TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 1, pp. 215-229, March 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

# ON THE SECOND FUNDAMENTAL FORMS OF THE INTERSECTION OF SUBMANIFOLDS

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**Abstract.** Let $G$ be a Lie group and $H$ its subgroup, and let $M^p$, $N^q$ be two submanifolds of dimensions $p$, $q$, respectively, in the Riemannian homogeneous space $G/H$. We study the relationships between the second fundamental forms of $M^p \cap gN^q$ and the second fundamental forms of $M^p$, $N^q$ for $g \in G$. We find that the second fundamental form of $M^p \cap gN^q$ can be expressed by the curvature functions of $M^p$, $N^q$ and the "*angle*" between $M^p$ and $N^q$. All results achieved are the generalizations of known results of the classical differential geometry in $\mathbf{R}^3$.

## 1. Introduction

Let $G$ be a Lie group (that is, a manifold equipped with group structure), which is assumed to have a left and also right invariant Riemannian metric. Let $H$ be a closed subgroup of $G$. Then $G/H$ is a Riemannian homogeneous space. Denote by $dg$ the kinematic density of $G$ (the Haar measure in geometric measure theory). Let $M^p$, $N^q$ be two submanifolds of dimensions $p$, $q$, respectively, in $G/H$. We assume that $M^p$ is fixed and $N^q$ is moving under the action $g \in G$. It is always assumed that $M^p$ and $N^q$ are in general positions, that is, for almost all $g \in G$, the dimension of $M^p \cap gN^q$ is $p + q - \dim(G/H) \geq 0$.

Let $I(M^p \cap gN^q)$ be an *integral invariant* of the submanifold $M^p \cap gN^q$ of dimension $p + q - n$. Evaluating the integral of type

$$\int_G I(M^p \cap gN^q)\, dg \tag{1.1}$$

Received January 16, 2005, revised August 22, 2005.
Communicated by Shu-Cheng Chang.
2000 *Mathematics Subject Classification*: Primary 52A22, 53C65; Secondary 51C16.
*Key words and phrases*: Second fundamental form, Mean curvature, Normal curvature, Euler-Meusnier formula.
Supported in part by the Chinese NSF (grant No. 10671159), Hong Kong Qiu Shi Foundation and Southwest University.

and expressing by the integral invariants of submanifolds $M^p$ and $N^q$ is called the kinematic formula for $I(M^p \cap gN^q)$ in integral geometry. For example, in the case that $G$ is the group of isometry of $\mathbf{R}^n$, $M^p$ and $N^q$ are submanifolds of $\mathbf{R}^n$, and $I(M^p \cap gN^q) = \mathrm{vol}(M^p \cap gN^q)$, the volume of $M^p \cap gN^q$, the evaluation of $\int_G I(M^p \cap gN^q)\, dg$ leads to formulas due to Poincaré, Blaschké, Santaló, Howard and others (see [9, 11, 12] for references). If $G$ is the unitary group $U(n+1)$ acting on complex projective space $\mathbf{CP}^n$, $M^p$ and $N^q$ are complex analytic submanifolds of $\mathbf{CP}^n$, and $I(M^p \cap gN^q)$ is the integral of a Chern class leads to the kinematic formula of Shifrin [14]. If $M$, $N$ are two domains of the Euclidean space $\mathbf{R}^n$ and $I(M \cap gN) = \chi(M \cap gN)$ is the Euler characteristic of the intersection of two domains $M$ and $N$ for rigid motion $g \in G$ of $\mathbf{R}^n$, then $\int_G \chi(M \cap gN)\, dg$ can be expressed explicitly by the integrals of elementary symmetric functions of principal curvatures over the boundaries and the Euler characteristics of the two domains $M$ and $N$. This well-known fundamental kinematic formula in integral geometry is due to S. S. Chern [5, 6]. Refer to [1-3, 7, 10, 13, 18, 19, 23] for literatures of kinematic formulas.

An important unsolved problem is that can an invariant $I(M^p \cap gN^q)$ (either intrinsic or extrinsic) be expressed by invariants of submanifolds $M^p$ and $N^q$. At least we are not aware of letting $I(M \cap gN) = \mathrm{diam}(M \cap gN)$, the diameter of intersection $M \cap gN$ of two domains $M$ and $N$ in $\mathbf{R}^n$. The classical Euler formula says that the curvature $\kappa$ of intersection curve $M \cap gN$ of two surfaces $M$ and $N$ in $\mathbf{R}^3$ can be expressed by their normal curvatures of surfaces and the angle between $M$ and $N$.

**Proposition 1.** *Let $M$ and $N$ be two surfaces in $\mathbf{R}^3$ with the normal curvatures $\kappa_n^M$ and $\kappa_n^N$. Let $\kappa$ be the curvature of the intersection curve $M \cap gN$ and $\phi$ be the angle between $M$ and $gN$. Then we have the following Euler formula ([4, 15])*

$$\kappa^2 \sin^2 \phi = \left(\kappa_n^M\right)^2 + \left(\kappa_n^N\right)^2 - 2\cos\phi\, \left(\kappa_n^M\right)\left(\kappa_n^N\right). \tag{1.2}$$

We used this formula to prove the C-S. Chen's kinematic formula ([3, 23]). Let $H_M$, $H_N$ be, respectively, mean curvatures of $M$, $N$, and let

$$\tilde{H}_M = \int_M H_M^2\, d\sigma, \quad \tilde{H}_N = \int_N H_N^2\, d\sigma. \tag{1.3}$$

Then we have the the following kinematic formula

$$\begin{aligned} &\int_G \left( \int_{M \cap gN} \kappa^2 ds \right) dg \\ &= 2\pi^2 \left\{ \left(3\tilde{H}_M - 2\pi\chi(M)\right) F_N + \left(3\tilde{H}_N - 2\pi\chi(N)\right) F_M \right\}, \end{aligned} \tag{1.4}$$

where $F_M$, $F_N$ are areas of $M$, $N$, respectively, and $\chi(.)$ is the Euler characteristic.

