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A Fully Discrete Spectral Method for the Nonlinear Time Fractional Klein-Gordon Equation

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Abstract. The numerical approximation of the nonlinear time fractional Klein-Gordon equation in a bounded domain is considered. The time fractional derivative is described in the Caputo sense with the order γ ($1 < \gamma < 2$). A fully discrete spectral scheme is proposed on the basis of finite difference discretization in time and Legendre spectral approximation in space. The stability and convergence of the fully discrete scheme are rigorously established. The convergence rate of the fully discrete scheme in H^1 norm is $\mathrm{O}(\tau^{3-\gamma} + N^{1-m})$, where τ, N and m are the time-step size, polynomial degree and regularity in the space variable of the exact solution, respectively. Numerical examples are presented to support the theoretical results.

1. Introduction

The following nonlinear Klein-Gordon equation is probably the simplest nonlinear relativistic equation of mathematical physics [17]:

$$\begin{gathered} u_{tt} - \Delta u + m^2 u + |u|^p u = 0, \quad x \in \mathbb{R}^3,\ m > 0,\ p > 0, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x). \end{gathered} \tag{1.1}$$

A complete understanding of it would illuminate our view of many other such equations, such as Shrödinger equation, Dirac equation, etc. The Klein-Gordon equation plays a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory [23]. There are lots of works to investigate the analytical and numerical aspects of the nonlinear Klein-Gordon equation, see [1, 7, 9, 11, 12, 17, 23, 24].

Fractional derivatives and integrals are the generalizations of the usual derivatives and integrals. Fractional differential equations are the equations involving the fractional derivatives of the unknown functions. One can refer to the book [5] for more details. In fact, many phenomena in physics and other sciences can be described more accurately

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using fractional calculus, such as anomalous diffusion [15], relaxation and reaction kinetics of polymers [8], image processing [3], bioengineering [14], continuous-time finance [16] and so on.

If we replace the second-order time derivative in equation (1.1) with a fractional derivative of order $1 < \gamma < 2$, we obtain the nonlinear time fractional Klein-Gordon equation. Golmankhaneh et al. [6] obtained approximate analytical solutions of the nonlinear time fractional Klein-Gordon equations using homotopy perturbation method. Demiray et al. [4] used the Generalized Kudryashov Method to obtain the exact solutions of the nonlinear time fractional Klein-Gordon equations. Vong and Wang [21] proposed a fourth order compact difference scheme for a nonlinear fractional Klein-Gordon equation, and proved the stability and convergence of the scheme using the energy method.

In this paper, we consider the following nonlinear time fractional Klein-Gordon equation in one-dimensional spatial domain:

$$ {}_0^C D_t^\gamma u(x,t) - u_{xx} + \beta(x)\,|u|^p\,u = f(x,t), \quad -1 < x < 1,\ 0 < t \le T \tag{1.2} $$

subject to the following initial and boundary conditions:

$$ u(x,0) = u_0(x), \quad u_t(x,0) = \psi(x), \quad x \in (-1,1), \tag{1.3} $$

$$ u(-1,t) = 0, \quad u(1,t) = 0, \quad 0 \le t \le T, \tag{1.4} $$

where

$$ {}_0^C D_t^\gamma u(x,t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{\partial^2 u(x,s)}{\partial s^2} \frac{\mathrm{d}s}{(t-s)^{\gamma-1}}, \quad 1 < \gamma < 2 $$

is the Caputo fractional derivative of order γ with respect to t, p is a positive number, $\beta \ge 0$, and f is a given function.

We propose a fully discrete spectral method to solve the (1.2)–(1.4) numerically. The proposed scheme is based on a finite difference method in the temporal direction and a Legendre spectral method in the spatial direction. More precisely, we use the $L1$ approximation coupled with a Crank-Nicolson technique to approximate the Caputo time fractional derivative, and use central difference to approximate the nonlinear term $|u|^p\,u$. We give a detailed analysis for the stability and convergence of the fully discrete scheme. The convergence order of the proposed scheme in H^1 norm is $\mathrm{O}(\tau^{3-\gamma} + N^{1-m})$, where τ, N, and m are the time-step size, polynomial degree, and regularity in the space variable of the exact solution, respectively.

The rest of the paper is organized as follows. In Section 2, some preliminaries and notations are shown. In Section 3, we present the formulation of the fully discrete spectral scheme, and give a priori estimate for the approximate solutions. Based on the a priori estimate, the existence and uniqueness of the approximate solutions are proved. In Section 4, we analyse the stability and convergence of the fully discrete scheme. We do some numerical experiments in Section 5. Finally, some conclusions are given in Section 6.

2. Preliminaries and notations

Let $\Lambda = (-1, 1)$. Throughout this paper, we use Sobolev spaces $W^{r,p}(\Lambda)$ with norm $\|\cdot\|_{r,p}$. When $p = 2$, we denote $W^{r,2}(\Lambda)$ and its inner product, semi-norm and norm by $H^r(\Lambda)$, $(\cdot\,,\cdot)_r$, $|\cdot|_r$, and $\|\cdot\|_r$ respectively. In particular, $(\cdot\,,\cdot) = (\cdot\,,\cdot)_0$, $\|\cdot\| = \|\cdot\|_0$. Furthermore,

$$H_0^1(\Lambda) = \left\{ v \in H^1(\Lambda) \mid v(\pm 1) = 0 \right\}.$$

We denote by $L^\infty(0, T; H^m(\Lambda))$ the space of the measurable functions $v\colon (0, T) \to H^m(\Lambda)$, such that

$$\|v\|_{L^\infty(H^m)} = \operatorname*{ess\,sup}_{0 \le t \le T} \|v(t)\|_m < \infty,$$

$C^k([0, T]; H^m(\Lambda))$ $(0 \le k < \infty)$ the space of k-times continuous differentiable functions $v\colon [0, T] \to H^m(\Lambda)$, such that

$$\|v\|_{C^k(H^m)} = \sum_{i=0}^{k} \max_{0 \le t \le T} \left\| v^{(i)}(t) \right\|_m < \infty.$$

For simplicity we denote $\partial_x^k v(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k} v(x)$. Throughout the paper, c denotes a generic positive constant.

Let N be a positive integer. We denote by $\mathbb{P}_N(\Lambda)$ the space of all polynomials of degree less than or equal to N. $\mathbb{P}_N^0 := \{\phi \in \mathbb{P}_N(\Lambda) \mid \phi(\pm 1) = 0\}$. Next we introduce some projection approximation results.

Let $\pi_N^{1,0}$ be the H_0^1-orthogonal projection operator from $H_0^1(\Lambda)$ into $\mathbb{P}_N^0$, such that for all $u \in H_0^1(\Lambda)$,

$$\left(\partial_x \pi_N^{1,0} u, \partial_x v_N \right) = (\partial_x u, \partial_x v_N), \quad \forall\, v_N \in \mathbb{P}_N^0.$$

For the projection operator $\pi_N^{1,0}$, one has the following approximation result:

Lemma 2.1. [2] *For all $u \in H_0^1(\Lambda) \cap H^m(\Lambda)$, we have*

$$\left\| u - \pi_N^{1,0} u \right\|_k \le C N^{k-m} \|u\|_m, \quad k = 0, 1,\ m \ge 1,$$

where C is a positive constant independent of N.

The following Poincaré's inequality is useful.

Lemma 2.2. *For any $u(x) \in C^1[-1, 1]$, with $u(-1) = u(1) = 0$, we have*

$$\|u\| \le \frac{1}{\sqrt{2}} \|\partial_x u\|.$$

Proof. The inequality can be obtained by a simple computation. □

We then give a discrete Grönwall's inequality.

Lemma 2.3. [10] *Let k, B, and a_μ, b_μ, c_μ, γ_μ, for integers $\mu \geq 0$, be nonnegative numbers such that*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq k\sum_{\mu=0}^{n}\gamma_\mu a_\mu + k\sum_{\mu=0}^{n} c_\mu + B, \quad n \geq 0.$$

Suppose that $k\gamma_\mu < 1$, for all μ, and set $\sigma_\mu \equiv (1-k\gamma_\mu)^{-1}$. Then

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq \exp\left(k\sum_{\mu=0}^{n}\sigma_\mu\gamma_\mu\right)\left\{k\sum_{\mu=0}^{n} c_\mu + B\right\}, \quad n \geq 0.$$

The following lemma will be used in the proof of the existence of approximation solutions.

