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Research Article

Nonlinear Integrable Couplings of Levi Hierarchy and WKI Hierarchy

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With the help of the known Lie algebra, a type of new 8-dimensional matrix Lie algebra is constructed in the paper. By using the 8-dimensional matrix Lie algebra, the nonlinear integrable couplings of the Levi hierarchy and the Wadati-Konno-Ichikawa (WKI) hierarchy are worked out, which are different from the linear integrable couplings. Based on the variational identity, the Hamiltonian structures of the above hierarchies are derived.

1. Introduction

The notion of integrable couplings was introduced when the study of Virasoro symmetric algebras [1, 2]. To find as many new integrable systems and their integrable couplings as possible and to elucidate in depth their algebraic and geometric properties are of both theoretical and practical value. During the past few years, some interesting integrable couplings and associated properties of some known interesting integrable hierarchies, such as the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy and the Kaup-Newell (KN) hierarchy, were obtained [3–13]. Here it is necessary to point out that the above mentioned integrable couplings are linear for the supplementary variable, so they are called linear integrable couplings.

Recently, Professor Ma proposed the notion of nonlinear integrable couplings and gave the general scheme to construct nonlinear integrable couplings of hierarchies [14]. Based on the general scheme of constructing nonlinear integrable couplings, Professor Zhang introduced some new explicit Lie algebras and obtained the nonlinear integrable couplings of the Giachetti-Johnson (GJ) hierarchy, the Yang hierarchy, and the classical Boussinesq-Burgers (CBB) hierarchy [15, 16].

The aim of the paper is to seek the nonlinear integrable couplings of the Levi hierarchy and the WKI hierarchy as well as their Hamiltonian structures. The plan of the paper is as follows. In Section 2, with the help of the Lie algebra $G = \left\{ \begin{pmatrix} g_1 & g_2 \\ 0 & g_1+g_2 \end{pmatrix} \mid g_1, g_2 \in sl(2) \right\}$, an 8-dimensional matrix Lie algebra is presented. It is different from the Lie algebras given in [14–16]. By employing the 8-dimensional matrix Lie algebra, the nonlinear integrable couplings of the Levi hierarchy and the WKI hierarchy are derived in Section 3. Furthermore, the corresponding Hamiltonian structures are worked out by virtue of the variational identity in Section 4. Finally, some conclusions are obtained in Section 5.

2. 8-Dimensional Matrix Lie Algebra

The Lie algebra is presented as $H = \text{span}\{h_1, h_2, h_3, h_4\}$ with the basis as follows:

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$h_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1}$$

equipped with the commutators

$$[h_1,h_2]=0,\qquad [h_1,h_3]=h_3,\qquad [h_1,h_4]=-h_4,$$
$$[h_2,h_3]=-h_3,\qquad [h_2,h_4]=h_4,\qquad [h_3,h_4]=h_1-h_2. \tag{2}$$

By virtue of the Lie algebra H, we construct an 8-dimensional matrix Lie algebra

$$G=\operatorname{span}\{g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8\} \tag{3}$$

with the basis as follows:

$$\begin{aligned}
g_1&=\begin{pmatrix} h_1 & 0\\ 0 & h_1\end{pmatrix}, && g_2=\begin{pmatrix} h_2 & 0\\ 0 & h_2\end{pmatrix},\\
g_3&=\begin{pmatrix} h_3 & 0\\ 0 & h_3\end{pmatrix}, && g_4=\begin{pmatrix} h_4 & 0\\ 0 & h_4\end{pmatrix},\\
g_5&=\begin{pmatrix} 0 & h_1\\ 0 & h_1\end{pmatrix}, && g_6=\begin{pmatrix} 0 & h_2\\ 0 & h_2\end{pmatrix},\\
g_7&=\begin{pmatrix} 0 & h_3\\ 0 & h_3\end{pmatrix}, && g_8=\begin{pmatrix} 0 & h_4\\ 0 & h_4\end{pmatrix},
\end{aligned} \tag{4}$$