Our main task of this paper is to find the Euler formula (1.2) in higher dimensions. We obtain a fundamental formula over the second fundamental form of $M^p \cap gN^q$, that is, the second fundamental form of the intersection $M^p \cap gN^q$ can be written as the linear combination of the second forms of $M^p$ and $N^q$. Since all curvature functions are determined by the second fundamental forms, our formula contains a great deal of curvature information in geometry.

The formulas we are pursuing can be applied to achieve more kinematic formulas in general homogeneous space $G/H$. In [18], we obtained a generalized Euler formula for hypersurfaces in $\mathbf{R}^n$ and as its applications we achieved the kinematic formulas for mean curvature powers of hypersurface. Moreover, we obtained an extension of Hadwiger's containment problem, i.e., a sufficient condition for one domain to contain another in the Euclidean space $\mathbf{R}^{2n}$. The significance of kinematic formulas are not just interested in their own light but also can be applied to other geometry branches. In their papers ([8, 11, 17, 18, 20-24]), Grinberg, Ren, Zhang, and Zhou obtained the sufficient conditions for Hadwiger's containment problem in high dimensions and the Willmore functional deficit estimate for convex surfaces in $\mathbf{R}^3$. As one see, our motivation of writing this paper clearly comes from the integral geometry.

## 2. Preliminaries

Let $X$ be a $p$-dimensional submanifold immersed in an $n$-dimensional Riemannian space $\mathbf{N}$. We choose a local field of orthonormal frames $e_1, \cdots, e_n$ in $\mathbf{N}$ such that, restricted to $X$, the vector $e_1, \cdots, e_p$ are tangent to $X$. We make use of the following convention on the ranges of indices:

$$\begin{aligned} 1 \leq & \ A, B, C, \cdots \leq n, \\ p+1 \leq & \ \ i, j, k, \cdots \ \ \leq n, \\ 1 \leq & \ \ \alpha, \beta, \gamma, \cdots \ \ \leq p. \end{aligned} \tag{2.1}$$

With respect to the frame field of $\mathbf{N}$ chosen above, let $\omega_1, \cdots, \omega_n$ be the field of dual frames. Then the structure equations of $N$ are given by

$$dx = \sum_A \omega_A e_A, \tag{2.2}$$

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.3}$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \Phi_{AB}, \;\; \Phi_{AB} = \frac{1}{2}\sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D. \tag{2.4}$$

$$\begin{aligned} K_{ABCD} = K_{CDAB}, \;\; K_{ABCD} &= -K_{ABDC} = -K_{BACD}, \\ K_{ABCD} + K_{ADBC} &+ K_{ACDB} = 0. \end{aligned} \tag{2.5}$$

If these are restricted to $X$, then

$$\omega_i = 0. \tag{2.6}$$

Since $0 = d\omega_i = -\sum_\alpha \omega_{i\alpha} \wedge \omega_\alpha$, by Cartan's lemma we can write

$$\omega_{i\alpha} = \sum_\beta h^i_{\alpha\beta}\, \omega_\beta, \;\; h^i_{\alpha\beta} = h^i_{\beta\alpha}. \tag{2.7}$$

From these formulas, we obtain

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \;\; \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0, \tag{2.8}$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \;\; \Omega_{\alpha\beta} = \frac{1}{2}\sum_{\gamma,\sigma} R_{\alpha\beta\gamma\sigma}\, \omega_\gamma \wedge \omega_\sigma, \tag{2.9}$$

$$\begin{aligned} R_{\alpha\beta\gamma\sigma} = R_{\gamma\sigma\alpha\beta}, \;\; R_{\alpha\beta\gamma\sigma} &= -R_{\alpha\beta\sigma\gamma} = -R_{\beta\alpha\gamma\sigma}, \\ R_{\alpha\beta\gamma\sigma} + R_{\alpha\sigma\beta\gamma} &+ R_{\alpha\gamma\sigma\beta} = 0. \end{aligned} \tag{2.10}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \;\; \Omega_{ij} = \frac{1}{2}\sum_{\alpha,\beta} R_{ij\alpha\beta}\, \omega_\alpha \wedge \omega_\beta. \tag{2.11}$$

$$R_{ij\alpha\beta}{=}R_{\alpha\beta ij}, \; R_{ij\alpha\beta}{=}{-}R_{ij\beta\alpha}{=}{-}R_{ji\alpha\beta}, \;\; R_{i\alpha\beta\gamma} + R_{i\gamma\alpha\beta} + R_{i\beta\gamma\alpha} = 0. \tag{2.12}$$

The Riemannian connection of $X$ is defined by $(\omega_{\alpha\beta})$. The form $(\omega_{ij})$ defines a connection in the normal bundle of $X$. We call

$$II = \sum_i II_i\, e_i = \sum_i < d^2x, e_i > \, e_i = \sum_{i,\alpha,\beta} h^i_{\alpha\beta}\, \omega_\alpha\, \omega_\beta\, e_i \tag{2.13}$$