Lemma 2.4. [20, Lemma 1.4, Ch. 2] *Let X be a finite-dimensional Hilbert space with scalar product $[\cdot\,,\cdot]$ and norm $[\,\cdot\,]$, and let P be a continuous mapping from X into itself such that*

$$[P(\xi), \xi] > 0 \quad \textit{for } [\xi] = k > 0.$$

Then there exists $\xi \in X$, $[\xi] \leq k$, such that

$$P(\xi) = 0.$$

3. The formulation of the fully discrete scheme

For a positive integer M, let $t_k = k\tau$, $k = 0, 1, \ldots, M$, where $\tau = T/M$ is the time-step length. Given a grid function $w = \left\{w^k \mid 0 \leq k \leq M\right\}$, we define

$$w^{k+1/2} = \frac{1}{2}\left(w^{k+1} + w^k\right), \quad \delta_t w^{k+1/2} = \frac{1}{\tau}\left(w^{k+1} - w^k\right).$$

For the discretization of the Caputo fractional derivative of order γ $(1 < \gamma < 2)$ in time, we use the $L1$ approximation coupled with the Crank-Nicolson technique as in [19]; see also [22].

Denote $\gamma_0 = \tau^{\gamma-1}\Gamma(3-\gamma)$ and $b_j = (j+1)^{2-\gamma} - j^{2-\gamma}$ for $j \geq 0$. For a differentiable function $v(t)$, let

$$L_t^\gamma v^{k+1/2} = \frac{1}{\gamma_0}\left(\delta_t v^{k+1/2} - \sum_{j=0}^{k-1}(b_j - b_{j+1})\delta_t v^{k-1-j+1/2} - b_k v'(t_0)\right),$$

where $k = 0, 1, \ldots, M-1$.

For the approximation of fractional derivative of order $\gamma \in (1, 2)$, we have the following lemma.

Lemma 3.1. [18] *Let*

$$D_t^\gamma v(t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{v''(s)}{(t-s)^{\gamma-1}}\,\mathrm{d}s, \quad 0 < t \le t_{k+1}.$$

Suppose $v(t) \in C^3[0, t_{k+1}]$ *(*$0 \le k \le M-1$*), then*

$$\begin{aligned}&\left|\frac{1}{2}\left(D_t^\gamma v(t_{k+1}) + D_t^\gamma v(t_k)\right) - L_t^\gamma v^{k+1/2}\right|\\ &\le \frac{1}{\Gamma(3-\gamma)}\left[\frac{2-\gamma}{12} + \frac{2^{3-\gamma}}{3-\gamma} - (1+2^{1-\gamma}) + \frac{1}{12}\right] \max_{0\le t\le t_{k+1}} \left|v'''(t)\right| \tau^{3-\gamma}.\end{aligned}$$

It is direct to check that

$$\begin{aligned}&b_j > 0, \quad j = 0, 1, \ldots, k;\\ &1 = b_0 > b_1 > \cdots > b_k, \quad b_k \to 0 \text{ when } k \to \infty;\\ &\sum_{j=0}^{k-1}(b_j - b_{j+1}) + b_k = 1.\end{aligned}$$

Moreover, we have

$$b_j = (j+1)^{2-\gamma} - j^{2-\gamma} = (2-\gamma)\int_j^{j+1} \frac{1}{t^{\gamma-1}}\,\mathrm{d}t \ge (2-\gamma)(j+1)^{1-\gamma}.$$

Let $t_{k+1/2} = (k+1/2)\tau$. For the nonlinear term $|u|^p u$, using Taylor's expansion, we have

$$\frac{1}{p+2}\frac{|u(t_{k+1})|^{p+2} - |u(t_k)|^{p+2}}{u(t_{k+1}) - u(t_k)} - \left|u(t_{k+1/2})\right|^p u(t_{k+1/2}) = \mathrm{O}(\tau^2).$$

We discretize the space using a Legendre spectral method. Let $u_N^j \in \mathbb{P}_N^0$ be the approximation of $u(x,t)$ at time $t = t_j$ for $j = 0, 1, \ldots, M$. Then the fully discrete scheme in weak formulation for (1.2) is as follows: find $u_N^{k+1} \in \mathbb{P}_N^0$, such that

$$\begin{aligned}&(\delta_t u_N^{k+1/2}, v_N) + \gamma_0(\partial_x u_N^{k+1/2}, \partial_x v_N) + \gamma_0\left(\frac{\beta}{p+2}\frac{\left|u_N^{k+1}\right|^{p+2} - \left|u_N^k\right|^{p+2}}{u_N^{k+1} - u_N^k}, v_N\right)\\ &= \gamma_0(f^{k+1/2}, v_N) + \sum_{j=0}^{k-1}(b_j - b_{j+1})(\delta_t u_N^{k-1-j+1/2}, v_N) + b_k(\psi, v_N), \quad \forall\, v_N \in \mathbb{P}_N^0,\end{aligned} \tag{3.1}$$

with $u_N^0 = \pi_N^{1,0} u_0$, where $k = 0, 1, \ldots, M-1$.

Before we prove the well-posedness of the scheme, we need a priori estimate for u_N^{k+1}.

Lemma 3.2. *Suppose u_N^{k+1} $(0 \le k \le M-1)$ is the solution of the problem* (3.1)*, then we have*

$$
\begin{aligned}
&\frac{\tau^{2-\gamma}}{\Gamma(3-\gamma)} \sum_{j=0}^{k} b_j \left\| \delta_t u_N^{k-j+1/2} \right\|^2 + \left\| \partial_x u_N^{k+1} \right\|^2 + \frac{2}{p+2} \int_{-1}^{1} \beta \left| u_N^{k+1} \right|^{p+2} \mathrm{d}x \\
&\le \left\| \partial_x u_N^0 \right\|^2 + \frac{2}{p+2} \int_{-1}^{1} \beta \left| u_N^0 \right|^{p+2} \mathrm{d}x \\
&\quad + \tau \sum_{j=0}^{k} 2\Gamma(2-\gamma) T^{\gamma-1} \left\| f^{j+1/2} \right\|^2 + \frac{2((k+1)\tau)^{2-\gamma}}{\Gamma(3-\gamma)} \|\psi\|^2.
\end{aligned}
$$

Proof. Taking $v_N = \delta_t u_N^{k+1/2}$ in (3.1) gives

$$
\begin{aligned}
&\left\| \delta_t u_N^{k+1/2} \right\|^2 + \gamma_0 (\partial_x u_N^{k+1/2}, \partial_x \delta_t u_N^{k+1/2}) \\
&\quad + \gamma_0 \left(\frac{\beta}{p+2} \frac{\left| u_N^{k+1} \right|^{p+2} - \left| u_N^k \right|^{p+2}}{u_N^{k+1} - u_N^k}, \delta_t u_N^{k+1/2} \right) \\
&= \gamma_0 (f^{k+1/2}, \delta_t u_N^{k+1/2}) + \sum_{j=0}^{k-1} (b_j - b_{j+1}) (\delta_t u_N^{k-1-j+1/2}, \delta_t u_N^{k+1/2}) + b_k (\psi, \delta_t u_N^{k+1/2}).
\end{aligned} \tag{3.2}
$$

For the second and third terms on the left-hand side of (3.2), we have

$$
\gamma_0 (\partial_x u_N^{k+1/2}, \partial_x \delta_t u_N^{k+1/2}) = \frac{\gamma_0}{2\tau} \left(\left\| \partial_x u_N^{k+1} \right\|^2 - \left\| \partial_x u_N^k \right\|^2 \right)
$$

and

$$
\begin{aligned}
&\gamma_0 \left(\frac{\beta}{p+2} \frac{\left| u_N^{k+1} \right|^{p+2} - \left| u_N^k \right|^{p+2}}{u_N^{k+1} - u_N^k}, \delta_t u_N^{k+1/2} \right) \\
&= \frac{\gamma_0}{(p+2)\tau} \left(\int_{-1}^{1} \beta \left| u_N^{k+1} \right|^{p+2} \mathrm{d}x - \int_{-1}^{1} \beta \left| u_N^k \right|^{p+2} \mathrm{d}x \right).
\end{aligned}
$$