which have the commutative relations

$$\begin{gathered}
[g_1,g_2]=0,\qquad [g_1,g_3]=g_3,\qquad [g_1,g_4]=-g_4,\\
[g_2,g_3]=-g_3,\qquad [g_2,g_4]=g_4,\qquad [g_3,g_4]=g_1-g_2,\\
[g_1,g_5]=0,\qquad [g_1,g_6]=0,\qquad [g_1,g_7]=g_7,\\
[g_1,g_8]=-g_8,\qquad [g_2,g_5]=0,\qquad [g_2,g_6]=0,\\
[g_2,g_7]=-g_7,\qquad [g_2,g_8]=g_8,\qquad [g_3,g_5]=-g_7,\\
[g_3,g_6]=g_7,\qquad [g_3,g_7]=0,\qquad [g_3,g_8]=g_5-g_6,\\
[g_4,g_5]=g_8,\qquad [g_4,g_6]=-g_8,\qquad [g_4,g_7]=g_6-g_5,\\
[g_4,g_8]=0,\qquad [g_5,g_6]=0,\qquad [g_5,g_7]=g_7,\\
[g_5,g_8]=-g_8,\qquad [g_6,g_7]=-g_7,\qquad [g_6,g_8]=g_8,\\
[g_7,g_8]=g_5-g_6.
\end{gathered} \tag{5}$$

Denoting $G_1=\operatorname{span}\{g_1,g_2,g_3,g_4\}$ and $G_2=\operatorname{span}\{g_5,g_6,g_7,g_8\}$, then we have

$$G=G_1\oplus G_2,\qquad G_1\cong H,\qquad [G_1,G_2]\subset G_2. \tag{6}$$

Here we need to emphasize that the subalgebras G_1 and G_2 are both nonsemisimple, which is very important for deriving nonlinear integrable couplings of hierarchies. By using the Lie algebra G, we can construct a few kinds of loop algebras $\widetilde{G}=G\otimes\lambda^{Nn+j}$, N and j stand for natural numbers. Among these loop algebras, the simplest one is

$$\widetilde{G}=\operatorname{span}\{g_i(n)\}_{i=1}^{8},\qquad g_i(n)=g_i\lambda^n, \tag{7}$$
$$i=1,2,3,4,5,6,7,8,$$

along with the commutators $[g_i(m),g_j(n)]=[g_i,g_j]\lambda^{m+n}$, $\deg(g_i(n))=n$, m, $n\in Z$, $1\le i$, and $j\le 8$.

In this section, by virtue of the Lie algebra H, we construct an 8-dimensional matrix Lie algebra G and corresponding loop algebra $\widetilde{G}$; in what follows we will generate the nonlinear integrable couplings of hierarchies by using the loop algebra $\widetilde{G}$.

3. Nonlinear Integrable Couplings of Hierarchies

In this section, based on the loop algebra $\widetilde{G}$, we construct two isospectral problems to generate the nonlinear integrable couplings of the Levi hierarchy and the WKI hierarchy, respectively.

3.1. Nonlinear Integrable Couplings of Levi Hierarchy. Take the following isospectral problem:

$$\phi_x=U\phi,\qquad \lambda_t=0,$$
$$\begin{aligned}
U={}&\frac12(-\lambda+u_1-u_2)(g_1(0)-g_2(0))+u_1g_3(0)+u_2g_4(0)\\
&+\frac12(u_3-u_4)(g_5(0)-g_6(0))+u_3g_7(0)+u_4g_8(0).
\end{aligned} \tag{8}$$

Set $V=v_1(g_1(0)-g_2(0))+v_2g_3(0)+v_3g_4(0)+v_4(g_5(0)-g_6(0))+v_5g_7(0)+v_6g_8(0)$, where $v_i=\sum_{m\ge0}v_{im}\lambda^{-m}$, $i=1,2,3,4,5,6$. Solving the stationary zero curvature equation $V_x=[U,V]$ gives rise to the recursion relation as follows:

