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(Φ, ρ)-MONOTONICITY AND GENERALIZED (Φ, ρ)-MONOTONICITY

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Abstract. In this paper, new concepts of monotonicity, namely (Φ, ρ)-monotonicity, (Φ, ρ)-pseudo-monotonicity and (Φ, ρ)-quasi-monotonicity are introduced for functions defined in Banach spaces. Series of necessary conditions are also given that relate (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity of the function with (Φ, ρ)-monotonicity and generalized (Φ, ρ)-monotonicity of its gradient.

1. INTRODUCTION

A concept closely related to the convexity of a real-valued function is the monotonicity of a vector-valued function. It is well known that the convexity of a real-valued function is equivalent to the monotonicity of the corresponding gradient function. It is worth noting that monotonicity has played a very important role in the study of the existence of solutions and numerical methods for the solution of variational inequality problems.

Just as convex functions are characterized by a monotone gradient, different kinds of generalized convex functions give rise to gradient maps with certain generalized monotonicity properties which are inherited from generalized convexity of the underlying function. Recently, various kinds of generalized monotonicity have been introduced for different classes of maps (see, for instance, [8, 9, 10, 15, 14], and the recent Hadjisavvas et al.'s handbook [6]).

In [8], Karamardian and Schaible defined seven kinds of monotone and generalized monotone maps. They showed in the case of a gradient map that generalized monotonicity corresponds to generalized convexity of the underlying function. However, for strongly pseudo-monotone maps, they have only established that strong pseudo-monotonicity of the gradient implies strong pseudo-convexity of the function. Later, Karamardian *et al.* [9] derived first-order characterizations of generalized monotone maps based on a geometrical analysis of generalized monotonicity. Komlosi [10]

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showed how quasi-convexity, pseudo-convexity, and strict pseudo-convexity of lower semicontinuous functions can be characterized via the quasi-monotonicity, pseudo-monotonicity, and strict pseudo-monotonicity of different types of generalized derivatives, including the Dini, Dini-Hadamard, Clarke, and Rockafellar derivatives. Zhu and Marcotte [18] introduced new classes of monotone and pseudo-monotone mappings. Further, they related them to previously introduced classes of (generalized) monotonicity. Yang *et al.* [16] studied generalized invariant monotonicity and its relationships with generalized invexity. They introduced several types of generalized invariant monotonicities which are generalizations of the (strict) monotonicity, (strict) pseudo-monotonicity, and quasi-monotonicity reported in Karamardian and Schaible [8]. Further, they established relations among generalized invariant monotonicities and generalized invexities.

On the other hand, there are some relationships between generalized invexity and generalized invariant monotonicity. Pini and Singh [12] analyzed some relationships between generalized invexity and generalized invariant monotonicity. Ruiz-Garzón *et al.* [13] defined the generalized invex monotone functions as an extension of monotone functions. They underlined the significance of these new classes of invariant monotonicity in order to characterize the solutions of the variational-like inequality problem and of a classical mathematical programming problem. Yang *et al.* [17] pointed out that some necessary conditions in [13] are wrong and corrected them with Condition C (see [11]). Further, Behera *et al.* [4] defined generalized (ρ, θ)-η-invariant-monotonicity and used them to characterize generalized (ρ, θ)-η-invexity.

In this paper, we introduce new classes of monotone, pseudomonotone and quasi-monotone mappings defined in a real Banach space. The so-called (strict) (Φ, ρ)-monotonicity, (strict) (Φ, ρ)-pseudo-monotonicity and (Φ, ρ)-quasi-monotonicity are analyzed and their properties and relationships with respect to other concepts of monotonicity by means of theoretical results, examples, and counterexamples are presented. We will connect (Φ, ρ)-invex and generalized (Φ, ρ)-invex functions (introduced in finite dimensional spaces by Caristi *et al.* [5]) to (Φ, ρ)-monotonicity and generalized (Φ, ρ)-monotonicity of their gradients through necessary conditions. We also give suitable counterexamples to show that, in general, the converse results may not hold. Further, we also show the relationships between (generalized) (Φ, ρ)-monotonicity and other generalized monotonicity notions existing in the literature.

2. (Φ, ρ)-MONOTONICITY

Let X be a real Banach space and S be a nonempty open convex subset of X. Throughout the paper, we denote by X^* the space of all continuous linear functionals on X (X^* is the dual space of X) and by $\langle x^*, x \rangle$ the duality pairing between $x \in X$ and $x^* \in X^*$. The symbols R and R_+ representing the sets of real and nonnegative real numbers. Further, let $\rho \in R$ and Φ be a function defined from $S \times S \times X^* \times R$ into R, where $\Phi(x, u, (\cdot, \cdot))$ is convex on $X^* \times R$, $\Phi(x, u, (0, a)) \geq 0$ for every $(x, u) \in S \times S$ and any $a \in R_+$.

Definition 1. [11]. Let V be a subset of X and $\eta : V \times V \to X$ be a given map. If the following relation

$$u + \lambda\eta(x, u) \in V$$

holds for all $x, u \in V$ and $\lambda \in [0, 1]$, then V is said to be an invex set (with respect to η).

In this section, we shall introduce a new concept of a generalized monotonicity of a function, that is, the so-called (Φ, ρ)-monotonicity. In this way, we extend the concept of a monotone function and the concept of an invariant monotone function to that is more general and we shall relate it to (Φ, ρ)-invex functions.

Definition 2. Let F be a function from S into X^*. Then F is said to be a (Φ, ρ)-monotone function if the relation

(1) $$\Phi(x, u, (F(u), \rho)) + \Phi(u, x, (F(x), \rho)) \leq 0$$

holds for all $x, u \in S$.

Remark 3. If the inequality (1) in Definition 2 is satisfied with $\rho < 0$, then F is called weakly (Φ, ρ)-monotone on S, whereas in the case when it is satisfied with $\rho > 0$, then F is called strongly (Φ, ρ)-monotone on S. In the case when $\rho = 0$, F is called Φ-monotone on S.