For the right-hand side of (3.2), using Hölder's inequality and Young's inequality, one has

$$
\begin{aligned}
&\sum_{j=0}^{k-1} (b_j - b_{j+1}) (\delta_t u_N^{k-1-j+1/2}, \delta_t u_N^{k+1/2}) \\
&\le \frac{1}{2} \sum_{j=0}^{k-1} (b_j - b_{j+1}) \left(\left\| \delta_t u_N^{k-1-j+1/2} \right\|^2 + \left\| \delta_t u_N^{k+1/2} \right\|^2 \right) \\
&= \frac{1}{2} \sum_{j=0}^{k-1} (b_j - b_{j+1}) \left\| \delta_t u_N^{k-1-j+1/2} \right\|^2 + \frac{1}{2} (1 - b_k) \left\| \delta_t u_N^{k+1/2} \right\|^2,
\end{aligned}
$$

$$b_k(\psi, \delta_t u_N^{k+1/2}) \leq b_k \left(\|\psi\|^2 + \frac{1}{4} \left\| \delta_t u_N^{k+1/2} \right\|^2 \right)$$

and

$$\gamma_0 (f^{k+1/2}, \delta_t u_N^{k+1/2}) \leq \frac{\gamma_0^2}{b_k} \left\| f^{k+1/2} \right\|^2 + \frac{1}{4} b_k \left\| \delta_t u_N^{k+1/2} \right\|^2 .$$

Substituting these inequalities into (3.2), and noticing that

$$\left\| \delta_t u_N^{k+1/2} \right\|^2 + \sum_{j=0}^{k-1} b_{j+1} \left\| \delta_t u_N^{k-1-j+1/2} \right\|^2 = \sum_{j=0}^{k} b_j \left\| \delta_t u_N^{k-j+1/2} \right\|^2 ,$$

we obtain

$$\begin{aligned} &\tau \sum_{j=0}^{k} b_j \left\| \delta_t u_N^{k-j+1/2} \right\|^2 + \gamma_0 \left\| \partial_x u_N^{k+1} \right\|^2 + \frac{2\gamma_0}{p+2} \int_{-1}^{1} \beta \left| u_N^{k+1} \right|^{p+2} \mathrm{d}x \\ &\leq \tau \sum_{j=0}^{k-1} b_j \left\| \delta_t u_N^{k-1-j+1/2} \right\|^2 + \gamma_0 \left\| \partial_x u_N^{k} \right\|^2 \\ &\quad + \frac{2\gamma_0}{p+2} \int_{-1}^{1} \beta \left| u_N^{k} \right|^{p+2} \mathrm{d}x + 2\tau b_k \|\psi\|^2 + \frac{2\tau\gamma_0^2}{b_k} \left\| f^{k+1/2} \right\|^2 . \end{aligned} \tag{3.3}$$

Let

$$F^0 = \gamma_0 \left\| \partial_x u_N^0 \right\|^2 + \frac{2\gamma_0}{p+2} \int_{-1}^{1} \beta \left| u_N^0 \right|^{p+2} \mathrm{d}x$$

and

$$F^{k+1} = \tau \sum_{j=0}^{k} b_j \left\| \delta_t u_N^{k-j+1/2} \right\|^2 + \gamma_0 \left\| \partial_x u_N^{k+1} \right\|^2 + \frac{2\gamma_0}{p+2} \int_{-1}^{1} \beta \left| u_N^{k+1} \right|^{p+2} \mathrm{d}x$$

for $0 \leq k \leq M-1$. Then from (3.3), we have

$$F^{k+1} \leq F^k + 2\tau b_k \|\psi\|^2 + \frac{2\tau\gamma_0^2}{b_k} \left\| f^{k+1/2} \right\|^2 ,$$

or equivalently

$$F^{k+1} \leq F^0 + \tau \sum_{j=0}^{k} \frac{2\gamma_0^2}{b_j} \left\| f^{j+1/2} \right\|^2 + 2\tau (k+1)^{2-\gamma} \|\psi\|^2 . \tag{3.4}$$

Since $b_j \geq (2-\gamma)(j+1)^{1-\gamma}$, $\Gamma(s+1) = s\Gamma(s)$ and $\gamma_0 = \Gamma(3-\gamma)\tau^{\gamma-1}$, we have

$$b_j^{-1} \leq \Gamma(2-\gamma) \frac{T^{\gamma-1}}{\gamma_0}, \quad \tau (k+1)^{2-\gamma} = \frac{\gamma_0 ((k+1)\tau)^{2-\gamma}}{\Gamma(3-\gamma)} .$$

Substituting these two inequalities into (3.4), we have

$$F^{k+1} \leq F^0 + \tau \sum_{j=0}^{k} 2\Gamma(2-\gamma) T^{\gamma-1} \gamma_0 \left\| f^{j+1/2} \right\|^2 + \frac{2\gamma_0 ((k+1)\tau)^{2-\gamma}}{\Gamma(3-\gamma)} \|\psi\|^2 .$$

Then we have

$$\begin{aligned}
&\frac{\tau^{2-\gamma}}{\Gamma(3-\gamma)}\sum_{j=0}^{k} b_j \left\|\delta_t u_N^{k-j+1/2}\right\|^2 + \left\|\partial_x u_N^{k+1}\right\|^2 + \frac{2}{p+2}\int_{-1}^{1}\beta\left|u_N^{k+1}\right|^{p+2}\mathrm{d}x \\
\le\, & \left\|\partial_x u_N^0\right\|^2 + \frac{2}{p+2}\int_{-1}^{1}\beta\left|u_N^0\right|^{p+2}\mathrm{d}x \\
&+\tau\sum_{j=0}^{k} 2\Gamma(2-\gamma)T^{\gamma-1}\left\|f^{j+1/2}\right\|^2 + \frac{2((k+1)\tau)^{2-\gamma}}{\Gamma(3-\gamma)}\|\psi\|^2. \qquad \square
\end{aligned}$$

For the existence of solution u_N^{k+1} for (3.1), we have the following theorem.

Theorem 3.3. *For $\left\{u_N^j\right\}_{j=0}^{k}$ given, there exists one u_N^{k+1} satisfying* (3.1).

Proof. Let $X = \mathbb{P}_N^0$; the scalar product on X is the usual inner product $(\cdot,\cdot)$ in $L^2(\Lambda)$. Define a mapping $P\colon X \to X$, such that

$$\begin{aligned}
[P(w), v] &= (P(w), v) \\
&= \left(\frac{w}{\tau}, v\right) + \gamma_0\left(\frac{\partial_x w + 2\partial_x u_N^k}{2}, \partial_x v\right) \\
&\quad + \gamma_0\left(\frac{\beta}{p+2}\frac{\left|w+u_N^k\right|^{p+2} - \left|u_N^k\right|^{p+2}}{w}, v\right) - \gamma_0(f^{k+1/2}, v) \\
&\quad - \sum_{j=0}^{k-1}(b_j - b_{j+1})(\delta_t u_N^{k-1-j+1/2}, v) - b_k(\psi, v), \quad \forall\, w, v \in X.
\end{aligned}$$

The continuity of the mapping P is obvious. Next, we have

$$\begin{aligned}
[P(w), w] = {}& \frac{\|w\|^2}{\tau} + \gamma_0\left(\frac{\|\partial_x w\|^2}{2} + (\partial_x u_N^k, \partial_x w)\right) \\
&+ \frac{\gamma_0}{p+2}\left(\int_{-1}^{1}\beta\left|w+u_N^k\right|^{p+2}\mathrm{d}x - \int_{-1}^{1}\beta\left|u_N^k\right|^{p+2}\mathrm{d}x\right) \\
&- \gamma_0(f^{k+1/2}, w) - \sum_{j=0}^{k-1}(b_j - b_{j+1})(\delta_t u_N^{k-1-j+1/2}, w) - b_k(\psi, w).
\end{aligned} \tag{3.5}$$