$$\begin{aligned}
v_{1mx}&=u_1v_{3m}-u_2v_{2m},\\
v_{2mx}&=-v_{2m+1}+(u_1-u_2)v_{2m}-2u_1v_{1m},\\
v_{3mx}&=v_{3m+1}-(u_1-u_2)v_{3m}+2u_2v_{1m},\\
v_{4mx}&=u_1v_{6m}-u_4v_{2m}-u_2v_{5m}+u_3v_{3m}+u_3v_{6m}-u_4v_{5m},\\
v_{5mx}&=(u_1-u_2)v_{5m}-2u_3v_{1m}-v_{5m+1}-2u_1v_{4m}\\
&\quad+(u_3-u_4)(v_{2m}+v_{5m})-2u_3v_{4m},\\
v_{6mx}&=2u_4v_{1m}-(u_1-u_2)v_{6m}+v_{6m+1}+2u_2v_{4m}\\
&\quad-(u_3-u_4)(v_{3m}+v_{6m})+2u_4v_{4m},
\end{aligned}$$
$$v_{10}=-\frac{\alpha}{2}\neq0,\qquad v_{20}=v_{30}=v_{40}=v_{50}=v_{60}=0,$$
$$v_{11}=0,\qquad v_{21}=\alpha u_1,\qquad v_{31}=\alpha u_2,$$
$$v_{41}=0,\qquad v_{51}=\alpha u_3,\qquad v_{61}=\alpha u_4,\qquad v_{12}=\alpha u_1u_2,$$

$$\begin{aligned}
v_{22} &= \alpha u_1 (u_1 - u_2) - \alpha u_{1x}, \\
v_{32} &= \alpha u_2 (u_1 - u_2) + \alpha u_{2x}, \\
v_{42} &= \alpha (u_1 u_4 + u_2 u_3 + u_3 u_4), \\
v_{52} &= \alpha u_3 (u_1 - u_2) - \alpha u_{3x} + \alpha (u_1 + u_3)(u_3 - u_4), \\
v_{62} &= \alpha u_4 (u_1 - u_2) + \alpha u_{4x} + \alpha (u_2 + u_4)(u_3 - u_4).
\end{aligned} \tag{9}$$

Denoting $V_+^{(n)} = \sum_{m=0}^{n} (v_{1m}, v_{2m}, v_{3m}, v_{4m}, v_{5m}, v_{6m})^T \lambda^{n-m}$ and $V_-^{(n)} = \lambda^n V - V_+^{(n)}$, it is easy to compute

$$\begin{aligned}
-V_{+x}^{(n)} + \left[U, V_+^{(n)}\right] &= v_{2n+1} g_3(0) - v_{3n+1} g_4(0) \\
&\quad + v_{5n+1} g_7(0) - v_{6n+1} g_8(0).
\end{aligned} \tag{10}$$

Take

$$\begin{aligned}
V^{(n)} &= V_+^{(n)} + \Delta_n, \\
\Delta_n &= \frac{1}{2}(v_{2n} - v_{3n} - 2v_{1n})(g_1(0) - g_2(0)) \\
&\quad + \frac{1}{2}(v_{5n} - v_{6n} - 2v_{4n})(g_5(0) - g_6(0)).
\end{aligned} \tag{11}$$

Thus, the zero curvature equation

$$U_t - V_x^{(n)} + \left[U, V^{(n)}\right] = 0 \tag{12}$$

leads to the following integrable system:

$$\begin{aligned}
U_t &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t \\
&= \begin{pmatrix} -v_{2n+1} + (v_{2n} - v_{3n} - 2v_{1n}) u_1 \\ v_{3n+1} - (v_{2n} - v_{3n} - 2v_{1n}) u_2 \\ -v_{5n+1} + (v_{2n} - v_{3n} - 2v_{1n}) u_3 + (v_{5n} - v_{6n} - 2v_{4n})(u_1 + u_3) \\ v_{6n+1} - (v_{2n} - v_{3n} - 2v_{1n}) u_4 - (v_{5n} - v_{6n} - 2v_{4n})(u_2 + u_4) \end{pmatrix} \\
&= \begin{pmatrix} v_{2nx} - v_{1nx} \\ v_{3nx} + v_{1nx} \\ v_{5nx} - v_{4nx} \\ v_{6nx} + v_{4nx} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & \partial \\ 0 & 0 & \partial & 0 \\ 0 & \partial & 0 & -\partial \\ \partial & 0 & -\partial & 0 \end{pmatrix} \begin{pmatrix} v_{1n} + v_{3n} + v_{4n} + v_{6n} \\ -v_{1n} + v_{2n} - v_{4n} + v_{5n} \\ v_{1n} + v_{3n} \\ -v_{1n} + v_{2n} \end{pmatrix} = J_1 P_n,
\end{aligned} \tag{13}$$