Definition 4. Let F be a function from S into X^*. Then F is said to be a strictly (Φ, ρ)-monotone function if the relation

(2) $$\Phi(x, u, (F(u), \rho)) + \Phi(u, x, (F(x), \rho)) < 0$$

holds for all $x, u \in S$, $x \neq u$.

Remark 5. If we take $X = R^n$, $\Phi(x, u, (\xi, \rho)) = [x - u]^T \xi$ and $\rho = 0$, then the above definition of a (Φ, ρ)-monotone function reduces to the definition of a monotone function $h : D \to R^n$, where $D \subset R^n$ is a nonempty open convex set, that is, the inequality

$$[x - u]^T (h(u) - h(x)) \leq 0,$$

holds for all $x, u \in D$, where T denotes transpose of a vector in R^n (see [8], [13]). If the inequality above is sharp for all $x, u \in D$, $x \neq u$, then h is strictly monotone on D.

Remark 6. If we take $X = R^n$, $\Phi(x,u,(\xi,\rho)) = [\eta(x,u)]^T \xi$, then the above definition of a (Φ,ρ)-monotone function reduces to the definition of an invariant monotone function $h : D \to R^n$ with respect to η, where $D \subset R^n$ is a nonempty open invex set with respect to $\eta : D \times D \to R^n$, that is, the relation

$$[\eta(x,u)]^T h(u) + [\eta(u,x)]^T h(x) \leq 0$$

holds for all $x, u \in D$ (see [13, 16, 17]).
If the inequality above is sharp for all $x, u \in D$, $x \neq u$, then h is strictly invariant monotone with respect to η on D.

Now, we give the definition of a Fréchet differentiable (Φ,ρ)-invex function and the definition of a Fréchet differentiable strictly (Φ,ρ)-invex function (see Caristi *et al.* [5] in finite dimensional spaces).

Definition 7. Let $f : S \to R$ be a Fréchet differentiable function on S and $u \in S$. Then f is said to be a (Φ,ρ)-invex function at u on S if the following inequality

$$f(x) - f(u) \geq \Phi(x,u,(\nabla f(u),\rho)) \tag{3}$$

holds for all $x \in S$.
If the inequality (3) is satisfied at every point u, then f is said to be a (Φ,ρ)-invex function on S.

In order to define an analogous class of (strictly) (Φ,ρ)-incave functions, the direction of the inequality in the definition of these functions should be changed to the opposite one.

Definition 8. Let $f : S \to R$ be a Fréchet differentiable function on S and $u \in S$. Then f is said to be a strictly (Φ,ρ)-invex function at u on S if the following inequality

$$f(x) - f(u) > \Phi(x,u,(\nabla f(u),\rho)) \tag{4}$$

holds for all $x \in S$, $x \neq u$.
If the inequality (4) is satisfied at every point u, $x \neq u$, then f is said to be a strictly (Φ,ρ)-invex function on S.

Remark 9. For other properties of a class of scalar differentiable (Φ,ρ)-invex functions, the readers are advised to consult [5], and, for a class of scalar locally Lipschitz (Φ,ρ)-invex functions, see [1]. In [2], moreover, Antczak established both parametric and nonparametric optimality conditions and several duality results in the sense of Mond-Weir and in the sense of Wolfe for a new class of nonconvex nonsmooth minimax programming problems with nondifferentiable (Φ,ρ)-invex functions.

He also showed that invexity and generalized invexity notions existing in the literature fail in proving the sufficiency of the Karush-Kuhn-Tucker necessary optimality conditions and duality results in the sense of Mond-Weir and in the sense of Wolfe for such a class of nonsmooth minimax programming problems. Furthermore, Antczak [3] introduced the concepts of vector (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity for strongly compact Lipschitz mappings in Banach spaces extending the definitions of (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity defined previously for optimization problems in finite-dimensional Euclidean spaces. Further, he used them to establish the sufficient optimality conditions for proper efficiency and duality results for nonsmooth vector minimization problems in which the involved functions belong to the class of (generalized) nondifferentiable (Φ, ρ)-invex functions defined in Banach spaces. As it follows from the results established in [2] and [3], the concepts of (Φ, ρ)-invex functions and generalized (Φ, ρ)-invex functions are useful to prove the sufficiency of Karush-Kuhn-Tucker necessary optimality conditions and various duality theorems for a larger class of nonconvex optimization problems than other generalized convexity concepts, for instance invexity introduced by Hanson [7].

Proposition 10. *Let $F : S \to X^*$ be an invariant monotone function on S with respect to η. Then, it is also a (Φ, ρ)-monotone function on S, where $\Phi(x, u, (\xi, \rho)) = \langle \xi, \eta(x, u) \rangle$ and $\rho = 0$.*

Proof. Let F be a monotone function on S with respect to η. If we set $\Phi(x, u, (\xi, \rho)) = \langle \xi, \eta(x, u) \rangle$ and $\rho = 0$, then, by Definition 2, it follows that f is a (Φ, ρ)-monotone function on S (see also Remark 6). ■

The converse result is, in general, not true.

Example 11. Let $X = R$, $S = R$ and we consider the function $F : R \to R$ defined by

$$F(x) = \frac{\sin^2 x}{x^2 + 1}.$$

Let $\eta : R \times R \to R$ be defined by

$$\eta(x, u) = (x + u)^2,$$

and the function Φ be defined by

$$\Phi(x, u, (\xi, \rho)) = \left(e^{-\xi} - 1\right) \eta(x, u),$$

where $\rho = 0$. Note that the function Φ defined above satisfies all conditions in the definition of a (Φ, ρ)-monotone function on R (see Definition 2). Indeed, $\Phi(x, u, (\cdot, \cdot))$ is convex on $R \times R$, $\Phi(x, u, (0, a)) \geq 0$ for every $(x, u) \in R \times R$ and any $a \in R_+$.