*the second fundamental form* of the immersed submanifold $X$. Sometimes we shall denote the second fundamental form by

$$II_i =< d^2x, e_i >= \sum_{\alpha,\beta} h^i_{\alpha\beta}\omega_\alpha\, \omega_\beta \; =< II, e_i > \tag{2.14}$$

or simply its components $h^i_{\alpha\beta}$. The length of the second fundamental form $II$ of $X$ is defined by

$$|II|^2 = \sum_i |II_i|^2 = \sum_i \sum_{\alpha,\beta} \left(h^i_{\alpha\beta}\right)^2 . \tag{2.15}$$

The *mean curvature vector* $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{p} \sum_i (\mathbf{trace}(II_i))\, e_i = \frac{1}{p} \sum_i \left( \sum_\alpha h^i_{\alpha\alpha} \right) e_i, \tag{2.16}$$

and its length $H$, that is,

$$H = \frac{1}{p} \left\{ \sum_i (\mathbf{trace}(II_i))^2 \right\}^{1/2} = \frac{1}{p} \left\{ \sum_i \left( \sum_\alpha h^i_{\alpha\alpha} \right)^2 \right\}^{1/2} \tag{2.17}$$

is called the *mean curvature* of $X$.

Let $X^p \subset Y^q \subset \mathbf{N}$ $(p \leq q < n)$ be two submanifolds. If we choose the frame

$$(e_1, \cdots, e_p, e_{p+1}, \cdots, e_q, e_{q+1}, \cdots, e_n) \tag{2.18}$$

such that $e_1, \cdots, e_p \in T(X^p)$ and $e_1, \cdots, e_q \in T(Y^q)$, then we have the mean curvature vector $\overrightarrow{H}_X$ of $X^p$, the mean curvature vector $\overrightarrow{H}_Y$ of $Y^q$, respectively, are

$$\begin{aligned} \overrightarrow{H}_X &= \frac{1}{p} \sum_{i=p+1}^{q} \left( \sum_{\alpha=1}^{p} h^i_{\alpha\alpha} \right) e_i + \frac{1}{p} \sum_{j=q+1}^{n} \left( \sum_{\alpha=1}^{p} h^j_{\alpha\alpha} \right) e_j \\ &= \overrightarrow{H}_{\mathbf{Geo}(X)} + \overrightarrow{H}_{\mathbf{Nor}(Y)}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \overrightarrow{H}_Y &= \frac{1}{q} \sum_{j=q+1}^{n} \left( \sum_{\rho=1}^{q} h^j_{\rho\rho} \right) e_j \\ &= \frac{1}{q} \sum_{j=q+1}^{n} \left( \sum_{\rho=1}^{p} h^j_{\rho\rho} \right) e_j + \frac{1}{q} \sum_{j=q+1}^{n} \left( \sum_{\rho=p+1}^{q} h^j_{\rho\rho} \right) e_j \\ &= \frac{p}{q} \overrightarrow{H}_{\mathbf{Nor}(Y)} + \frac{1}{q} \sum_{j=q+1}^{n} \left( \sum_{\rho=p+1}^{q} h^j_{\rho\rho} \right) e_j. \end{aligned} \tag{2.20}$$

Therefore

$$\overrightarrow{H}_{\mathbf{Nor}(Y)} = \frac{1}{p}\left\{ q\overrightarrow{H}_Y - \sum_{j=q+1}^{n}\left(\sum_{\rho=p+1}^{q} h^j_{\rho\rho}\right) e_j \right\}. \tag{2.21}$$

If $p = n-2$, $q = n-1$ then we have

$$\overrightarrow{H}_X = \frac{1}{n-2}\sum_{\alpha=1}^{n-2} h^{n-1}_{\alpha\alpha}\, e_{n-1} + \frac{1}{n-2}\sum_{\alpha=1}^{n-2} h^{n}_{\alpha\alpha}\, e_{n}. \tag{2.22}$$

It follows that $\overrightarrow{H}_{\mathbf{Nor}(Y)}$ only depends on $Y$ (normal bundle of $X$). Where $\overrightarrow{H}_{\mathbf{Geo}(X)}$ is defined as the *geodesic curvature vector* at $x \in Y$ (related to $X$) and $\overrightarrow{H}_{\mathbf{Nor}(Y)}$ the *normal curvature vector* at $x \in Y$ (relative to $X$). Their lengths, i.e., $|\overrightarrow{H}_{\mathbf{Geo}(X)}| = \kappa_g(X)$, $|\overrightarrow{H}_{\mathbf{Nor}(Y)}| = \kappa_n(Y)$ are called, respectively, the *geodesic curvature* of $X$ at $x \in X$ (relative to $Y$), *normal curvature* of $Y$ at $x \in X$. It is obviously (by (2.21)) that the normal curvature is determined by the mean curvature $H_Y$ and the trace of the second fundamental forms $(h^j_{\alpha\beta})$ of $X$ $(\alpha, \beta = 1, \cdots, p; j = q+1, \cdots, n)$ and it is an (extrinsic) invariant. Therefore the geodesic curvature is also an (extrinsic) invariant . These $h^j_{\alpha\beta}$ $(j = p+1, \cdots, q)$ are called the *geodesic curvature components* at $x \in Y$ (relative to $X$) and those $h^j_{\alpha\beta}$ $(j = q+1, \cdots, n)$ are called the *normal curvature components* at $x \in Y$ (relative to $X$). It is obvious that *two submanifolds $Y$ and $Y'$ of the same dimension which are tangent at submanifold $X$ have the same normal curvature* (*relative to $X$.*)