where J_1 is a Hamiltonian operator and $P_{n+1} = L_1 P_n$, the recurrence operator L_1 is given from (9) by

$$L_1 = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ 0 & 0 & L_{33} & L_{34} \\ 0 & 0 & L_{43} & L_{44} \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned}
L_{11} &= \partial - (u_2 + u_4) + \partial^{-1}(u_1 + u_3)\partial, \\
L_{12} &= (u_2 + u_4) + \partial^{-1}(u_2 + u_4)\partial, \\
L_{13} &= -u_4 + \partial^{-1} u_3 \partial, \\
L_{14} &= u_4 + \partial^{-1} u_4 \partial, \\
L_{21} &= -(u_1 + u_3) - \partial^{-1}(u_1 + u_3)\partial, \\
L_{22} &= -\partial + (u_1 + u_3) - \partial^{-1}(u_2 + u_4)\partial, \\
L_{23} &= -u_3 - \partial^{-1} u_3 \partial, \\
L_{24} &= u_3 - \partial^{-1} u_4 \partial,
\end{aligned} \tag{15}$$

$$\begin{aligned}
L_{33} &= \partial - u_2 + \partial^{-1} u_1 \partial, \\
L_{34} &= u_2 + \partial^{-1} u_2 \partial, \\
L_{43} &= -u_1 - \partial^{-1} u_1 \partial, \\
L_{44} &= u_1 - \partial^{-1} u_2 \partial - \partial.
\end{aligned} \tag{16}$$

Therefore, the system (13) can be written as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = J_1 L_1^{n-1} \begin{pmatrix} v_{11} + v_{31} + v_{41} + v_{61} \\ -v_{11} + v_{21} - v_{41} + v_{51} \\ v_{11} + v_{31} \\ -v_{11} + v_{21} \end{pmatrix}. \tag{17}$$

When $u_3 = u_4 = 0$, the system (13) reduces to the Levi hierarchy; therefore, in terms of the definition of integrable coupling, we conclude that the system (13) is an integrable coupling of the Levi hierarchy. Especially taking $n = 2$, we have the following reduced equations:

$$\begin{aligned}
u_{1t} &= \alpha\left(u_1^2 - 2u_1 u_2\right)_x - \alpha u_{1xx}, \\
u_{2t} &= \alpha\left(-u_2^2 + 2u_1 u_2\right)_x + \alpha u_{2xx}, \\
u_{3t} &= 2\alpha\left(u_1 u_3 - u_2 u_3 - u_1 u_4 - u_3 u_4\right)_x - \alpha u_{3xx} + \alpha\left(u_3^2\right)_x, \\
u_{4t} &= 2\alpha\left(u_1 u_4 - u_2 u_4 + u_3 u_4 + u_2 u_3\right)_x + \alpha u_{4xx} - \alpha\left(u_4^2\right)_x.
\end{aligned} \tag{18}$$

Obviously, (18) are nonlinear equations in u_3 and u_4, so we call (13) the nonlinear integrable coupling of the Levi hierarchy.

3.2. Nonlinear Integrable Couplings of WKI Hierarchy. Consider an isospectral problem

$$\phi_x = U\phi, \qquad \lambda_t = 0,$$

$$U = i\left[g_2(1) - g_1(1)\right] + u_1 g_3(1) + u_2 g_4(1) + u_3 g_7(1) + u_4 g_8(1). \tag{19}$$

Set $V = \sum_{m\geq 0} \widetilde{V}$, where

$$\begin{aligned}\widetilde{V} = {} & [\lambda v_{1m} g_1(-m) - \lambda v_{1m} g_2(-m) + (v_{2mx} + i\lambda u_1 v_{1m}) \\ & \times g_3(-m) + (v_{3mx} + i\lambda u_2 v_{1m}) g_4(-m) \\ & + \lambda v_{4m} g_5(-m) - \lambda v_{4m} g_6(-m) \\ & + (v_{5mx} + i\lambda u_3 v_{1m} + i\lambda u_1 v_{4m} + i\lambda u_3 v_{4m}) g_7(-m) \\ & + (v_{6mx} + i\lambda u_4 v_{1m} + i\lambda u_2 v_{4m} + i\lambda u_4 v_{4m}) g_8(-m)].\end{aligned} \tag{20}$$