Then, we have

$$\Phi\left(x,u,\left(F\left(u\right),\rho\right)\right)+\Phi\left(u,x,\left(F\left(x\right),\rho\right)\right)=$$
$$\left(e^{-\frac{\sin^{2}u}{u^{2}+1}}-1\right)\left(x+u\right)^{2}+\left(e^{-\frac{\sin^{2}x}{x^{2}+1}}-1\right)\left(x+u\right)^{2}\leq 0.$$

Thus, by Definition 2, it follows that F is a (Φ,ρ)-monotone function on R with respect to Φ and ρ given above.
Now, we show that F is not an invariant monotone function on R with respect to η given above. Indeed, we have

$$\left[\eta\left(x,u\right)\right]^{T}F\left(u\right)+\left[\eta\left(u,x\right)\right]^{T}F\left(x\right)=$$
$$\left(x+u\right)^{2}\tfrac{\sin^{2}u}{x^{2}+1}+\left(x+u\right)^{2}\tfrac{\sin^{2}x}{x^{2}+1}=\left(x+u\right)^{2}\left(\tfrac{\sin^{2}x}{x^{2}+1}+\tfrac{\sin^{2}u}{u^{2}+1}\right)\geq 0.$$

Then, we conclude, by the inequality above, that F is not an invariant monotone function with respect to η on R.

Remark 12. The case when $\rho=0$ considered in Example 11 is not unique, for which the converse result is not true. Now, we show, for an another ρ, that the converse to the result in Proposition 10 is not true.

Example 13. Let $X=R$, $S=\left(-\sqrt{2},\sqrt{2}\right)\subset R$ and we consider the function $F:S\to R$ defined by

$$F\left(x\right)=\ln\left(2-x^{2}\right).$$

Let $\eta:S\times S\to R$ be defined by

$$\eta\left(x,u\right)=\left(x-u\right)^{2},$$

and the function Φ by

$$\Phi\left(x,u,\left(\xi,\rho\right)\right)=\eta\left(x,u\right)e^{\xi}+\rho\left(x-u\right)^{2},$$

and, moreover, we set $\rho=-2$. Note that the above defined function Φ satisfies all conditions in the definition of a (Φ,ρ)-monotone function on R (see Definition 2). Indeed, $\Phi\left(x,u,\left(\cdot,\cdot\right)\right)$ is convex on $R\times R$, $\Phi\left(x,u,\left(0,a\right)\right)\geq 0$ for every $(x,u)\in S\times S$ and any $a\in R_{+}$. Then, we have

$$\Phi\left(x,u,\left(F\left(u\right),\rho\right)\right)+\Phi\left(u,x,\left(F\left(x\right),\rho\right)\right)=$$
$$=\left(x-u\right)^{2}e^{\ln\left(2-u^{2}\right)}-2\left(x-u\right)^{2}+\left(x-u\right)^{2}e^{\ln\left(2-x^{2}\right)}-2\left(x-u\right)^{2}$$
$$=\left(x-u\right)^{2}\left(4-u^{2}-x^{2}\right)-4\left(x-u\right)^{2}=-\left(u^{2}+x^{2}\right)\left(x-u\right)^{2}\leq 0.$$

Thus, by Definition 2, it follows that F is a (Φ, ρ)-monotone function on S with respect to Φ and ρ given above (more exactly, as it follows from Remark 3, F is a weakly (Φ, ρ)-monotone function on S with respect to Φ and ρ given above).
Now, we show that F is not an invariant monotone function on S with respect to η given above. We set $x = \frac{1}{2}$ and $u = \frac{1}{4}$. Indeed, by definition, we have

$$\begin{aligned}[\eta(x,u)]^T F(u) + [\eta(u,x)]^T F(x) &= (x-u)^2 \ln\left(2-u^2\right) + (x-u)^2 \ln\left(2-x^2\right) \\ &= \frac{1}{16}\left(\ln\left(\frac{7}{4}\right) + \ln\left(\frac{31}{16}\right)\right) > 0.\end{aligned}$$

This means, by definition, that F is not an invariant monotone function on S with respect to η.

Proposition 14. *Let $\eta : S \times S \to X$ and $F : S \to R$ be a strictly invariant monotone function on S with respect to η. Then, it is also a strictly (Φ, ρ)-monotone function on S, where $\Phi(x, u, (\xi, \rho)) = \langle \xi, \eta(x,u) \rangle$ and $\rho = 0$.*

Proof. Proof is the same as proof of Proposition 10. ∎

Remark 15. The converse result to that presented in Proposition 14 is, in general, not true. There exist strictly (Φ, ρ)-monotone functions on S not being strictly invariant monotone on S with respect to every function η defined by $\eta : S \times S \to X$. For instance, we consider the function $F : S \to R$ defined by $F(x) = 1 - x^2$, where $S = R$. It is not difficult to show, by Definition 4, that F is strictly (Φ, ρ)-monotone on R, for example, with respect to $\Phi(x, u, (\xi, \rho)) = \xi u^2 + \rho\left(x^2 + u^2\right)$, where $\rho = -\frac{1}{2}$. However, F is not a strictly invariant monotone function on R with respect to every function η defined by $\eta : S \times S \to R$.

Theorem 16. *A necessary condition for $f : S \to R$ to be a (Φ, ρ)-invex function on S is that ∇f is (Φ, ρ)-monotone on S.*

Proof. Assume that $f : S \to R$ is a (Φ, ρ)-invex function on S. Then, by Definition 7, it follows that the following inequalities

$$f(x) - f(u) \geq \Phi(x, u, (\nabla f(u), \rho)),$$

$$f(u) - f(x) \geq \Phi(u, x, (\nabla f(x), \rho))$$

hold. Adding both sides of the above inequalities, we obtain

$$\Phi(x, u, (\nabla f(u), \rho)) + \Phi(u, x, (\nabla f(x), \rho)) \leq 0.$$

Therefore, by Definition 2, we conclude that ∇f is (Φ, ρ)-monotone with respect to Φ and ρ on S. ∎

In general, the converse result is not true. Now, we give an example of such a function f, for which ∇f is (Φ, ρ)-monotone on R, but f is not a (Φ, ρ)-invex function on R.