The above result actually is the classic Meusnier's theorem when $X \equiv \Gamma$ is a smooth curve containing in a surface $Y \equiv \Sigma \subset \mathbf{R}^3$. That is, let $\kappa$ be the curvature at $x \in \Gamma$, $T$ and $N$ be, respectively, the tangent and the normal of $\Gamma$, and $\kappa_g$ and $\kappa_n$ be, respectively, the geodesic curvature and the normal curvature of $\Sigma$ at $x$ along $T$. Let $n$ be the normal of $\Sigma$ and $\mu = n \wedge T$, then we have the following Meusnier's formula

$$\kappa N = \kappa_g \mu + \kappa_n n. \tag{2.23}$$

Let $V$ and $W$ be vector subspaces of dimensional $p$ and $q$, respectively. Let $v_{p+1}, \ldots, v_n$ be an orthonormal basis of $N(V)$ and $w_{q+1}, \ldots, w_n$ an orthonormal basis of $N(W)$, that is,

$$\begin{aligned} N(V) &= \mathbf{span}\{v_{p+1}, \cdots, v_n\}; \\ N(W) &= \mathbf{span}\{w_{q+1}, \cdots, w_n\}, \end{aligned} \tag{2.24}$$

the normal spaces to $V$, $W$, respectively. The angle between subspaces $V$ and $W$ is defined by

$$\Delta(V, W) = \| v_{p+1} \wedge \cdots \wedge v_n \wedge w_{q+1} \wedge \cdots \wedge w_n \|, \tag{2.25}$$

where

$$\|x_1 \wedge \cdots \wedge x_k\|^2 = |\mathbf{det}(< x_i,\, x_s >)|. \tag{2.26}$$

If $V,\ W$ are both $(n-1)$-dimensional then $\Delta(V, W) = |\sin\theta|$, where $\theta$ is the angle between normals of $V$ and $W$. It is obvious that

$$0 \le \Delta(V, W) \le 1,$$

with

$$\begin{aligned} \Delta(V, W) = 0 &\quad \text{if and only if} \quad V \cap W \neq \{0\}, \\ \Delta(V, W) = 1 &\quad \text{if and only if} \quad V \perp W. \end{aligned} \tag{2.27}$$

Also if $g$ is an isometry of $E^n$, then $\Delta(gV, gW) = \Delta(V, W)$.

Let $G$ be a Lie group (a smooth submanifold which is also a group in such a way that the group operations are smooth) acting on a left coset space $G/H$ by left multiplication, where $H$ is a closed subgroup of $G$. We assume that $G/H$ has an invariant Riemannian metric. Let $M^p$, $N^q$ be submanifolds in $G/H$, of dimensions $p$, $q$, respectively.

Let us list indices that we will use very often through the rest of this paper in the following table:

$$\begin{gathered} 1 \le A, B, C \le n; \quad 1 \le \alpha, \beta \le p+q-n; \quad p+q-n+1 \le i, j \le n; \\ p+q-n+1 \le a, b \le p; \quad 1 \le e, f \le p; \quad p+1 \le \lambda, \mu \le n; \\ p+q-n+1 \le h, l \le q; \quad 1 \le u, v \le q; \quad q+1 \le \rho, \sigma \le n. \end{gathered} \tag{2.28}$$

Let $xe_A$ be orthonormal frames, so that $x \in M^p$ and $e_1, \cdots, e_p$ are tangent to $M^p$ at $x$. Similarly, let $x'e'_A$ be frames, such that $x' \in gN^q$ and $e'_1, \cdots, e'_q$ are tangent to $gN^q$ at $x'$. Suppose $g$ be generic, so that $M^p \cap gN^q$ is of dimension $p+q-n$. We restrict the above families of frames by the condition

$$x = x', \qquad e_\alpha = e'_\alpha. \tag{2.29}$$

Geometrically the latter means that $x \in M^p \cap gN^q$ and $e_\alpha$ are tangent to $M_p \cap gN^q$ at $x$. The two submanifolds $M^p$ and $N^q$ at $x$ have a scalar invariant, which is also called the "angle" between $M^p$ and $N^q$, i.e.,

$$\Delta^2 = |\mathbf{det}(e_\lambda, e'_\rho)| = |\mathbf{det}(e_a, e'_h)|. \tag{2.30}$$

In the case of that $M^p$ and $N^q$ are both hypersurfaces $(p = q = n-1)$ it is the absolute value of the cosine of the angle between their normal vectors.

The second fundamental forms are all symmetric bilinear functions on $T_xM \times T_xM$ for all $x$ in $M$. That is, the second fundamental form of $M$ at $x \in M$ is a symmetric bilinear mapping

$$h_x^M : M_x \times M_x \longrightarrow M_x^\perp, \tag{2.31}$$

where $M_x$ is the tangent bundle of $M$ and $M_x^\perp$ is the normal bundle of $M$ at $x$. If $e_1, \cdots, e_n$ is orthonormal basis of $\mathbf{N}$ such that $e_1, \cdots, e_p$ is a basis of $M_x$ and $e_{p+1}, \cdots, e_n$ is a basis of $M_x^\perp$, then the components of $h_x^M$ in this basis are the numbers $\left(h_x^M\right)^i_{\alpha\beta} =< h_x^M(e_\alpha, e_\beta), e_i >$, $1 \le \alpha, \beta \le p$, $p+1 \le i \le n$.

## 3. The Euler-meusnier Formulas

Let $G$ be the isometry group acting on the $n$-dimensional Riemannian space $\mathbf{N}$. Let $M^p$, $N^q$ be a pair of submanifolds $\mathbf{N}$, where $p+q-n \ge 0$ so that generically $M^p \cap gN^q$ is always a submanifold of dimension $p+q-n$ for almost all $g \in G$. Our goal is to express the second fundamental forms of the intersection of $p+q-n$ dimensional manifold $M_g^{p+q-n} = M^p \cap gN^q$ in terms of those of $M^p$ and $gN^q$ and the "angle" between $M^p$ and $gN^q$.