Because every term in U includes λ, V is different from the common form and includes potentials u_1, u_2, u_3, and u_4 and v_{2mx}, v_{3mx}, v_{5mx}, v_{6mx}, and so on. Then the zero curvature equation $V_x = [U, V]$ yields

$$\begin{gathered} v_{1mx} = u_1 v_{3mx} - u_2 v_{2mx}, \\ i(u_1 v_{1m+1})_x + v_{2mxx} = -2i v_{2m+1x}, \\ i(u_2 v_{1m+1})_x + v_{3mxx} = 2i v_{3m+1x}, \\ v_{4mx} = u_1 v_{6mx} - u_2 v_{5mx} + u_3 v_{3mx} - u_4 v_{2mx} + u_3 v_{6mx} - u_4 v_{5mx}, \\ i(u_3 v_{1m+1} + u_1 v_{4m+1} + u_3 v_{4m+1})_x + v_{5mxx} = -2i v_{5m+1x}, \\ i(u_4 v_{1m+1} + u_2 v_{4m+1} + u_4 v_{4m+1})_x + v_{6mxx} = 2i v_{6m+1x}, \\ v_{10} = \alpha_1, \qquad v_{20} = \alpha_2, \qquad v_{30} = \alpha_3, \\ v_{40} = \alpha_4, \qquad v_{50} = \alpha_5, \qquad v_{60} = \alpha_6, \\ v_{11} = \frac{2}{p}, \qquad v_{21} = \frac{-u_1}{p}, \qquad v_{31} = \frac{u_2}{p}, \\ v_{41} = -\frac{2}{p} - \frac{2}{p'}, \qquad v_{51} = \frac{u_1}{p} + \frac{u_1 + u_3}{p'}, \\ v_{61} = -\frac{u_2}{p} - \frac{u_2 + u_4}{p'}, \qquad p = \sqrt{1 - u_1 u_2}, \\ p' = \sqrt{1 - (u_1 + u_3)(u_2 + u_4)}. \end{gathered} \tag{21}$$

Denoting $V_+^{(n)} = \sum_{m=0}^{n} \widetilde{V} = \lambda^n V - V_-^{(n)}$, then we have $-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]$. A direct calculation reads

$$\begin{aligned} -V_{+x}^{(n)} + \left[U, V_+^{(n)}\right] = {} & -\lambda v_{2n-1xx} g_3(0) - \lambda v_{3n-1xx} g_4(0) \\ & - \lambda v_{5n-1xx} g_7(0) - \lambda v_{6n-1xx} g_8(0). \end{aligned} \tag{22}$$

Therefore, the zero curvature equation

$$U_t - V_x^{(n)} + \left[U, V^{(n)}\right] = 0 \tag{23}$$

admits

$$\begin{aligned} U_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} v_{2n-1xx} \\ v_{3n-1xx} \\ v_{5n-1xx} \\ v_{6n-1xx} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \partial^2 \\ 0 & 0 & -\partial^2 & 0 \\ 0 & \partial^2 & 0 & -\partial^2 \\ -\partial^2 & 0 & \partial^2 & 0 \end{pmatrix} \\ \times \begin{pmatrix} -v_{3n-1} - v_{6n-1} \\ v_{2n-1} + v_{5n-1} \\ -v_{3n-1} \\ v_{2n-1} \end{pmatrix} = J_2 Q_{n-1}. \end{aligned} \tag{24}$$