For the right-hand side of (3.5), using Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
\sum_{j=0}^{k-1}(b_j - b_{j+1})(\delta_t u_N^{k-1-j+1/2}, w) &\le \frac{1}{2}\sum_{j=0}^{k-1}(b_j - b_{j+1})\left(\tau\left\|\delta_t u_N^{k-1-j+1/2}\right\|^2 + \frac{\|w\|^2}{\tau}\right) \\
&= \frac{\tau}{2}\sum_{j=0}^{k-1}(b_j - b_{j+1})\left\|\delta_t u_N^{k-1-j+1/2}\right\|^2 + \frac{1}{2}(1-b_k)\frac{\|w\|^2}{\tau},
\end{aligned}$$

$$b_k(\psi, w) \le b_k \left(\tau \|\psi\|^2 + \frac{1}{4} \frac{\|w\|^2}{\tau} \right)$$

and

$$\gamma_0(f^{k+1/2}, w) \le \frac{\gamma_0^2}{b_k} \tau \left\| f^{k+1/2} \right\|^2 + \frac{1}{4} b_k \frac{\|w\|^2}{\tau}.$$

Substituting the above inequalities into (3.5), and using the a priori estimate in Lemma 3.2, we get

$$\begin{aligned}
[P(w), w] \ge& \frac{1}{2} \frac{\|w\|^2}{\tau} - \gamma_0 \frac{\left\|\partial_x u_N^k\right\|^2}{2} + \frac{\gamma_0}{p+2} \left(\int_{-1}^{1} \beta \left|w + u_N^k\right|^{p+2} \mathrm{d}x - \int_{-1}^{1} \beta \left|u_N^k\right|^{p+2} \mathrm{d}x \right) \\
&- \frac{\gamma_0^2}{b_k} \tau \left\| f^{k+1/2} \right\|^2 - \frac{1}{2} \sum_{j=0}^{k-1} (b_j - b_{j+1}) \tau \left\| \delta_t u_N^{k-1-j+1/2} \right\|^2 - b_k \tau \|\psi\|^2 \\
\ge& \frac{1}{2} \frac{\|w\|^2}{\tau} - \gamma_0 \frac{\left\|\partial_x u_N^k\right\|^2}{2} - \frac{\gamma_0}{p+2} \int_{-1}^{1} \beta \left|u_N^k\right|^{p+2} \mathrm{d}x \\
&- \frac{\gamma_0^2}{b_k} \tau \left\| f^{k+1/2} \right\|^2 - \frac{1}{2} \sum_{j=0}^{k-1} b_j \tau \left\| \delta_t u_N^{k-1-j+1/2} \right\|^2 - b_k \tau \|\psi\|^2 \\
=& \frac{1}{2} \frac{\|w\|^2}{\tau} - \frac{\gamma_0}{2} \Bigg(\left\|\partial_x u_N^k\right\|^2 + \frac{2}{p+2} \int_{-1}^{1} \beta \left|u_N^k\right|^{p+2} \mathrm{d}x \\
&\qquad + \frac{\tau^{2-\gamma}}{\Gamma(3-\gamma)} \sum_{j=0}^{k-1} b_j \left\| \delta_t u_N^{k-1-j+1/2} \right\|^2 + \frac{2\gamma_0}{b_k} \tau \left\| f^{k+1/2} \right\|^2 + \frac{2b_k}{\gamma_0} \tau \|\psi\|^2 \Bigg) \\
\ge& \frac{\gamma_0}{2} \left(\frac{\|w\|^2}{\tau\gamma_0} - E \right),
\end{aligned}$$

where

$$E = \left\|\partial_x u_N^0\right\|^2 + \frac{2}{p+2} \int_{-1}^{1} \beta \left|u_N^0\right|^{p+2} \mathrm{d}x + \tau \sum_{j=0}^{k} 2\Gamma(2-\gamma) T^{\gamma-1} \left\| f^{j+1/2} \right\|^2 + \frac{2T^{2-\gamma}}{\Gamma(3-\gamma)} \|\psi\|^2 .$$

Then it follows that $[P(w), w] > 0$ for $\|w\| = K$, $K > (\tau\gamma_0 E)^{1/2}$. By virtue of Lemma 2.4, there exists $w^{k+1} \in X$ satisfying $P(w^{k+1}) = 0$. Set $u_N^{k+1} = w^{k+1} + u_N^k$, thus the existence of u_N^{k+1} is proved. □

Next we give the uniqueness theorem.

Theorem 3.4. *The solution* u_N^{k+1} *for* (3.1) *is unique.*

Proof. Suppose $\left\{u_*^j\right\}_{j=0}^M$, $\left\{u_{**}^j\right\}_{j=0}^M$ are the solutions of the problem (3.1), with the same initial condition. Let $u^j = u_*^j - u_{**}^j$, then we have

$$\begin{aligned}
&(\delta_t u^{k+1/2}, v_N) + \gamma_0(\partial_x u^{k+1/2}, \partial_x v_N) + \gamma_0 \left(\beta(G^{k+1/2} - \overline{G}^{k+1/2}), v_N \right) \\
&= \sum_{j=0}^{k-1} (b_j - b_{j+1})(\delta_t u^{k-1-j+1/2}, v_N), \quad \forall\, v_N \in \mathbb{P}_N^0,\ k = 0, 1, \ldots, M-1,
\end{aligned} \tag{3.6}$$

where

$$G^{k+1/2}=\frac{1}{p+2}\frac{\left|u_*^{k+1}\right|^{p+2}-\left|u_*^k\right|^{p+2}}{u_*^{k+1}-u_*^k},\quad \overline{G}^{k+1/2}=\frac{1}{p+2}\frac{\left|u_{**}^{k+1}\right|^{p+2}-\left|u_{**}^k\right|^{p+2}}{u_{**}^{k+1}-u_{**}^k}.$$

Denote $g(s)=|s|^p s$, then we have [1]

$$G^{k+1/2}=\int_0^1 g(\theta u_*^{k+1}+(1-\theta)u_*^k)\,\mathrm{d}\theta,\quad \overline{G}^{k+1/2}=\int_0^1 g(\theta u_{**}^{k+1}+(1-\theta)u_{**}^k)\,\mathrm{d}\theta.$$

By virtue of a priori estimate, the u_*^j, u_{**}^j are uniformly bounded, therefore,

$$\begin{aligned}\left|G^{k+1/2}-\overline{G}^{k+1/2}\right| &= \int_0^1 g(\theta u_*^{k+1}+(1-\theta)u_*^k)-g(\theta u_{**}^{k+1}+(1-\theta)u_{**}^k)\,\mathrm{d}\theta\\ &\le c_1\int_0^1 \theta\left|u^{k+1}\right|+(1-\theta)\left|u^k\right|\mathrm{d}\theta\\ &=\frac{c_1}{2}\left(\left|u^{k+1}\right|+\left|u^k\right|\right).\end{aligned}$$

Let $v_N=\delta_t u^{k+1/2}$ in (3.6), we have

$$\begin{aligned}&\left\|\delta_t u^{k+1/2}\right\|^2+\frac{\gamma_0}{2\tau}\left(\left\|\partial_x u^{k+1}\right\|^2-\left\|\partial_x u^k\right\|^2\right)\\ &=\sum_{j=0}^{k-1}(b_j-b_{j+1})(\delta_t u^{k-1-j+1/2},\delta_t u^{k+1/2})-\gamma_0\left(\beta(G^{k+1/2}-\overline{G}^{k+1/2}),\delta_t u^{k+1/2}\right).\end{aligned}\tag{3.7}$$

For the right-hand side of (3.7), using Hölder's inequality and Young's inequality, one has

$$\begin{aligned}&\sum_{j=0}^{k-1}(b_j-b_{j+1})(\delta_t u^{k-1-j+1/2},\delta_t u^{k+1/2})\\ &\le\frac12\sum_{j=0}^{k-1}(b_j-b_{j+1})\left(\left\|\delta_t u^{k-1-j+1/2}\right\|^2+\left\|\delta_t u^{k+1/2}\right\|^2\right)\\ &=\frac12\sum_{j=0}^{k-1}(b_j-b_{j+1})\left\|\delta_t u^{k-1-j+1/2}\right\|^2+\frac12(1-b_k)\left\|\delta_t u^{k+1/2}\right\|^2\end{aligned}$$

and

$$-\gamma_0\left(\beta(G^{k+1/2}-\overline{G}^{k+1/2}),\delta_t u^{k+1/2}\right)\le\frac{c_1^2\gamma_0^2}{4b_k}\left(\left\|u^{k+1}\right\|^2+\left\|u^k\right\|^2\right)+\frac12 b_k\left\|\delta_t u_N^{k+1/2}\right\|^2.$$