Example 17. Let $X = R$ and we consider a function $f : R \to R$ defined by

$$f(x) = -\frac{1}{3}x^3 - x.$$

We set

$$\Phi(x, u, (\xi, \rho)) = x^2\xi + \left(\rho^2 - \rho + \frac{1}{2}\right)\left(x^2 + u^2\right)$$

and $\rho = 1$. Then, we have

$$\begin{aligned} &\Phi(x, u, (\nabla f(u), \rho)) + \Phi(u, x, (\nabla f(x), \rho)) \\ &= x^2\left(-u^2 - 1\right) + \frac{1}{2}\left(x^2 + u^2\right) + u^2\left(-x^2 - 1\right) + \frac{1}{2}\left(x^2 + u^2\right) \\ &= -2x^2u^2 - x^2 - u^2 + x^2 + u^2 = -2x^2u^2 \leq 0. \end{aligned}$$

Then, by Definition 2, we conclude that ∇f is (Φ, ρ)-monotone on R. Now, we show that f is not a (Φ, ρ)-invex function on R with respect to the same functional Φ and the same scalar ρ given above. Indeed, by Definition 7, the following inequality

$$f(x) - f(u) \geq \Phi(x, u, (\nabla f(u), \rho))$$

is not satisfied for all $x, u \in R$. We have

$$-\frac{1}{3}x^3 - x - \left(-\frac{1}{3}u^3 - u\right) \geq x^2\left(-u^2 - 1\right) + \frac{1}{2}\left(x^2 + u^2\right).$$

It is not difficult to show that the above inequality is not satisfied for all $x, u \in R$. Indeed, if we set $u = 0$, then the inequality $-\frac{1}{3}x^3 + \frac{1}{2}x^2 - x \geq 0$ is not satisfied for all $x \in R$.

Remark 18. Note that the function f considered in Example 17 is such a function, for which ∇f is (Φ, ρ)-monotone on R, where $\Phi(x, u, (\nabla f(u), \rho)) = [\eta(x, u)]^T \nabla f(u) + \left(\rho^2 - \rho + \frac{1}{2}\right) \|\theta(x, u)\|^2$, where $\theta : S \times S \to X$, but ∇f is not (ρ, θ)-η-invariant monotone on R in the sense of Definition 2.3 [4]. Indeed, by Definition 2.3 [4], the following inequality

$$[\eta(x, u)]^T \nabla f(u) + [\eta(u, x)]^T \nabla f(x) + \rho\left[\|\theta(x, u)\|^2 + \|\theta(u, x)\|^2\right] \leq 0$$

is not satisfied for all $x, u \in R$. We have

$$\begin{aligned} &[\eta(x, u)]^T \nabla f(u) + [\eta(u, x)]^T \nabla f(x) + \rho\left[\|\theta(x, u)\|^2 + \|\theta(u, x)\|^2\right] \\ &= x^2\left(-u^2 - 1\right) + u^2\left(-x^2 - 1\right) + 2\left(x^2 + u^2\right) \\ &= -2x^2u^2 + x^2 + u^2 = (x - u)^2 \geq 0. \end{aligned}$$

Thus, ∇f is not (ρ, θ)-η-invariant monotone with respect to η and θ on R, although ∇f is (Φ, ρ)-monotone on R, where $\Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) = \left[\eta\left(x, u\right)\right]^T \nabla f\left(u\right) + \left(\rho^2 - \rho + \frac{1}{2}\right) \left\|\theta\left(x, u\right)\right\|^2$.

Theorem 19. *A necessary condition for $f : S \to R$ to be a strictly (Φ, ρ)-invex function on S is that ∇f be strictly (Φ, ρ)-monotone on S.*

Proof. The proof is similar to that of Theorem 16. ■

3. (Φ, ρ)-Quasi-monotonicity

In this section, we introduce the new concept of (Φ, ρ)-quasi monotonicity which is the generalization of the corresponding definition in the literature (see [13], [16]). We will give a necessary condition for (Φ, ρ)-quasi-invexity.

Definition 20. Let F be a function from S into X^*. Then F is said to be a (Φ, ρ)-quasi-monotone function if the following implication

$$\Phi\left(u, x, \left(F\left(x\right), \rho\right)\right) > 0 \Longrightarrow \Phi\left(x, u, \left(F\left(u\right), \rho\right)\right) \leq 0 \tag{5}$$

holds for all $x, u \in S$.

Remark 21. If the relation (5) in Definition 20 is satisfied with $\rho < 0$, then F is called a weakly (Φ, ρ)-quasi-monotone function on S, whereas in the case when it is satisfied with $\rho > 0$, then F is called a strongly (Φ, ρ)-quasi-monotone function on S. In the case when $\rho = 0$, F is called Φ-quasi-monotone on S.

Remark 22. If we take $X = R^n$, $\Phi\left(x, u, \left(\xi, \rho\right)\right) = \left[x - u\right]^T \xi$ and $\rho = 0$, then the above definition of (Φ, ρ)-quasi-monotonicity with respect to Φ and ρ reduces to the definition of a quasi-monotone function $h : D \to R^n$, where $D \subset R^n$ is a nonempty open convex set, that is, the following implication

$$\left[u - x\right]^T h\left(x\right) > 0 \Longrightarrow \left[x - u\right]^T h\left(u\right) \leq 0$$

holds for all $x, u \in D$.

Remark 23. If we take $X = R^n$, $\Phi\left(x, u, \left(\xi, \rho\right)\right) = \left[\eta\left(x, u\right)\right]^T \xi$ and $\rho = 0$, then the above definition of a (Φ, ρ)-quasi-monotone function reduces to the definition of an invariant quasi-monotone function $h : D \to R^n$ with respect to η, where $D \subset R^n$ is a nonempty open invex set with respect to $\eta : D \times D \to R^n$ that is, the following implication

$$\left[\eta\left(u, x\right)\right]^T h\left(x\right) > 0 \Longrightarrow \left[\eta\left(x, u\right)\right]^T h\left(u\right) \leq 0$$

holds for all $x, u \in D$ (see [16, 17]).

Remark 24. In general, the given function can be (Φ, ρ)-quasi-monotone with respect to more than one function Φ and the scalar ρ. Now, we give an example of such a function.