We choose orthonormal frames $\{e_A\}$ and $\{e'_B\}$ such that:

(1) $e_\alpha = e'_\alpha$;
(2) $e_1, \cdots, e_{p+q-n} \in T(M^p \cap gN^q)$;
(3) $e_1, \cdots, e_p \in T(M^p)$;
(4) $e_1, \cdots, e_{p+q-n}, e'_{p+q-n+1}, \cdots, e'_q \in T(gN^q)$;
(5) $e_{p+1}, \cdots, e_n \in N(M^p)$, the normal bundle of $M^p$;
(6) $e'_{q+1}, \cdots, e'_n \in N(gN^q)$, the normal bundle of $gN^q$;
(7) $\mathrm{span}\{e_{p+1}, \cdots, e_n, e'_{q+1}, \cdots, e'_n\} = \mathrm{span}\{e_{p+q-n+1}, \cdots, e_p, e'_{p+q-n+1}, \cdots, e'_q\} = N(M^p \cap gN^q)$, the normal bundle of $M^p \cap gN^q$.

For the families of frames $xe_A$ and $xe'_A$, let

$$\omega_A = (dx, e_A), \quad \omega'_A = (dx', e'_A), \tag{3.1}$$

$$\omega_{AB} = (de_A, e_B), \quad \omega'_{AB} = (de'_A, e'_B), \tag{3.2}$$

so that

$$\omega_{AB} + \omega_{BA} = 0, \quad \omega'_{AB} + \omega'_{BA} = 0. \tag{3.3}$$

When restricted to $M^p$, $N^q$ we have, respectively,

$$\omega_\lambda = 0, \quad \omega'_\rho = 0. \tag{3.4}$$

And restricted to $M_g^{p+q-n}$, we have

$$\omega_{\alpha\lambda} = \sum_\beta h^\lambda_{\alpha\beta}\omega_\beta, \quad \omega'_{\alpha\rho} = \sum_\beta h'^{\rho}_{\alpha\beta}\omega_\beta, \tag{3.5}$$

where

$$h^\lambda_{\alpha\beta} = h^\lambda_{\beta\alpha}, \quad h'^{\rho}_{\alpha\beta} = h'^{\rho}_{\beta\alpha}, \tag{3.6}$$

The second fundamental forms $II^g$ of $M_g^{p+q-n} = M^p \cap gN^q$

$$II^g = \sum_i II^g_i e_i = \sum_i < d^2x, e_i > e_i = \sum_{i,\alpha,\beta} h^i_{\alpha\beta}\omega_\alpha\omega_\beta e_i, \tag{3.7}$$

related to frames $\{e_A\}, \ \{e'_A\}$ are, respectively

$$\begin{aligned} II^g = & \sum_a II_a\, e_a + \sum_\lambda II_\lambda\, e_\lambda; \\ II^g = & \sum_h II'_h\, e'_h + \sum_\rho II'_\rho\, e'_\rho, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} II_a = (d^2x, e_a) = \sum_{\alpha,\beta} h^a_{\alpha\beta}\,\omega_\alpha\,\omega_\beta; \quad & II_\lambda = (d^2x, e_\lambda) = \sum_{\alpha,\beta} h^\lambda_{\alpha\beta}\,\omega_\alpha\,\omega_\beta; \\ II'_h = (d^2x, e'_h) = \sum_{\alpha,\beta} h'^{h}_{\alpha\beta}\,\omega_\alpha\,\omega_\beta; \quad & II'_\rho = (d^2x, e'_\rho) = \sum_{\alpha,\beta} h'^{\rho}_{\alpha\beta}\,\omega_\alpha\,\omega_\beta. \end{aligned} \tag{3.9}$$

The submanifolds $M^p$ and $gN^q$ have a scalar invariant, which is the "angle" between $M^p$ and $gN^q$,

$$\Delta^2 = |\mathbf{det}(e_a, e'_\rho)| = |\mathbf{det}(a_{\rho a})| = |\mathbf{det}(e_\lambda, e'_h)| = |\mathbf{det}(b_{\lambda h})|, \tag{3.10}$$

$a_{\rho a}$ and $b_{\lambda h}$ are the angle elements between $M^p$ and $N^q$.

For a pair of hypersurfaces $(p = q = n-1)$ it is clearly the absolute value of the sine of the angle between their normal vectors.

We are now in the position to prove our theorems.

**Theorem 1.** *Let $M^p$, $N^q$ be, respectively, a pair of submanifolds of dimensions $p$, $q$ in an $n$-dimensional Riemannian space $\mathbf{N}$ with $p+q-n \geq 0$. Let $h^\lambda_{\alpha\beta}$, $h'^{\rho}_{\alpha\beta}$ be the second fundamental forms of $M^p$, $N^q$, respectively. Let $\Delta$ be the angle between $M^p$ and $gN^q$, for $g \in G$, the group of isometry of $\mathbf{N}$. Let $II^g$ be the* *second fundamental form of the intersection submanifold $M_g^{p+q-n} = M^p \cap gN^q$. Then we have*

$$\tag{3.11} \begin{aligned} \Delta^2\, II^g = & \sum_{\lambda,\alpha,\beta} \left( h^{\lambda}_{\alpha\beta} - \sum_{\sigma} a_{\lambda\sigma}\, h'^{\sigma}_{\alpha\beta} \right) \omega_\alpha\, \omega_\beta\, e_\lambda \\ & + ds \textstyle\sum_{\rho,\alpha,\beta} \left( h'^{\rho}_{\alpha\beta} - \sum_{\mu} b_{\rho\mu}\, h^{\mu}_{\alpha\beta} \right) \omega_\alpha\, \omega_\beta\, e'_\rho, \end{aligned}$$