Then we have

$$\begin{aligned}&\tau\sum_{j=0}^{k}b_j\left\|\delta_t u^{k-j+1/2}\right\|^2+\gamma_0\left\|\partial_x u^{k+1}\right\|^2\\ &\le\tau\sum_{j=0}^{k-1}b_j\left\|\delta_t u^{k-1-j+1/2}\right\|^2+\gamma_0\left\|\partial_x u_N^k\right\|^2+\frac{\tau c_1^2\gamma_0^2}{2b_k}\left(\left\|u^{k+1}\right\|^2+\left\|u^k\right\|^2\right).\end{aligned}$$

Following the same lines as in the proof of Lemma 3.2, and using Poincaré's inequality, we have

$$\begin{aligned}\left\|\partial_x u^{k+1}\right\|^2 &\le \left\|\partial_x u^0\right\|^2 + \tau\sum_{j=0}^{k}\frac{c_1^2}{2}\Gamma(2-\gamma)T^{\gamma-1}\left(\left\|u^{j+1}\right\|^2+\left\|u^j\right\|^2\right)\\ &\le \left\|\partial_x u^0\right\|^2 + \tau\sum_{j=0}^{k+1} c_2\left\|\partial_x u^j\right\|^2.\end{aligned}$$

Then suppose $\tau < 1/c_2$, using Grönwall's inequality in Lemma 2.3, we have

$$\left\|\partial_x u^{k+1}\right\|^2 \le c\left\|\partial_x u^0\right\|^2 = 0.$$

Finally, using Poincaré's inequality, we have $\left\|u^{k+1}\right\| = 0$, that is $u_*^{k+1} - u_{**}^{k+1} \equiv 0$. The uniqueness is proved. □

4. Stability and convergence of the fully discrete scheme

Suppose $\left\{\widetilde{u}_N^j\right\}_{j=0}^M$ are the solutions of the following:

$$\begin{aligned}&(\delta_t\widetilde{u}_N^{k+1/2}, v_N) + \gamma_0(\partial_x\widetilde{u}_N^{k+1/2}, \partial_x v_N) + \gamma_0\left(\frac{\beta}{p+2}\frac{\left|\widetilde{u}_N^{k+1}\right|^{p+2}-\left|\widetilde{u}_N^k\right|^{p+2}}{\widetilde{u}_N^{k+1}-\widetilde{u}_N^k}, v_N\right)\\ &= \gamma_0(\widetilde{f}^{k+1/2}, v_N) + \sum_{j=0}^{k-1}(b_j - b_{j+1})(\delta_t\widetilde{u}_N^{k-1-j+1/2}, v_N) + b_k(\widetilde{\psi}, v_N), \quad \forall\, v_N \in \mathbb{P}_N^0,\end{aligned}$$

with an initial condition $\widetilde{u}_N^0$, where $k = 0, 1, \ldots, M-1$.

Following the same lines as in the proof of Lemma 3.2 and Theorem 3.4, we can obtain the following stability result.

Theorem 4.1. *Suppose $\left\{u_N^j\right\}_{j=0}^M$ are the solutions of problem (3.1), with an initial condition u_N^0. Let $\eta_N^j = u_N^j - \widetilde{u}_N^j$, then we have*

$$\begin{aligned}&\left\|\partial_x\eta_N^{k+1}\right\|^2\\ &\le c\left(\left\|\partial_x\eta_N^0\right\|^2 + \tau\sum_{j=0}^{k}2\Gamma(2-\gamma)T^{\gamma-1}\left\|f^{k+1/2}-\widetilde{f}^{k+1/2}\right\|^2 + \frac{2T^{2-\gamma}}{\Gamma(3-\gamma)}\left\|\psi-\widetilde{\psi}\right\|^2\right).\end{aligned}$$

For the convergence of the fully discrete scheme (3.1), we have

Theorem 4.2. *Let u be the exact solution of (1.2)–(1.4), $\left\{u_N^k\right\}_{k=0}^M$ be the solution of the problem (3.1) with the initial condition $u_N^0 = \pi_N^{1,0}u_0(x)$. Suppose $u \in C^3([0,T]; H^1(\Lambda)) \cap$*

$L^\infty(0,T;H^m(\Lambda))$, $D_t^\gamma u \in L^\infty(0,T;H^m(\Lambda))$, $\psi \in H^m(\Lambda)$, $m \geq 1$, *then for* $j = 0,1,\ldots,M$, *we have*

$$\left\|\partial_x u(t_j) - \partial_x u_N^j\right\|^2 \leq c2\Gamma(2-\gamma)T^\gamma \left(N^{-2m}\left\|D_t^\gamma u\right\|^2_{L^\infty(H^m)} + \tau^{6-2\gamma}\left\|u\right\|^2_{C^3(H^1)}\right) + \frac{c2T^{2-\gamma}}{\Gamma(3-\gamma)}N^{-2m}\left\|\psi\right\|_m^2 + cN^{2-2m}\left\|u\right\|^2_{L^\infty(H^m)}.$$

Proof. Let $e_N^j = u(t_j) - u_N^j$, $\widetilde{e}_N^j = \pi_N^{1,0}u(t_j) - u_N^j$, $\widehat{e}_N^j = u(t_j) - \pi_N^{1,0}u(t_j)$, thus we have $e_N^j = \widetilde{e}_N^j + \widehat{e}_N^j$, in particular $e_N^0 = u^0 - \pi_N^{1,0}u^0 = \widehat{e}_N^0$, $\widetilde{e}_N^0 = 0$. From the initial equation (1.2) and the fully discrete scheme (3.1), we have the following error equation

$$\begin{aligned}&(\delta_t e_N^{k+1/2}, v_N) + \gamma_0(\partial_x e_N^{k+1/2}, \partial_x v_N) + \gamma_0\left(\beta(G^{k+1/2} - \overline{G}^{k+1/2}), v_N\right)\\ &= \sum_{j=0}^{k-1}(b_j - b_{j+1})(\delta_t e_N^{k-1-j+1/2}, v_N) + \gamma_0(R_t^{k+1/2}, v_N), \quad \forall\, v_N \in \mathbb{P}_N^0,\end{aligned}$$

where $R_t^{k+1/2} = L_t^\gamma u^{k+1/2} - \frac{1}{2}(D_t^\gamma u(t_{k+1}) + D_t^\gamma u(t_k)) + c\tau^2$,

$$G^{k+1/2} = \frac{1}{p+2}\frac{\left|u^{k+1}\right|^{p+2} - \left|u^k\right|^{p+2}}{u^{k+1} - u^k}, \quad \overline{G}^{k+1/2} = \frac{1}{p+2}\frac{\left|u_N^{k+1}\right|^{p+2} - \left|u_N^k\right|^{p+2}}{u_N^{k+1} - u_N^k}.$$

As $e_N^j = \widetilde{e}_N^j + \widehat{e}_N^j$, and by virtue of the definition of the projection operator $\pi_N^{1,0}$, we get

$$\begin{aligned}&(\delta_t \widetilde{e}_N^{k+1/2}, v_N) + \gamma_0(\partial_x \widetilde{e}_N^{k+1/2}, \partial_x v_N) + \gamma_0\left(\beta(G^{k+1/2} - \overline{G}^{k+1/2}), v_N\right)\\ &= \sum_{j=0}^{k-1}(b_j - b_{j+1})(\delta_t \widetilde{e}_N^{k-1-j+1/2}, v_N) + \gamma_0(R_t^{k+1/2}, v_N) + \sum_{i=1}^{2} R_i^{k+1}, \quad \forall\, v_N \in \mathbb{P}_N^0,\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}R_1^{k+1} &= -\left(\delta_t \widehat{e}_N^{k+1/2} - \sum_{j=0}^{k-1}(b_j - b_{j+1})\delta_t \widehat{e}_N^{k-1-j+1/2} - b_k(\psi - \pi_N^{1,0}\psi), v_N\right),\\ R_2^{k+1} &= -b_k(\psi - \pi_N^{1,0}\psi, v_N).\end{aligned}$$