Here J_2 is a Hamiltonian operator and $Q_n = L_2 Q_{n-1}$, the recurrence operator L_2 is given from (21) by

$$L_2 = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ 0 & 0 & L_{33} & L_{34} \\ 0 & 0 & L_{43} & L_{44} \end{pmatrix}, \tag{25}$$

where

$$\begin{aligned} L_{11} &= -\frac{i}{2}\partial - \frac{i}{4}\frac{u_2 + u_4}{p'}\partial^{-1}\frac{u_1 + u_3}{p'}\partial^2, \\ L_{12} &= \frac{i}{4}\frac{u_2 + u_4}{p'}\partial^{-1}\frac{u_2 + u_4}{p'}\partial^2, \\ L_{13} &= -\frac{i}{4}\left[\frac{u_2 + u_4}{p'}\partial^{-1}\frac{1}{p'}\left(-u_1\partial^2 + q\partial\frac{q}{p}\partial^{-1}\frac{u_1}{p}\partial^2\right) + \frac{u_2 + u_4}{p}\partial^{-1}\frac{u_1}{p}\partial^2\right], \\ L_{14} &= \frac{i}{4}\left[\frac{u_2 + u_4}{p'}\partial^{-1}\frac{1}{p'}\left(-u_2\partial^2 + q\partial\frac{q}{p}\partial^{-1}\frac{u_2}{p}\partial^2\right) + \frac{u_2 + u_4}{p}\partial^{-1}\frac{u_2}{p}\partial^2\right], \end{aligned}$$

$$L_{21} = -\frac{i}{4}\frac{u_1+u_3}{p'}\partial^{-1}\frac{u_1+u_3}{p'}\partial^2,$$

$$L_{22} = \frac{i}{2}\partial + \frac{i}{4}\frac{u_1+u_3}{p'}\partial^{-1}\frac{u_2+u_4}{p'}\partial^2,$$

$$L_{23} = -\frac{i}{4}\left[\frac{u_1+u_3}{p'}\partial^{-1}\frac{1}{p'}\left(-u_1\partial^2 + q\partial\frac{q}{p}\partial^{-1}\frac{u_1}{p}\partial^2\right)\right.$$
$$\left.+\frac{u_1+u_3}{p}\partial^{-1}\frac{u_1}{p}\partial^2\right],$$

$$L_{24} = \frac{i}{4}\left[\frac{u_1+u_3}{p'}\partial^{-1}\frac{1}{p'}\partial^2\frac{1}{p'}\partial^{-1}\right.$$
$$\times\frac{1}{p'}\left(-u_2\partial^2 + q\partial\frac{q}{p}\partial^{-1}\frac{u_2}{p}\partial^2\right)$$
$$\left.+\frac{u_1+u_3}{p}\partial^{-1}\frac{u_2}{p}\partial^2\right],$$

$$L_{33} = -\frac{i}{2}\partial - \frac{i}{4}\frac{u_2}{p}\partial^{-1}\frac{u_1}{p}\partial^2,$$

$$L_{34} = \frac{i}{4}\frac{u_2}{p}\partial^{-1}\frac{u_2}{p}\partial^2,$$

$$L_{43} = -\frac{i}{4}\frac{u_1}{p}\partial^{-1}\frac{u_1}{p}\partial^2,$$

$$L_{44} = \frac{i}{2}\partial + \frac{i}{4}\frac{u_1}{p}\partial^{-1}\frac{u_2}{p}\partial^2. \tag{26}$$

Here p, p' are given in (21) and $q = \sqrt{u_1u_4 + u_2u_3 + u_3u_4}$. Hence, the system (24) can be written as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = J_2L_2^{n-2}\begin{pmatrix} v_{31}+v_{61} \\ v_{21}+v_{51} \\ v_{31} \\ v_{21} \end{pmatrix}. \tag{27}$$

When $u_3 = u_4 = 0$, the system (24) is just the WKI hierarchy. Taking $n = 2$, the system (24) reduces the following equations:

$$u_{1t} = \left(-\frac{u_1}{\sqrt{1-u_1u_2}}\right)_{xx},$$

$$u_{2t} = \left(\frac{u_2}{\sqrt{1-u_1u_2}}\right)_{xx},$$

$$u_{3t} = \left(\frac{u_1}{\sqrt{1-u_1u_2}} + \frac{u_1+u_3}{\sqrt{1-(u_1+u_3)(u_2+u_4)}}\right)_{xx},$$

$$u_{4t} = \left(-\frac{u_2}{\sqrt{1-u_1u_2}} - \frac{u_2+u_4}{\sqrt{1-(u_1+u_3)(u_2+u_4)}}\right)_{xx}. \tag{28}$$

It is easy to find that (28) are nonlinear equations in u_3 and u_4, so we call (24) the nonlinear integrable coupling of the WKI hierarchy.