Example 25. Let $X = R^2$, $S = R^2$ and $F : R^2 \to R^2$ be defined by

$$F(x_1, x_2) = \left(\ln\left(2 - \sin^2(x_1 + x_2)\right), \ln\left(2 - \cos^2(x_1 + x_2)\right)\right).$$

We set

$$\Phi_1\left(x, u, (\xi, \rho_1)\right) = \begin{cases} e^{\xi_1} + e^{\xi_2} - 2 + \rho_1\left(\cos^2(x_1 + x_2) - \cos^2(u_1 + u_2)\right) \\ \qquad\qquad\qquad \text{if } \cos^2(x_1 + x_2) > \cos^2(u_1 + u_2) \\ e^{\xi_1} + e^{\xi_2} - 2 + \rho_1 \cos^2(u_1 + u_2) \\ \qquad\qquad\qquad \text{if } \cos^2(x_1 + x_2) \leq \cos^2(u_1 + u_2) \end{cases}$$

and $\rho_1 = -2$. Then, by Definition 20, F is a (Φ_1, ρ_1)-quasi-monotone function on R^2. However, if we set

$$\Phi_2\left(x, u, (\xi, \rho_2)\right) = \begin{cases} e^{-\xi_1} + e^{-\xi_2} - 2 + \rho_2 \frac{1 + \sin^2(u_1 + u_2)\cos^2(x_1 + x_2)}{\left(2 - \sin^2(u_1 + u_2)\right)\left(2 - \cos^2(u_1 + u_2)\right)} \\ \qquad\qquad\qquad \text{if } \cos^2(x_1 + x_2) > \cos^2(u_1 + u_2) \\ e^{-\xi_1} + e^{-\xi_2} - 2 + \rho_2 \frac{\frac{1}{2} + \sin^2(u_1 + u_2)\cos^2(x_1 + x_2)}{\left(2 - \sin^2(u_1 + u_2)\right)\left(2 - \cos^2(u_1 + u_2)\right)} \\ \qquad\qquad\qquad \text{if } \cos^2(x_1 + x_2) \leq \cos^2(u_1 + u_2) \end{cases}$$

and $\rho_2 = 2$, then, by Definition 20, F is a (Φ_2, ρ_2)-quasi-monotone function on R^2.

Remark 26. Every (Φ, ρ)-monotone function is a (Φ, ρ)-quasi-monotone function, but the converse is not necessarily true.

Now, we illustrate this result in the next example.

Example 27. Let $X = R^2$, $S = R^2$ and $F : R^2 \to R^2$ be defined by

$$F(x) = \left(\ln\left(2 - \sin(x_1 + x_2)\right), \ln\left(2 - \sin(x_1 + x_2)\right)\right).$$

We set

$$\Phi\left(x, u, (\xi, \rho)\right) = \begin{cases} \frac{1}{2}\left(e^{\xi_1} + e^{\xi_2} + \rho\right)\left(1 + \sin(u_1 + u_2)\right), \\ \qquad\qquad\qquad \text{if } \sin(u_1 + u_2) > \sin(x_1 + x_2) \\ \frac{1}{2}\left(e^{\xi_1} + e^{\xi_2} - 2\right)\left(1 + \sin(u_1 + u_2)\right) + \rho\cos^2(x_1 + x_2), \\ \qquad\qquad\qquad \text{if } \sin(u_1 + u_2) \leq \sin(x_1 + x_2) \end{cases}$$

and $\rho = -2$. Then, by Definition 20, F is a (Φ, ρ)-quasi-monotone function with respect to Φ and ρ on R^2. However, F is not a (Φ, ρ)-monotone function on R^2, since the inequality $\Phi\left(u, x, \left(F\left(x\right), \rho\right)\right) + \Phi\left(x, u, \left(F\left(u\right), \rho\right)\right) \leq 0$ is not satisfied for all $x, u \in R^2$. Indeed, if we set $x = (0, 0)$ and $u = (0, 0)$, then the above inequality is not satisfied. We conclude, by Definition 2, that F is not a (Φ, ρ)-monotone function on R^2.

Now, we give the definition of a Fréchet differentiable (Φ, ρ)-quasi-invex function with respect to Φ and ρ.

Definition 28. Let $f : S \to R$ be a Fréchet differentiable function on S and $u \in S$. Then f is said to be a (Φ, ρ)-quasi-invex function at u on S if the following implication

$$f(x) \leq f(u) \Longrightarrow \Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) \leq 0 \tag{6}$$

holds for all $x \in S$.
If the implication (6) is satisfied at any point u, then f is said to be a (Φ, ρ)-quasi-invex function on S.

Theorem 29. *A necessary condition for $f : S \to R$ to be a (Φ, ρ)-quasi-invex function with respect to Φ and ρ on S is that ∇f is (Φ, ρ)-quasi-monotone on S.*

Proof. Proof follows directly from Definition 28 and Definition 20. ■

In general, the converse result is not true. Now, we give an example of such a function f, for which ∇f is (Φ, ρ)-quasi-monotone on S, but f is not a (Φ, ρ)-quasi-invex function on S.

Example 30. Let $X = R$ and consider a function $f : R \to R$ defined by

$$f\left(x\right) = \frac{1}{1 + e^x}.$$

We set

$$\Phi\left(x, u, \left(\xi, \rho\right)\right) = \begin{cases} \left(1 + e^u\right)^2 \xi + \left(2^\rho - 1\right) e^x & \text{if} \quad x \geq u \\ \left(1 + e^u\right)^2 \xi + \left(2^\rho - 1\right) e^u & \text{if} \quad x < u \end{cases}$$

and $\rho = 1$. Then, we have

$$\Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) + \Phi\left(u, x, \left(\nabla f\left(x\right), \rho\right)\right) = -e^u + e^x - e^x + e^u = 0 \leq 0.$$

Therefore, by Definition 20, we conclude that ∇f is (Φ, ρ)-quasi-monotone on R. Now, we show that f is not a (Φ, ρ)-quasi-invex function on R. Indeed, by Definition 28, the following implication

$$f(x) \leq f(u) \Longrightarrow \Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) \leq 0$$

is not satisfied for all $x, u \in R$. Let $f(x) \leq f(u)$. It is not difficult to show that this inequality holds for $0 \leq u \leq x$ or $u < x \leq 0$. Now we show that the inequality $\Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) \leq 0$ is not satisfied for $u \leq x$. Indeed, we have

$$\Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) = \left(1 + e^{u}\right)^{2} \frac{-e^{u}}{\left(1 + e^{u}\right)^{2}} + e^{x} = -e^{u} + e^{x} \geq 0.$$

Hence, if we set $u = 0$ and $x = 1$, then the inequality $\Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) \leq 0$ is not satisfied. This means that f is not a (Φ, ρ)-quasi-invex function on R.