Denote $\widehat{e}_N(x,t) = u - \pi_N^{1,0}u$, we have

$$\begin{aligned}&\delta_t \widehat{e}_N^{k+1/2} - \sum_{j=0}^{k-1}(b_j - b_{j+1})\delta_t \widehat{e}_N^{k-1-j+1/2} - b_k(\psi - \pi_N^{1,0}\psi)\\ &= \gamma_0 L_t^\alpha \widehat{e}_N^{k+1/2}\\ &= \gamma_0 \frac{1}{2}\left(D_t^\gamma \widehat{e}_N(t_{k+1}) + D_t^\gamma \widehat{e}_N(t_k)\right) + \gamma_0 \widehat{R}_t^{k+1/2}.\end{aligned}$$

Then according to the Lemmas 2.1, 2.2 and 3.1, we have

$$\left|R_1^{k+1}\right| \le \frac{c\gamma_0^2}{b_k}\left(N^{-2m}\left\|D_t^\gamma u\right\|_{L^\infty(H^m)}^2 + \tau^{6-2\gamma}\left\|u\right\|_{C^3(H^1)}\right) + \frac{1}{8}b_k\left\|v_N\right\|^2,$$
$$\left|R_2^{k+1}\right| \le cb_kN^{-2m}\left\|\psi\right\|_m^2 + \frac{1}{8}b_k\left\|v_N\right\|^2,$$

and

$$\gamma_0(R_t^{k+1/2}, v_N) \le \frac{c\gamma_0^2}{b_k}\tau^{6-2\gamma}\left\|u\right\|_{C^3(L^2)} + \frac{1}{8}b_k\left\|v_N\right\|^2.$$

We also have

$$\begin{aligned}\sum_{j=0}^{k-1}(b_j - b_{j+1})(\delta_t\widetilde{e}_N^{k-1-j+1/2}, v_N) &\le \frac{1}{2}\sum_{j=0}^{k-1}(b_j - b_{j+1})\left(\left\|\delta_t\widetilde{e}_N^{k-1-j+1/2}\right\|^2 + \left\|v_N\right\|^2\right)\\ &= \frac{1}{2}\sum_{j=0}^{k-1}(b_j - b_{j+1})\left\|\delta_t\widetilde{e}_N^{k-1-j+1/2}\right\|^2 + \frac{1}{2}(1-b_k)\left\|v_N\right\|^2.\end{aligned}$$

Denote $g(s) = |s|^p s$, then we have

$$G^{k+1/2} = \int_0^1 g(\theta u^{k+1} + (1-\theta)u^k)\,\mathrm{d}\theta, \quad \overline{G}^{k+1/2} = \int_0^1 g(\theta u_N^{k+1} + (1-\theta)u_N^k)\,\mathrm{d}\theta.$$

Therefore,

$$\begin{aligned}\left|G^{k+1/2} - \overline{G}^{k+1/2}\right| &= \int_0^1 g(\theta u^{k+1} + (1-\theta)u^k) - g(\theta u_N^{k+1} + (1-\theta)u_N^k)\,\mathrm{d}\theta\\ &\le c_1\int_0^1 \theta\left|e_N^{k+1}\right| + (1-\theta)\left|e_N^k\right|\mathrm{d}\theta\\ &= \frac{c_1}{2}\left(\left|e_N^{k+1}\right| + \left|e_N^k\right|\right)\end{aligned}$$

and

$$\gamma_0\left(\beta(G^{k+1/2} - \overline{G}^{k+1/2}), v_N\right) \le \frac{c_1^2\gamma_0^2}{b_k}\left(\left\|e_N^{k+1}\right\|^2 + \left\|e_N^k\right\|^2\right) + \frac{1}{8}b_k\left\|v_N\right\|^2.$$

Substituting these above inequalities into (4.1), and taking $v_N = \delta_t\widetilde{e}_N^{k+1/2}$, we get

$$\begin{aligned}&\tau\sum_{j=0}^{k} b_j\left\|\delta_t\widetilde{e}_N^{k-j+1/2}\right\|^2 + \gamma_0\left\|\partial_x\widetilde{e}_N^{k+1}\right\|^2\\ &\le \tau\sum_{j=0}^{k-1} b_j\left\|\delta_t\widetilde{e}_N^{k-1-j+1/2}\right\|^2 + \gamma_0\left\|\partial_x\widetilde{e}_N^{k}\right\|^2 + 2\tau b_kcN^{-2m}\left\|\psi\right\|_m^2\\ &\quad + \frac{2\tau\gamma_0^2}{b_k}c\left(N^{-2m}\left\|D_t^\gamma u\right\|_{L^\infty(H^m)}^2 + \tau^{6-2\gamma}\left\|u\right\|_{C^3(H^1)}\right) + \frac{2\tau\gamma_0^2c_1^2}{b_k}\left(\left\|e_N^{k+1}\right\|^2 + \left\|e_N^k\right\|^2\right).\end{aligned}$$

Then following the same lines as in the proof of Lemma 3.2, we obtain

$$\begin{aligned}\left\|\partial_x\widetilde{e}_N^{k+1}\right\|^2 \le{}& \tau\sum_{j=0}^{k} c2\Gamma(2-\gamma)T^{\gamma-1}\left(N^{-2m}\left\|D_t^\gamma u\right\|_{L^\infty(H^m)}^2+\tau^{6-2\gamma}\left\|u\right\|_{C^3(H^1)}\right)\\ &+\tau\sum_{j=0}^{k} c_1^2 2\Gamma(2-\gamma)T^{\gamma-1}\left(\left\|e_N^{j+1}\right\|^2+\left\|e_N^{j}\right\|^2\right)+\frac{c2T^{2-\gamma}}{\Gamma(3-\gamma)}N^{-2m}\left\|\psi\right\|_m^2.\end{aligned}$$

Finally, using the triangular inequality $\left\|\partial_x e_N^{k+1}\right\| \le \left\|\partial_x\widetilde{e}_N^{k+1}\right\|+\left\|\partial_x\widehat{e}_N^{k+1}\right\|$ and Lemma 2.1, we have

$$\begin{aligned}\left\|\partial_x e_N^{k+1}\right\|^2 \le{}& cN^{2-2m}\left\|u\right\|_{L^\infty(H^m)}^2+\tau\sum_{j=0}^{k+1} c_1^2 2\Gamma(2-\gamma)T^{\gamma-1}\left\|\partial_x e_N^{j}\right\|^2\\ &+c\Gamma(2-\gamma)T^{\gamma}\left(N^{-2m}\left\|D_t^\gamma u\right\|_{L^\infty(H^m)}^2+\tau^{6-2\gamma}\left\|u\right\|_{C^3(H^1)}\right)\\ &+\frac{c2T^{2-\gamma}}{\Gamma(3-\gamma)}N^{-2m}\left\|\psi\right\|_m^2.\end{aligned}$$

Suppose $\tau c_1^2 2\Gamma(2-\gamma)T^{\gamma-1}<1$, then according to Grönwall's inequality in Lemma 2.3, we have

$$\begin{aligned}\left\|\partial_x e_N^{k+1}\right\|^2 \le{}& c2\Gamma(2-\gamma)T^{\gamma}\left(N^{-2m}\left\|D_t^\gamma u\right\|_{L^\infty(H^m)}^2+\tau^{6-2\gamma}\left\|u\right\|_{C^3(H^1)}^2\right)\\ &+\frac{c2T^{2-\gamma}}{\Gamma(3-\gamma)}N^{-2m}\left\|\psi\right\|_m^2+cN^{2-2m}\left\|u\right\|_{L^\infty(H^m)}^2. \qquad \square\end{aligned}$$

5. Numerical experiment

5.1. Implementation

For the implementation of our methods, we follow the similar lines as in paper [13]. Both of the integrals are evaluated by using numerical quadratures. We use the Legendre-Gauss-Lobatto quadrature to compute the integrals. Let $L_n(x)$ denote the Legendre polynomial with degree n. $\{x_j\}_{j=0}^N$ are the zeros of $(1-x^2)L_N'(x)$, and the weights are expressed by

$$\omega_j=\frac{2}{N(N+1)}\frac{1}{[L_N(x_j)]^2},\quad 0\le j\le N.$$

$\{x_j,\omega_j\}_{j=0}^N$ are referred to as the Legendre-Gauss-Lobatto quadrature nodes and weights, such that the following quadrature holds:

$$\int_{-1}^{1} p(x)\,\mathrm{d}x=\sum_{j=0}^{N} p(x_j)\omega_j,\quad \forall\, p\in\mathbb{P}_{2N-1}(\Lambda).$$

Define the discrete inner product as follows:

$$(\phi,\psi)_N=\sum_{j=0}^{N}\phi(x_j)\psi(x_j)\omega_j,\quad \forall\,\phi,\psi\in C^0(\overline{\Lambda}),$$

and the associated discrete norm $\|\phi\|_N := (\phi,\phi)_N^{1/2}$.

