 \\
&= \sum_{\varphi\in\phi}\int_{S_{L^\perp}} | < \mathcal{T}\varphi(\xi), \mathcal{T}f(\xi) >_{l^2(L^\perp)} |^2 d\xi \\
&= \int_{S_{L^\perp}} < \widetilde{G}_\phi(\xi)\mathcal{T}f(\xi), \mathcal{T}f(\xi) > d\xi.
\end{aligned} \tag{3.6}
$$

From (3.5) and (3.6) we obtain $R = \widetilde{G}_\phi$ a.e. ■

Example 3.8. Let G be the second countable LCA group $\mathbb{R}^n \times \mathbb{Z}^n \times \mathbb{T}^n \times Z_n$, for $n \in \mathbb{N}$, where Z_n is the finite abelian group $\{1, 2, ..., n\}$ of residues modulo n. Then $L = \mathbb{Z}^n \times \mathbb{Z}^n \times \{1\} \times Z_n$ is a uniform lattice in G and $L^\perp = \widehat{G/L} = \mathbb{Z}^n \times \{1\} \times \mathbb{Z}^n \times \{1\}$. Let π be the left regular representation of G on $L^2(G)$ and $\psi \in L^2(G)$ be admissible (see [7]). Then the continuous wavelet transform, $V_\psi : L^2(G) \to L^2(G)$, defined by $V_\psi\varphi(x) =< \varphi, \pi(x)\psi >$ is obviously a SP operator, so by Theorem 2.1 there is a range operator R such that for every $f \in L^2(G)$, $R(\xi)(\mathcal{T}f(\xi)) = (\mathcal{T} \circ V_\psi)f(\xi) = ((\widehat{V_\psi f}(\xi\eta))_{\eta\in L^\perp} = (\hat{f}(\xi\eta)\overline{\hat{\psi}}(\xi\eta))_{\eta\in L^\perp}$.

Example 3.9. Define $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Uf(x) = f(x) - f(x - 1)$. Obviously U is a SP operator. By Theorem 2.1 there exists a range operator R so that $R(\xi)(\mathcal{T}f(\xi)) = (\mathcal{T} \circ U)f(\xi) = (\widehat{Uf}(\xi+k))_{k\in\mathbb{Z}} = (1+\exp(i\xi))(\hat{f}(\xi+k))_{k\in\mathbb{Z}}$, for every $f \in L^2(\mathbb{R})$.

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