4. (Φ, ρ)-Pseudo-monotonicity

In this section, we introduce the concept of (Φ, ρ)-pseudo monotonicity which is the generalization of the corresponding definitions in the literature (see [13], [16]).

Definition 31. Let F be a function from S into X^{*}. Then F is said to be a (Φ, ρ)-pseudo-monotone function if the following implication

$$\Phi\left(u, x, \left(F\left(x\right), \rho\right)\right) \geq 0 \Longrightarrow \Phi\left(x, u, \left(F\left(u\right), \rho\right)\right) \leq 0 \tag{7}$$

holds for all $x, u \in S$.

Remark 32. If the relation (7) in Definition 31 is satisfied with $\rho < 0$, then F is called a weakly (Φ, ρ)-pseudo-monotone function on S, whereas in the case when it is satisfied with $\rho > 0$, then F is called a strongly (Φ, ρ)-pseudo-monotone function on S. In the case when $\rho = 0$, F is called Φ-pseudo-monotone on S.

Remark 33. If we take $X = R^{n}$, $\Phi\left(x, u, \left(\xi, \rho\right)\right) = \left[x - u\right]^{T} \xi$ and $\rho = 0$, then the above definition of a (Φ, ρ)-pseudo-monotone function reduces to the definition of a pseudo-monotone function $h : D \rightarrow R^{n}$, where $D \subset R^{n}$ is a nonempty open convex set, that is, the following implication

$$\left[u - x\right]^{T} h\left(x\right) \geq 0 \Longrightarrow \left[x - u\right]^{T} h\left(u\right) \leq 0$$

holds for all $x \in D$.

Remark 34. If we take $X = R^{n}$, $\Phi\left(x, u, \left(\xi, \rho\right)\right) = \left[\eta\left(x, u\right)\right]^{T} \xi$ and $\rho = 0$, then the above definition of a (Φ, ρ)-pseudo-monotone function reduces to the definition of an invariant pseudo-monotone function $h : D \rightarrow R^{n}$ with respect to η, where $D \subset R^{n}$ is a nonempty open invex set with respect to $\eta : D \times D \rightarrow R^{n}$, that is, the implication

$$\left[\eta\left(u, x\right)\right]^{T} h\left(x\right) \geq 0 \Longrightarrow \left[\eta\left(x, u\right)\right]^{T} h\left(u\right) \leq 0$$

holds for all $x \in D$.

Remark 35. In general, there exist more than one function Φ and the scalar ρ with respect to which the given function F is (Φ, ρ)-pseudo-monotone. Now, we give an example of such a function.

Example 36. Let $X = R$, $S = R$ and $F : R \to R$ be defined by

$$F(x) = \ln\left(\frac{1}{1+e^x}\right).$$

We set

$$\Phi_1(x, u, (\xi, \rho_1)) = \begin{cases} \left(\frac{e^x}{1+e^u}\right) e^{-\xi} + \rho_1 e^x & \text{if} \quad x < u \\ \left(\frac{e^u}{1+e^u}\right) e^{-\xi} + \rho_1 e^u & \text{if} \quad x \geq u \end{cases}$$

and $\rho_1 = -1$. Then, by Definition 31, F is a (Φ_1, ρ_1)-pseudo-monotone function on R. However, if we set

$$\Phi_2(x, u, (\xi, \rho_2)) = \begin{cases} e^u(1+e^u) e^{\xi} + \rho_2 e^x & \text{if} \quad x < u \\ e^u(1+e^u) e^{\xi} + \rho_2 e^u & \text{if} \quad x \geq u \end{cases}$$

and $\rho_2 = 1$, then, by Definition 31, F is also a (Φ_2, ρ_2)-pseudo-monotone function on R.

Proposition 37. *Every (Φ, ρ)-monotone function is a (Φ, ρ)-pseudo-monotone function.*

Proof. Assume that $F : S \to X^*$ is a (Φ, ρ)-monotone function with respect to Φ and ρ on S. Then, by Definition 2, it follows that the inequality

$$\Phi(x, u, (F(u), \rho)) + \Phi(u, x, (F(x), \rho)) \leq 0 \tag{8}$$

holds for all $x, u \in S$. Let $x, u \in S$ be such that

$$\Phi(u, x, (F(x), \rho)) \geq 0. \tag{9}$$

Combining (8) and (9), we get that the inequality

$$\Phi(x, u, (F(u), \rho)) \leq 0$$

holds. This means, by Definition 31, that F is a (Φ, ρ)-pseudo-monotone function on S. ∎

The converse result is, in general, not true.

Example 38. Let $X = R$, $S = R$ and $F : R \to R$ be defined by

$$F(x) = \sin^2 x + 1.$$

We set

$$\Phi\left(x, u, (\xi, \rho)\right) = \xi\left(\sin x - \sin u\right)$$

and $\rho = 0$. Then, by Definition 31, F is a (Φ, ρ)-pseudo-monotone function with respect to Φ and ρ on R. However, F is not a (Φ, ρ)-monotone function on R, since the inequality $\Phi\left(u, x, \left(F\left(x\right), \rho\right)\right) + \Phi\left(x, u, \left(F\left(u\right), \rho\right)\right) \leq 0$ is not satisfied for all $x, u \in R$. Indeed, if we set $x = -\frac{\pi}{6}$ and $u = 0$, then the above inequality is not satisfied. Therefore, we conclude, by Definition 2, that F is not a (Φ, ρ)-monotone function on R.