4. Hamiltonian Structures

In this section, we will seek the Hamiltonian structures of the nonlinear integrable couplings of the Levi hierarchy (13) and the WKI hierarchy (24) by virtue of the variational identity. First, we construct a linear map $G \to R^8$, $g = \sum_{i=1}^{8} v_i g_i(0) \to v = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)^T$, $g \in G$, $v \in R^8$. We can conclude that the linear map is an isomorphism from G to R^8. Let $a, b \in R^8$; matrix $R(b)$ is determined by [9]

$$[a,b]^T = a^T R(b), \tag{29}$$

where $a = (a_1, \ldots, a_8)^T$ and $b = (b_1, \ldots, b_8)^T$. From (29), we have

$$R(b) = \begin{pmatrix} 0 & 0 & b_3 & -b_4 & 0 & 0 & b_7 & -b_8 \\ 0 & 0 & -b_3 & b_4 & 0 & 0 & -b_7 & b_8 \\ b_4 & -b_4 & b_2-b_1 & 0 & b_8 & -b_8 & b_6-b_5 & 0 \\ -b_3 & b_3 & 0 & b_1-b_2 & -b_7 & b_7 & 0 & b_5-b_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_3+b_7 & -b_4-b_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -b_3-b_7 & b_4+b_8 \\ 0 & 0 & 0 & 0 & b_4+b_8 & -b_4-b_8 & b_2-b_1-b_5+b_6 & 0 \\ 0 & 0 & 0 & 0 & -b_3-b_7 & b_3+b_7 & 0 & b_1-b_2+b_5-b_6 \end{pmatrix}. \tag{30}$$

Solving the matrix equation for the constant matrix F, $R(b)F = -(R(b)F)^T$, $F^T = F$,

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{31}$$

Then in terms of F, define a linear functional in the R^8

$$\begin{aligned}\{a, b\} = {}& a^T F b = (a_1 + a_5) b_1 + (a_2 + a_6) b_2 + (a_4 + a_8) b_3 \\ &+ (a_3 + a_7) b_4 + a_1 b_5 + a_2 b_6 + a_4 b_7 + a_3 b_8.\end{aligned} \tag{32}$$

It is easy to find that $\{a, b\}$ satisfies the variational identity

$$\frac{\delta}{\delta u} \int \{V, U_\lambda\}\, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \{V, U_u\}. \tag{33}$$

Rewrite the Lax pair of nonlinear integrable coupling of the Levi hierarchy as follows:

$$\begin{aligned} U = {}& \left(\frac{1}{2}(u_1 - u_2 - \lambda), \frac{1}{2}(\lambda - u_1 + u_2), u_1, u_2, \right. \\ & \left. \frac{1}{2}(u_3 - u_4), \frac{1}{2}(u_4 - u_3), u_3, u_4\right)^T, \\ V = {}& (v_1, -v_1, v_2, v_3, v_4, -v_4, v_5, v_6)^T. \end{aligned} \tag{34}$$

By using (32), we have

$$\begin{aligned} &\{V, U_\lambda\} = -v_1 - v_4, \qquad \{V, U_{u_1}\} = v_1 + v_3 + v_4 + v_6, \\ &\{V, U_{u_2}\} = -v_1 + v_2 - v_4 + v_5, \\ &\{V, U_{u_3}\} = v_1 + v_3, \qquad \{V, U_{u_4}\} = -v_1 + v_2. \end{aligned} \tag{35}$$

According the variational identity (33), we have

$$\begin{aligned} &\frac{\delta}{\delta u} \int (-v_1 - v_4)\, dx \\ &= \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} (v_1 + v_3 + v_4 + v_6, -v_1 + v_2 \\ &\quad - v_4 + v_5, v_1 + v_3, -v_1 + v_2)^T. \end{aligned} \tag{36}$$