*where $a_{\lambda\sigma}$ and $b_{\rho\mu}$ are angle elements between $M^p$ and $N^q$.*

*Proof.* We wish to express $(d^2x, e_a)$ as a linear combination of $II_\lambda$ and $II'_\rho$. Therefore we set

$$\tag{3.12} e'_\rho = \sum_a a_{\rho a} e_a + \sum_\lambda a_{\rho\lambda} e_\lambda$$

so that

$$\tag{3.13} a_{\rho a} = (e'_\rho, e_a), \quad a_{\rho\lambda} = (e'_\rho, e_\lambda).$$

Under our hypothesis $\Delta = |\mathbf{det}(a_{\rho a})| \neq 0$. let $(b_{b\sigma})$ be the inverse matrix of $(a_{\rho a})$, so that

$$\tag{3.14} \sum_\sigma b_{b\sigma} a_{\sigma a} = \delta_{ba}, \quad \sum_a a_{\rho a} b_{a\sigma} = \delta_{\rho\sigma}.$$

Then we have

$$\tag{3.15} e_a = \sum_\rho b_{a\rho} e'_\rho + \sum_\lambda b_{a\lambda} e_\lambda,$$

where

$$\tag{3.16} b_{a\lambda} = -\sum_\rho b_{a\rho} a_{\rho\lambda}.$$

The condition $(e'_\rho, e'_\sigma) = \delta_{\rho\sigma}$ is expressed by

$$\tag{3.17} \sum_a a_{\rho a} a_{\sigma a} + \sum_\lambda a_{\rho\lambda} a_{\sigma\lambda} = \delta_{\rho\sigma}.$$

Therefore we have

$$\tag{3.18} II_a = (d^2x, e_a) = \sum_{\alpha,\beta} h^a_{\alpha\beta}\omega_\alpha\omega_\beta = \sum_\rho b_{a\rho} II'_\rho + \sum_\lambda b_{a\lambda} II_\lambda.$$

By the same way, We wish to express $(d^2x, e'_h)$ as a linear combination of $II_\lambda$ and $II'_\rho$. Therefore we set

$$e_\lambda = \sum_h b_{\lambda h} e'_h + \sum_\sigma b_{\lambda\sigma} e'_\sigma, \tag{3.19}$$

so that

$$b_{\lambda h} = (e_\lambda, e'_h), \quad b_{\lambda\sigma} = (e_\lambda, e'_\sigma). \tag{3.20}$$

Under our hypothesis $\Delta = |\mathbf{det}(b_{\lambda h})| \neq 0$. let $(a_{l\mu})$ be the inverse matrix of $(b_{\lambda h})$, so that

$$\sum_\lambda a_{h\lambda} b_{\lambda l} = \delta_{hl}, \quad \sum_h b_{\lambda h} a_{h\mu} = \delta_{\lambda\mu}. \tag{3.21}$$

Then we have

$$e'_h = \sum_\lambda a_{h\lambda} e_\lambda + \sum_\sigma a_{h\sigma} e'_\sigma, \tag{3.22}$$

where

$$a_{h\sigma} = -\sum_\lambda a_{h\lambda} b_{\lambda\sigma}. \tag{3.23}$$

The condition $(e_\lambda, e_\mu) = \delta_{\lambda\mu}$ is expressed by

$$\sum_l b_{\lambda l} b_{\mu l} + \sum_\sigma b_{\lambda\sigma} b_{\mu\sigma} = \delta_{\lambda\mu}. \tag{3.24}$$

Therefore we have

$$II'_h = (d^2x, e'_h) = \sum_{\alpha,\beta} h'^h_{\alpha\beta} \omega_\alpha \omega_\beta = \sum_\sigma a_{h\sigma} II'_\sigma + \sum_\lambda a_{h\lambda} II_\lambda. \tag{3.25}$$

To express the second fundamental forms of $M_g^{p+q-n}$ as a linear combination of $II_\lambda$ and $II'_\rho$, we set

$$II^g = \sum_{\lambda,\alpha,\beta} X^\lambda_{\alpha\beta}\, \omega_\alpha\, \omega_\beta\, e_\lambda + \sum_{\rho,\alpha,\beta} Y^\rho_{\alpha\beta}\, \omega_\alpha\, \omega_\beta\, e'_\rho, \tag{3.26}$$

where $X^\lambda_{\alpha\beta}$ and $Y'^\rho_{\alpha\beta}$ are to be determined.

Therefore, by (3.12) we have

$$
II^g = \sum_{\lambda,\alpha,\beta} X^{\lambda}_{\alpha\beta}\omega_\alpha\omega_\beta e_\lambda + \sum_{\rho,\alpha,\beta} Y^{\rho}_{\alpha\beta}\omega_\alpha\omega_\beta \left(\sum_a a_{\rho a} e_a + \sum_\lambda a_{\rho\lambda} e_\lambda\right) \tag{3.27}
$$
$$
= \sum_{a,\alpha,\beta}\left(\sum_\rho a_{\rho a} Y^{\rho}_{\alpha\beta}\right)\omega_\alpha\omega_\beta e_a + \sum_{\lambda,\alpha,\beta}\left(X^{\lambda}_{\alpha\beta} + \sum_\rho a_{\rho\lambda} Y^{\rho}_{\alpha\beta}\right)\omega_\alpha\omega_\beta e_\lambda.
$$