Remark 39. There exist more than one function Φ and a scalar ρ with respect to which the considered function F is (Φ, ρ)-pseudo-monotone, but it might not be (Φ, ρ)-monotone.

Theorem 40. *If $f : S \to R$ is a (Φ, ρ)-invex function on S, then ∇f is (Φ, ρ)-pseudo-monotone on S.*

Proof. By Theorem 16 and as (Φ, ρ)-monotonicity implies (Φ, ρ)-pseudo-monotonicity (see Proposition 37), the theorem is proved. ■

Now, we give the definition of a Fréchet differentiable (Φ, ρ)-pseudo-invex function.

Definition 41. Let $f : S \to R$ be a Fréchet differentiable function on S and $u \in S$. Then f is said to be a (Φ, ρ)-pseudo-invex function at u on S if the following implication

$$\Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) \geq 0 \Longrightarrow f(x) \geq f(u) \tag{10}$$

holds for all $x \in S$.
If the implication (10) is satisfied at any point u, then f is said to be a (Φ, ρ)-pseudo-invex function on S.

Theorem 42. *Let $f : S \to R$ be a (Φ, ρ)-pseudo-invex function with respect to Φ and ρ on S and a (Φ, ρ)-quasi-invex function on S. Then ∇f is (Φ, ρ)-pseudo-monotone on S.*

Proof. Assume that $f : S \to R$ is a (Φ, ρ)-pseudo-invex function on S. Assume that for $x, u \in S$, we have

$$\Phi\left(u, x, \left(\nabla f\left(x\right), \rho\right)\right) \geq 0.$$

Since f is a (Φ, ρ)-pseudo-invex function with respect to Φ and ρ on S, then the above inequality implies

$$f(u) \geq f(x). \tag{11}$$

In order to prove (Φ, ρ)-pseudo-monotonicity with respect to Φ and ρ of ∇f on S, we have to show that

$$\Phi(x, u, (\nabla f(u), \rho)) \leq 0.$$

We proceed by contradiction. Suppose that

$$\Phi(x, u, (\nabla f(u), \rho)) > 0.$$

By assumption, f is a (Φ, ρ)-quasi-invex function with on S. Hence, by Definition 28, the above inequality implies

$$f(x) > f(u),$$

contradicting (11). ∎

Now, we give an example of such a function f, for which ∇f is (Φ, ρ)-pseudo-monotone on S, but f is not a (Φ, ρ)-pseudo-invex function on S.

Example 43. Let $X = R$, $S = R_+$ and we consider a function $f : R_+ \to R$ defined by

$$f(x) = x(\ln x - 1).$$

We set

$$\Phi(x, u, (\xi, \rho)) = \frac{1}{2}e^{\xi} + (2^{\rho} - 1)x$$

and $\rho = -1$. It is not difficult to show, by Definition 31, that ∇f is (Φ, ρ)-monotone on R_+. Now, we show that f is not a (Φ, ρ)-pseudo-invex function on S. Indeed, by Definition 41, we have to show that the following implication

$$\Phi(x, u, (\nabla f(u), \rho)) \geq 0 \Longrightarrow f(x) \geq f(u)$$

is not satisfied for all $x, u \in R$. Let $x = 1$ and $u = 2$. Then $\Phi(x, u, (\nabla f(u), \rho)) \geq 0$, but

$$f(x) = f(1) = -1 \ngeq 2(\ln 2 - 1) = f(2) = f(u).$$

This means, by Definition 41, that f is not a (Φ, ρ)-pseudo-invex function on R_+.

5. Strictly (Φ, ρ)-Pseudo-monotonicity

Finally, we introduce the concept of strictly (Φ, ρ)-pseudo monotonicity, which is an extension of corresponding definitions in the literature (see [13], [16]). We will give a necessary condition for strictly (Φ, ρ)-pseudo-invexity.

Definition 44. Let F be a function from S into X^*. Then F is said to be a strictly (Φ, ρ)-pseudo-monotone function with respect to Φ and ρ if the following implication

$$\Phi(u, x, (F(x), \rho)) \geq 0 \Longrightarrow \Phi(x, u, (F(u), \rho)) < 0 \tag{12}$$

holds for all points $x, u \in S$, $x \neq u$.

Remark 45. If we take $X = R^n$, $\Phi(x, u, (\xi, \rho)) = [x - u]^T \xi$ and $\rho = 0$, then the above definition of strictly (Φ, ρ)-pseudo-monotonicity with respect to Φ and ρ reduces to the definition of a strictly pseudo-monotone function $h : D \to R^n$, where $D \subset R^n$ is a nonempty open convex set, that is, the implication

$$[u - x]^T h(x) \geq 0 \Longrightarrow [x - u]^T h(u) < 0$$

is satisfied for all $x \in D$, $x \neq u$.

Remark 46. If we take $X = R^n$, $\Phi(x, u, (\xi, \rho)) = [\eta(x, u)]^T \xi$ and $\rho = 0$, then the above definition of strictly (Φ, ρ)-pseudo-monotonicity reduces to the definition of a strictly invariant pseudo-monotone function $h : D \to R^n$ with respect to η, where $D \subset R^n$ is a nonempty open invex set with respect to $\eta : D \times D \to R^n$, that is, the implication

$$[\eta(u, x)]^T h(x) \geq 0 \Longrightarrow [\eta(x, u)]^T h(u) < 0$$

is satisfied for all $x \in D$, $x \neq u$.

Remark 47. Every strictly (Φ, ρ)-pseudo-monotone map is (Φ, ρ)-pseudo-monotone with, but the converse is not necessarily true.

Now, we give an example of a function which is strictly (Φ, ρ)-pseudo-monotone, but it is not (Φ, ρ)-pseudo-monotone.

Example 48. Let $X = R$, $S = R$ and $F : R \to R$ be defined by

$$F(x) = \ln\left(\sin^2 x \cos^2 x + 1\right).$$

We set

$$\Phi(x, u, (\xi, \rho)) = \frac{1}{2}\left(e^{\xi} - 1\right) + (2^{\rho} - 1)\sin^2 x \cos^2 x$$

and $\rho = -1$. Then, by Definition 31, F is a (Φ, ρ)-pseudo-monotone function on S. However, as it follows from Definition 44, F is not a strictly (Φ, ρ)-pseudo-monotone function on S.