Let $\alpha_0 = \tau\gamma_0/2$, we rewrite (3.1) in the form

$$(u_N^{k+1}, v_N)_N + \alpha_0(\partial_x u_N^{k+1}, \partial_x v_N)_N + \frac{2\alpha_0}{p+2}\left(\beta\frac{\left|u_N^{k+1}\right|^{p+2} - \left|u_N^k\right|^{p+2}}{u_N^{k+1} - u_N^k}, v_N\right) = F_N^{k+1}(v_N)$$

for all $v_N \in \mathbb{P}_N^0$, where

$$\begin{aligned} F_N^{k+1}(v_N) = {} & (u_N^k, v_N)_N - \alpha_0(\partial_x u_N^k, \partial_x v_N)_N + \alpha_0(f^{k+1} + f^k, v_N)_N \\ & + \sum_{j=0}^{k-1}(b_j - b_{j+1})(u_N^{k-j} - u_N^{k-1-j}, v_N)_N + \tau b_k(\psi, v_N)_N. \end{aligned}$$

We express the function u_N^{k+1} in terms of the Lagrangian interpolants based on the Legendre-Gauss-Lobatto points x_j, $j = 0, 1, \ldots, N$,

$$u_N^{k+1}(x) = \sum_{j=0}^{N} \widehat{u}_j^{k+1} h_j(x),$$

where $\widehat{u}_j^{k+1} := u_N^{k+1}(x_j)$, are unknowns of the discrete solution, $h_j(x)$ is the Lagrangian polynomial defined in Λ. So we have

$$h_i(x_j) = \delta_{ij}, \quad \forall\, i, j \in \{0, 1, \ldots, N\},$$

here δ_{ij} denotes the Kronecker-delta function.

As $u_N^{k+1}(\pm 1) = 0$, then choosing each test function v_N to be $h_i(x)$, $i = 1, 2, \ldots, N-1$, we obtain

$$\begin{aligned} & \sum_{j=1}^{N-1}(h_j, h_i)_N \widehat{u}_j^{k+1} + \alpha_0 \sum_{j=1}^{N-1}(\partial_x h_j, \partial_x h_i)_N \widehat{u}_j^{k+1} + \frac{2\alpha_0}{p+2}\left(\beta\frac{\left|u_N^{k+1}\right|^{p+2} - \left|u_N^k\right|^{p+2}}{u_N^{k+1} - u_N^k}, h_i\right)_N \\ & = F_N^{k+1}(h_i). \end{aligned}$$

Using the definition of the discrete inner product, we have

$$(h_i, h_j)_N = \sum_{l=0}^{N} h_i(x_l) h_j(x_l)\omega_l = w_i \delta_{ij},$$

$$(\partial_x h_i, \partial_x h_j)_N = \sum_{l=0}^{N} \partial_x h_i(x_l)\partial_x h_j(x_l)\omega_l$$

and

$$\begin{aligned} \left(\beta\frac{\left|u_N^{k+1}\right|^{p+2} - \left|u_N^k\right|^{p+2}}{u_N^{k+1} - u_N^k}, h_i\right)_N & = \sum_{l=0}^{N} \beta(x_l)\frac{\left|u_N^{k+1}(x_l)\right|^{p+2} - \left|u_N^k(x_l)\right|^{p+2}}{u_N^{k+1}(x_l) - u_N^k(x_l)} h_i(x_l)\omega_l \\ & = \beta(x_i)\frac{\left|\widehat{u}_i^{k+1}\right|^{p+2} - \left|\widehat{u}_i^k\right|^{p+2}}{\widehat{u}_i^{k+1} - \widehat{u}_i^k}\omega_i. \end{aligned}$$

Thus we obtain the following system of nonlinear equations:

$$\widehat{u}_i^{k+1}\omega_i + \alpha_0 \sum_{j=1}^{N-1}\sum_{l=0}^{N} \partial_x h_i(x_l)\partial_x h_j(x_l)\omega_l \widehat{u}_j^{k+1} + \frac{2\alpha_0}{p+2}\beta(x_i)\frac{\left|\widehat{u}_i^{k+1}\right|^{p+2} - \left|\widehat{u}_i^{k}\right|^{p+2}}{\widehat{u}_i^{k+1} - \widehat{u}_i^{k}}\omega_i$$
$$= F_N^{k+1}(h_i), \quad i = 1, 2, \ldots, N-1.$$

We adopt the Newton iteration method to solve it.

5.2. Numerical results

We carry out some numerical experiments and present some results to confirm our theoretical statements.

Example 5.1. We consider the problem (1.2)–(1.4) with an exact analytical solution:

$$u(x,t) = t^{2+\gamma}\sin(\pi x),$$

$\beta(x) = 4\cos(x)$, $p = 2$. The corresponding forcing term is

$$f(x,t) = \frac{\Gamma(3+\gamma)}{2}t^2\sin(\pi x) + \pi^2 t^{2+\gamma}\sin(\pi x) + 4\cos(x)t^{6+3\gamma}\sin^3(\pi x).$$

Example 5.2. We consider the problem (1.2)–(1.4) with an exact solution which has limited regularity:

$$u(x,t) = t^{2+\gamma}(1-x^2)x^{16/3},$$

(one can verify $u \in H^5(\Lambda)$, but $\notin H^6(\Lambda)$), $\beta(x) = 1$, $p = 1/2$. The corresponding forcing term is

$$f(x,t) = \frac{\Gamma(3+\gamma)}{2}t^2(1-x^2)x^{16/3} + t^{2+\gamma}\left(\frac{418}{9}x^{16/3} - \frac{208}{9}x^{10/3}\right) + t^{3+1.5\gamma}(1-x^2)^{3/2}x^8.$$

To confirm the temporal accuracy, we choose N big enough to eliminate the error caused by spatial discretization. For Example 5.1 we take $N = 15$, while for Example 5.2 we take $N = 100$. Tables 5.1 and 5.2 show the errors $\left\|u(T) - u_N^M\right\|$ and the corresponding temporal convergence rates, which are consistent with our theoretical analysis. Here $T = 1$. The convergence rate is given by the formula: Rate $= \log_{\tau_1/\tau_2}(e_1/e_2)$ (e_i is the error corresponding to τ_i).

τ	$\alpha = 1.01$		$\alpha = 1.5$		$\alpha = 1.9$	
	Error	Rate	Error	Rate	Error	Rate
1/10	1.7360e-02	1.9699	3.9470e-02	1.6353	2.3262e-01	1.0527
1/20	4.4313e-03	1.9921	1.2706e-02	1.6061	1.1214e-01	1.0761
1/40	1.1139e-03	1.9976	4.1737e-03	1.5795	5.3188e-02	1.0886
1/80	2.7895e-04	1.9989	1.3966e-03	1.5586	2.5010e-02	1.0947
1/160	6.9790e-05	1.9992	4.7409e-04	1.5428	1.1711e-02	1.0976
1/320	1.7457e-05	*	1.6272e-04	*	5.4726e-03	*

Table 5.1: H^1 errors and temporal convergence rates for Example 5.1.

τ	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
	Error	Rate	Error	Rate	Error	Rate
1/10	2.2529e-04	1.9083	1.8471e-03	1.4957	1.2132e-02	1.0847
1/20	6.0019e-05	1.9092	6.5500e-04	1.4988	5.7202e-03	1.0910
1/40	1.5979e-05	1.9109	2.3177e-04	1.5004	2.6853e-03	1.0952
1/80	4.2492e-06	1.9172	8.1922e-05	1.5022	1.2569e-03	1.0976
1/160	1.1251e-06	1.9315	2.8920e-05	1.5107	5.8734e-04	1.0988
1/320	2.9494e-07	*	1.0149e-05	*	2.7423e-04	*

Table 5.2: H^1 errors and temporal convergence rates for Example 5.2.