From (3.9), and (3.27) we have

$$
\begin{cases} h^a_{\alpha\beta} = \sum_\rho a_{\rho a} Y^{\rho}_{\alpha\beta}; \\ h^\lambda_{\alpha\beta} = X^{\lambda}_{\alpha\beta} + \sum_\rho a_{\rho\lambda} Y^{\rho}_{\alpha\beta}. \end{cases} \tag{3.27}
$$

Similarly, by (3.9), (3.26) and (3.27) we have

$$
II^g = \sum_{\lambda,\alpha,\beta} X^{\lambda}_{\alpha\beta}\,\omega_\alpha\,\omega_\beta \left(\sum_h b_{\lambda h}\, e'_h + \sum_\sigma b_{\lambda\sigma} e_\sigma\right) + \sum_{\sigma,\alpha,\beta} Y^{\sigma}_{\alpha\beta}\,\omega_\alpha\,\omega_\beta\, e'_\sigma \tag{3.29}
$$
$$
= \sum_{h,\alpha,\beta}\left(\sum_\lambda b_{\lambda h}\, X^{\lambda}_{\alpha\beta}\right)\omega_\alpha\,\omega_\beta\, e'_h + \sum_{\sigma,\alpha,\beta}\left(Y^{\sigma}_{\alpha\beta} + \sum_\lambda b_{\lambda\sigma}\, X^{\lambda}_{\alpha\beta}\right)\omega_\alpha\,\omega_\beta\, e'_\sigma,
$$

and

$$
\begin{cases} {h'}^h_{\alpha\beta} = \sum_\lambda b_{\lambda h}\, X^{\lambda}_{\alpha\beta}; \\ {h'}^\sigma_{\alpha\beta} = Y^{\sigma}_{\alpha\beta} + \sum_\lambda b_{\lambda\sigma}\, X^{\lambda}_{\alpha\beta}. \end{cases} \tag{3.30}
$$

Combining (3.28) and (3.30) together gives

$$
\begin{cases} h^\lambda_{\alpha\beta} = X^{\lambda}_{\alpha\beta} + \sum_\rho a_{\lambda\rho}\, Y^{\rho}_{\alpha\beta}; \\ {h'}^\rho_{\alpha\beta} = Y^{\rho}_{\alpha\beta} + \sum_\lambda b_{\rho\lambda}\, X^{\lambda}_{\alpha\beta}, \end{cases} \tag{3.31}
$$

or

$$
\begin{pmatrix} h^\lambda_{\alpha\beta} \\ {h'}^\rho_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} (I_{\lambda\lambda}) & (a_{\lambda\rho}) \\ (b_{\rho\lambda}) & (I_{\rho\rho}) \end{pmatrix} \begin{pmatrix} X^\lambda_{\alpha\beta} \\ Y^\rho_{\alpha\beta} \end{pmatrix}. \tag{3.32}
$$

Finally, the equations (3.32) lead to

$$
\begin{pmatrix} X^\lambda_{\alpha\beta} \\ Y^\rho_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} (I_{\lambda\lambda}) & (a_{\lambda\rho}) \\ (b_{\rho\lambda}) & (I_{\rho\rho}) \end{pmatrix}^{-1} \begin{pmatrix} h^\lambda_{\alpha\beta} \\ {h'}^\rho_{\alpha\beta} \end{pmatrix} \tag{3.33}
$$
$$
= \frac{1}{\Delta^2}\begin{pmatrix} (I_{\lambda\lambda}) & (-a_{\lambda\rho}) \\ (-b_{\rho\lambda}) & (I_{\rho\rho}) \end{pmatrix} \begin{pmatrix} h^\lambda_{\alpha\beta} \\ {h'}^\rho_{\alpha\beta} \end{pmatrix},
$$

where $\Delta^2 = \mathbf{det}\begin{pmatrix} (I_{\lambda\lambda}) & (a_{\lambda\rho}) \\ (b_{\rho\lambda}) & (I_{\rho\rho}) \end{pmatrix}$.

That is

$$\begin{aligned} \Delta^2 X^{\lambda}_{\alpha\beta} &= h^{\lambda}_{\alpha\beta} - \sum_{\sigma} a_{\lambda\sigma} h'^{\sigma}_{\alpha\beta}, \\ \Delta^2 Y^{\rho}_{\alpha\beta} &= h'^{\rho}_{\alpha\beta} - \sum_{\mu} b_{\rho\mu} h^{\mu}_{\alpha\beta}. \end{aligned} \tag{3.34}$$

Inserting (3.34) into (3.26) we complete the proof of our Theorem 1.

Let $M$, $N$ be two hypersurfaces in the Euclidean space $\mathbf{R}^n$. We choose the frames $\{e_A\}$ and $\{e'_A\}$ such that $e_1 = e_1, \cdots, e_{n-2} = e'_{n-2}$ are tangent to $\Sigma_g = M \cap gN$ and $e_n$, $e'_n$ are, respectively, the normal vector of $M$, $N$. The angle between $M$ and $N$ is $\Delta = |\sin\phi|$ and $a_{\lambda\sigma} = b_{\rho\mu} = \cos\phi$. Then we have the following

**Theorem 2.** *Let $M$, $N$ be two hypersurfaces of class $C^2$ in the Euclidean space $\mathbf{R}^n$ and let $h^n_{ij}$, $h'^n_{ij}$ be the normal curvatures of $M$, $N$, respectively. Then we have*

$$\sin^2\phi\, II_{\Sigma_g} = \left(\sum_{i,j} h^n_{ij} - \cos\phi \sum_{i,j} h'^n_{ij}\right) e_n + \left(\sum_{i,j} h'^n_{ij} - \cos\phi \sum_{i,j} h^n_{ij}\right) e'_n, \tag{3.35}$$

*where* $\cos\phi = (e_n, e'_n)$.