Now, we give the definition of a Fréchet differentiable strictly (Φ, ρ)-pseudo-invex function with respect to Φ and ρ.

Definition 49. Let $f : S \to R$ be a Fréchet differentiable function on S and $u \in S$. Then f is said to be a strictly (Φ, ρ)-pseudo-invex function at u on S if the following implication

$$\Phi(x, u, (\nabla f(u), \rho)) \geq 0 \Longrightarrow f(x) > f(u) \tag{13}$$

is satisfied for all $x \in S$, $x \neq u$.
If the implication (13) is satisfied at every point u, then f is said to be a strictly (Φ, ρ)-pseudo-invex function on S.

Theorem 50. *A necessary condition for $f : S \to R$ to be a strictly (Φ, ρ)-pseudo-invex function on S is that ∇f is strictly (Φ, ρ)-pseudo-monotone on S.*

Proof. Assume that $f : S \to R$ is a strictly (Φ, ρ)-pseudo-invex function with respect to Φ and ρ on S. Let $x, u \in S$, $x \neq u$, be such that

$$\Phi\left(u, x, \left(\nabla f\left(x\right), \rho\right)\right) \geq 0.$$

By assumption, $f : S \to R$ is a strictly (Φ, ρ)-pseudo-invex function on S. Thus, by Definition 49, it follows that

$$f\left(u\right) > f\left(x\right). \tag{14}$$

We need to show that

$$\Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) < 0.$$

On the contrary, suppose that

$$\Phi\left(x, u, \left(\nabla f\left(u\right), \rho\right)\right) \geq 0.$$

Since $f : S \to R$ is a strictly (Φ, ρ)-pseudo-invex function on S, then, by Definition 49, the above inequality implies

$$f\left(x\right) > f\left(u\right),$$

contradicting (14). Thus, ∇f is strictly (Φ, ρ)-pseudo-monotone on S. ■

6. Conclusion

In this paper, we have introduced the concepts of (Φ, ρ)-monotone and generalized (Φ, ρ)-monotone functions defined in a Banach space. Therefore, the concepts of monotonicity and generalized monotonicity previously defined in the literature have been extended to these new ones. We have managed to prove that there are relationships between (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity of the function f and (Φ, ρ)-monotonicity and generalized (Φ, ρ)-monotonicity of the function ∇f, respectively, by way of the necessary conditions. Further, we have shown that (Φ, ρ)-monotonicity and generalized (Φ, ρ)-monotonicity are proper generalizations of monotonicity and generalized monotonicity, and also invariant monotonicity and generalized invariant monotonicity.

References

1. T. Antczak and A. Stasiak, (Φ, ρ)-invexity in nonsmooth optimization, *Numerical Functional Analysis and Optimization*, **32** (2011), 1-25.

2. T. Antczak, Nonsmooth minimax programming under locally Lipschitz (Φ, ρ)-invexity, *Applied Mathematics and Computation*, **217** (2011), 9606-9624.

3. T. Antczak, Proper efficiency conditions and duality results for nonsmooth vector optimization in Banach spaces under (Φ, ρ)-invexity, *Nonlinear Analysis*, **75** (2012), 3107-3121.

4. N. Behera, C. Nahak and S. Nanda, Generalized (ρ, θ)-η-invexity and generalized (ρ, θ)-η-invariant-monotonicity, *Nonlinear Analysis*, (2008), 2495-2506.

5. G. Caristi, M. Ferrara and A. Stefanescu, Mathematical programming with (Φ,ρ)-invexity; in: *Generalized Convexity and Related Topics*, I. V. Konnov, D. T. Luc and A. M. Rubinov (eds.), Lecture Notes in Economics and Mathematical Systems Vol. 583, Springer, Berlin Heidelberg, New York, 2006, pp. 167-176.

6. N. Hadjisavvas, S. Komlosi and S. Schaible (eds.), *Handbook of Generalized Convexity and Generalized Monotonicity*, Nonconvex Optimization and Its Applications, Vol. 76, Springer, New York, 2005.

7. M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *Journal of Mathematical Analysis and Applications*, **80** (1981), 545-550.

8. S. Karamardian and S. Schaible, Seven kinds of monotone maps, *Journal of Optimization Theory and Applications*, **66** (1990), 37-46.

9. S. Karamardian, S. Schaible and J. P. Crouzeix, Characterizations of generalized monotone maps, *Journal of Optimization Theory and Applications*, **76** (1993), 399-413.

10. S. Komlosi, Generalized monotonicity and generalized convexity, *Journal of Optimization Theory and Applications*, **84** (1995), 361-376.

11. S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, *Journal of Mathematical Analysis and Applications*, **189** (1995), 901-908.

12. R. Pini and C. Singh, Generalized convexity and generalized monotonicity, *Journal of Information and Optimization Sciences*, **20** (1999), 215-233.

13. G. Ruiz-Garzón, R. Osuna-Gómez and A. Rufián-Lizana, Generalized invex monotonicity, *European Journal of Operational Research*, **144** (2003), 501-512.

14. S. Schaible, Generalized monotonicity - a survey, in: *Generalized convexity* (Pécs 1992), Springer, Berlin, 1994, pp. 229-249.

15. S. Schaible, Criteria for generalized monotonicity, in: *New trends in mathematical programming*, F. Giannessi *et al.*, (ed.), Kluwer Academic Publishers, Dordrecht, Holland, 1998, pp. 277-288.

16. X. M. Yang, X. Q. Yang and K. L. Teo, Generalized invexity and generalized invariant monotonicity, *Journal of Optimization Theory and Applications*, **117** (2003), 607-625.

17. X. M. Yang, X. Q. Yang and K. L. Teo, Criteria for generalized invex monotonicities, *European Journal of Operational Research*, **164** (2005), 115-119.

18. D. L. Zhu and P. Marcotte, New classes of generalized monotonicity, *Journal of Optimization Theory and Applications*, **87** (1995), 457-471.

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