Next we check the spatial accuracy with respect to the polynomial degree N. By fixing the time step small enough to avoid the contamination of the temporal error. We take the case $\tau = 0.001$, $\gamma = 1.5$ to illustrate.

Figure 5.1 shows the errors corresponding to the polynomial degree N in a semi-log scale for Example 5.1. From which, we can see the errors decay exponentially. That is the so-called spectral accuracy.

Figure 5.2 shows the errors with respect to the polynomial degree N in a log-log scale for Example 5.2. Since its solution belongs to $H^5(\Lambda)$, but $\notin H^6(\Lambda)$, we can see from Figure 5.2 the convergence rate is between N^{-4} and N^{-5}, which conforms with our theoretical analysis.

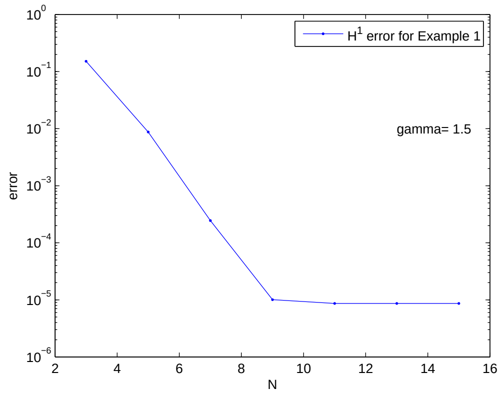

Figure 5.1: $\gamma = 1.5$ for Example 5.1

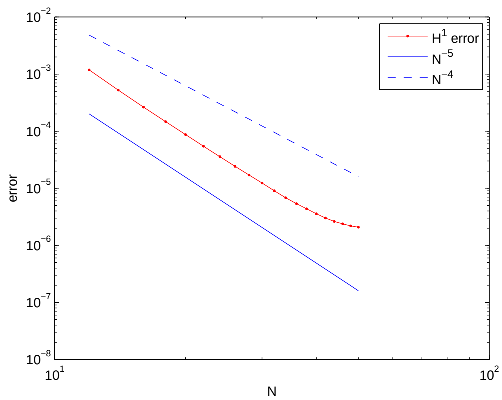

Figure 5.2: $\gamma = 1.5$ for Example 5.2

6. Conclusion

We have presented a fully discrete spectral scheme for the nonlinear time fractional Klein-Gordon equation in a bounded domain. The priori estimate for the approximate solution is derived. We have proved the well-posedness of the fully discrete scheme based on the priori estimate. The stability and convergence of the fully discrete scheme have been rigorously established. We have carried out some numerical experiments to confirm the theoretical results.

In the future, we will try to solve some other nonlinear time-fractional partial differential equation, such as Schrödinger equation, Dirac equation, etc.

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References

[1] W. Bao and X. Dong, *Analysis and comparison of numerical methods for the Klein-Gordon equation in the nonrelativistic limit regime*, Numer. Math. **120** (2012), no. 2, 189–229. https://doi.org/10.1007/s00211-011-0411-2

[2] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods: Fundamentals in Single Domains*, Scientific Computation, Springer-Verlag, Berlin, 2006. https://doi.org/10.1007/978-3-540-30726-6

[3] E. Cuesta, M. Kirane and S. A. Malik, *Image structure preserving denoising using generalized fractional time integrals*, Signal Process. **92** (2012), no. 2, 553–563. https://doi.org/10.1016/j.sigpro.2011.09.001

[4] S. T. Demiray, Y. Pandir and H. Bulut, *The investigation of exact solutions of nonlinear time-fractional Klein-Gordon equation by using generalized Kudryashov method*, AIP Conf. Proc. **1637** (2014), 283–289. https://doi.org/10.1063/1.4904590

[5] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-oriented Exposition Using Differential Operators of Caputo Type*, Lecture Notes in Mathematics **2004**, Springer-Verlag, Berlin, 2010. https://doi.org/10.1007/978-3-642-14574-2

[6] A. K. Golmankhaneh, A. K. Golmankhaneh and D. Baleanu, *On nonlinear fractional Klein-Gordon equation*, Signal Process. **91** (2011), no. 3, 446–451. https://doi.org/10.1016/j.sigpro.2010.04.016

[7] A. M. Grundland and E. Infeld, *A family of nonlinear Klein-Gordon equations and their solutions*, J. Math. Phys. **33** (1992), no. 7, 2498–2503. https://doi.org/10.1063/1.529620

[8] W. G. Glöckle and T. F. Nonnenmacher, *A fractional calculus approach to self-similar protein dynamics*, Biophys. J. **68** (1995), no. 1, 46–53. https://doi.org/10.1016/s0006-3495(95)80157-8

[9] B.-Y. Guo and Z.-Q. Wang, *A collocation method for generalized nonlinear Klein-Gordon equation*, Adv. Comput. Math. **40** (2014), no. 2, 377–398. https://doi.org/10.1007/s10444-013-9312-5

[10] J. G. Heywood and R. Rannacher, *Finite-element approximation of the nonstationary Navier-Stokes problem, IV: Error analysis for second-order time discretization*, SIAM J. Numer. Anal. **27** (1990), no. 2, 353–384. https://doi.org/10.1137/0727022

[11] J. W. Jerome, *The methods of lines and the nonlinear Klein-Gordon equation*, J. Differential Equations **30** (1978), no. 1, 20–30.
https://doi.org/10.1016/0022-0396(78)90020-7

[12] S. Jiménez and L. Vázquez, *Analysis of four numerical schemes for a nonlinear Klein-Gordon equation*, Appl. Math. Comput. **35** (1990), no. 1, 61–94.
https://doi.org/10.1016/0096-3003(90)90091-g

[13] Y. Lin, X. Li and C. Xu, *Finite difference/spectral approximations for the fractional cable equation*, Math. Comp. **80** (2011), no. 275, 1369–1396.
https://doi.org/10.1090/s0025-5718-2010-02438-x

[14] R. L. Magin, *Fractional calculus in bioengineering*, Begell House Publishers, Redding, 2006.

[15] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 1–77.
https://doi.org/10.1016/s0370-1573(00)00070-3

[16] E. Scalas, R. Gorenflo and F. Mainardi, *Fractional calculus and continuous-time finance*, Phys. A **284** (2000), no. 1-4, 376–384.
https://doi.org/10.1016/s0378-4371(00)00255-7

[17] W. Strauss and L. Vazquez, *Numerical solution of a nonlinear Klein-Gordon equation*, J. Comput. Phys. **28** (1978), no. 2, 271–278.
https://doi.org/10.1016/0021-9991(78)90038-4

[18] Z.-Z. Sun, *The Method of Order Reduction and Its Application to the Numerical Solutions of Partial Differential Equations*, Science Press, Beijing, 2009.

[19] Z.-Z. Sun and X. Wu, *A fully discrete difference scheme for a diffusion-wave system*, Appl. Numer. Math. **56** (2006), no. 2, 193–209.
https://doi.org/10.1016/j.apnum.2005.03.003

[20] R. Temam, *Navier-Stokes Equations: Theory and numerical analysis*, Studies in Mathematics and its Applications **2**, North-Holland Publishing, Amsterdam, 1977.
https://doi.org/10.1016/s0168-2024(09)x7004-9

[21] S. Vong and Zhibo Wang, *A high-order compact scheme for the nonlinear fractional Klein-Gordon equation*, Numer. Methods Partial Differential Equations **31** (2015), no. 3, 706–722. https://doi.org/10.1002/num.21912

[22] Y.-M. Wang, *A compact finite difference method for a class of time fractional convection-diffusion-wave equations with variable coefficients*, Numer. Algorithms **70** (2015), no. 3, 625–651. https://doi.org/10.1007/s11075-015-9965-x

[23] A.-M. Wazwaz, *New travelling wave solutions to the Boussinesq and the Klein-Gordon equations*, Commun. Nonlinear Sci. Numer. Simul. **13** (2008), no. 5, 889–901. https://doi.org/10.1016/j.cnsns.2006.08.005

[24] Y. S. Wong, Q. Chang and L. Gong, *An initial-boundary value problem of a nonlinear Klein-Gordon equation*, Appl. Math. Comput. **84** (1997), no. 1, 77–93. https://doi.org/10.1016/s0096-3003(96)00065-3

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