By taking the normal of (3.35) we have the following generalized Euler formula

**Theorem 3.** *Let $M$, $N$ be two hypersurfaces of class $C^2$ in $\mathbf{R}^n$ and let $h^n_{ij}$, $h'^n_{ij}$ be the normal curvatures of $M$, $N$, respectively. Then we have*

$$\sin^2\phi\, \left|II_{\Sigma_g}\right|^2 = \left(\sum_{i,j} h^n_{ij}\right)^2 + \left(\sum_{i,j} h'^n_{ij}\right)^2 - 2\cos\phi \left(\sum_{i,j} h^n_{ij}\right)\left(\sum_{i,j} h'^n_{ij}\right), \tag{3.36}$$

*where* $\cos\phi = (e_n, e'_n)$.

If $M$, $N \subset \mathbf{R}^3$ are two smooth surfaces, we choose the frames $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$ such that $e_1 = e'_1$, the tangent of the curve $\Gamma_g = M \cap gN$, for rigid motion $g \in G$, and $e_3$, $e'_3$ are, respectively the normal of $M$, $N$. Let $\kappa^M_n$ and $\kappa^N_n$ be, respectively the normal curvatures of $M$ and $N$. Then we immediately obtain (also see [19])

**Theorem 4.** *Let $M$, $N$ be two smooth surfaces in $\mathbf{R}^3$ and let $\kappa^M_n$, $\kappa^N_n$ be the normal curvatures of $M$, $N$, respectively. Then we have*

$$\sin^2\phi\, II_{\Gamma_g} = \left(\kappa^M_n - \kappa^N_n \cos\phi\right) e_3 + \left(\kappa^N_n - \kappa^M_n \cos\phi\right) e'_3, \tag{3.37}$$

*where* $\cos\phi = (e_3, e_3')$.

Note $\left|II_{\Gamma_g}\right| = \kappa$, the curvature of $\Gamma_g$. Then by taking the norm of (3.37) we immediately obtain the known classical Euler formula (1.2).

## Acknowledgment

The work was partially supported by the Guizhou Province Sciences and Technologies Foundation. We would like to thank Professor Pengcheng Wu and Professor Chuanhan Lu for their supports. We would also like to thank Professor Hua Chen for his hospitalities and supports during our visiting to Wuhan University. Finally, we would like to thank anonymous referees for valuable comments.

## References

1. J. E. Brothers, Integral geometry in homogeneous space, *Trans. Amer. Math. Soc.*, **124** (1966), 480-517.
2. B-Y Chen, *Geometry of Submanifolds*, Marcel Dekker. Inc., New York, 1973.
3. C-S. Chen, On the kinematic formula of square of mean curvature, *Indiana Univ. Math. J.*, **22** (1972-73), 1163-1169.
4. F. Chen, X. Zhao & J. Zhou, An anlogue of the Euler formula, *J. Math.*, **25** (2005) 1-4.
5. S. S. Chern, On the kinematic formula in the euclidean space of n dimensions, *Amer. J. Math.*, **74** (1952), 227-236.
6. S. S. Chern, On the kinematic formula in integral geometry, *J. Math. Mech.*, **16** (1966), 101-118.
7. H. Federer, Curvature measure, *Trans. Amer. Math. Soc.*, **69** (1959), 418-491.
8. E. Grinberg, D. Ren & J. Zhou, The Symmetric Isoperimetric Deficit and the Containment Problem in a Plan of Constant Curvature, preprint.
9. R. Howard, The kinematic formula in riemannian geometry, *Memoirs Amer. Math. Soc.*, **509**, 1993.
10. D. Klain and G-C. Rota, *Introduction to Geometric Probability*, Cambridge University Press, 1997.
11. D. Ren, *Topics in Integral Geometry, Series in Pure Mathematics*, World Scientific (Singapore), 19 1994.
12. L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, Reading, Mass, 1976.
13. R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, New York: Cambridge University Press, 1993.

14. T. Shifrin, The kinematic formula in complex integral geometry, *Trans. Amer. Math. Soc.*, **264(2)** (1981), 255-293.

15. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish Inc., 1979.

16. H. Weyl, On the volume of tubes, *Amer. J. Math.*, **61**(1939), 461-472.

17. G. Zhang, A Sufficient Condition for one Convex Body Containing Another, *Chin. Ann. of Math.*, **9B-4** (1988), 447-451.

18. J. Zhou, Kinematic formulas for mean curvature powers of hypersurfaces and analogous of Hadwiger's theorem in $\mathbf{R}^{2n}$, *Trans. Amer. Math. Soc.*, **345** (1994), 243-262.

19. J. Zhou, Kinematic formula for square mean curvature of hypersurfaces, *Bull. of the Institute of Math.* (*Academia Sinica*), **22** (1994), 31-47.

20. J. Zhou, On the Willmore deficit of convex surfaces, *Lectures in Applied Mathematics of Amer. Math. Soc.*, **30** (1994), 279-287.

21. J. Zhou, When can one domain enclose another in $\mathbf{R}^3$? *J. Austral. Math. Soc.* (*Series A*), **59** (1995), 266-272.

22. J. Zhou, The sufficient condition for a convex domain to contain another in $\mathbf{R}^4$, *Proc. Amer. Math. Soc.*, **121** (1994), 907-913.

23. J. Zhou, A kinematic formula and analogous of Hadwiger's theorem in space, *Contemporary Mathematics*, **140** (1992), 159-167.

24. J. Zhou, Sufficient conditions for one domain to contain another in a space of constant curvature, *Proc. Amer. Math. Soc.*, **126** (1998), 2797-2803.

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