Comparing the coefficients of λ^{-n-2} yields

$$\frac{\delta}{\delta u} \int (-v_{1n+1} - v_{4n+1})\, dx = (\gamma - n) \times \begin{pmatrix} v_{1n} + v_{3n} + v_{4n} + v_{6n} \\ -v_{1n} + v_{2n} - v_{4n} + v_{5n} \\ v_{1n} + v_{3n} \\ -v_{1n} + v_{2n} \end{pmatrix}. \tag{37}$$

Taking $n = 1$ gives rise to $\gamma = 0$. Therefore,

$$P_n = \frac{\delta H_n}{\delta u}, \qquad H_n = \int \frac{a_{n+1} + d_{n+1}}{n} dx, \quad n \geq 1. \tag{38}$$

Hence, the nonlinear integrable coupling of the Levi hierarchy has the following Hamiltonian structure:

$$U_t = J_1 \frac{\delta H_n}{\delta u}, \quad n \geq 1. \tag{39}$$

Similar to (34), in order to deduce to the Hamiltonian structure of the nonlinear integrable coupling of the WKI hierarchy, we rewrite the Lax pair as follows:

$$\begin{aligned} U = {}& (-i\lambda, i\lambda, \lambda u_1, \lambda u_2, 0, 0, \lambda u_3, \lambda u_4)^T \\ V = {}& (\lambda v_1, -\lambda v_1, v_{2x} + i\lambda u_1 v_1, v_{3x} + i\lambda u_2 v_1, \lambda v_4, \\ & - \lambda v_4, v_{5x} + i\lambda u_3 (v_1 + v_4) \\ & + i\lambda u_1 v_4, v_{6x} + i\lambda u_4 (v_1 + v_4) + i\lambda u_2 v_4)^T. \end{aligned} \tag{40}$$

Repeat the above procedure; we have

$$\begin{aligned} \{V, U_\lambda\} = {}& -2i(v_1 + v_4) + 2iu_1(v_3 + v_6) \\ & - 2iu_2(v_2 + v_5) + 2iu_3 v_3 - 2iu_4 v_2, \\ \{V, U_{u_1}\} = {}& 2i(v_3 + v_6), \qquad \{V, U_{u_2}\} = -2i(v_2 + v_5), \\ \{V, U_{u_3}\} = {}& 2iv_3, \qquad \{V, U_{u_4}\} = -2iv_2, \\ \frac{\delta}{\delta u} \int & 2i[-(v_{1n-1} + v_{4n-1}) + u_1(v_{3n-1} + v_{6n-1}) \\ & - u_2(v_{2n-1} + v_{5n-1}) + u_3 v_{3n-1} - u_4 v_{2n-1}]\, dx \\ = {}& 2i(2 + \gamma - n) \begin{pmatrix} -v_{3n-1} - v_{6n-1} \\ v_{2n-1} + v_{5n-1} \\ -v_{3n-1} \\ v_{2n-1} \end{pmatrix}. \end{aligned} \tag{41}$$

Taking $n = 2$ in above equation gives $\gamma = -1$. Therefore, $Q_{n-1} = (\delta \widetilde{H}_{n-1} / \delta u)$, $n \geq 2$, where

$$\begin{aligned} \widetilde{H}_{n-1} = {}& \int ((v_{1n-1} - v_{4n-1}) - u_1(v_{3n-1} - v_{6n-1}) \\ & + u_2(v_{2n-1} - v_{5n-1}) - u_3 v_{3n-1} \\ & + u_4 v_{2n-1}) \\ & \times (n-1)^{-1} dx. \end{aligned} \tag{42}$$

Hence, the nonlinear integrable coupling of the WKI hierarchy has the following Hamiltonian structure:

$$U_t = J_2 \frac{\delta \widetilde{H}_{n-1}}{\delta u}, \quad n \geq 2. \tag{43}$$

5. Conclusions

In this paper, we presented a set of new 8-dimensional matrix Lie algebra by virtue of the Lie algebra given in [14–16]. With the help of the Lie algebra, we obtain the nonlinear integrable couplings of the Levi hierarchy and the WKI hierarchy. Their Hamiltonian structures are also worked out by the variational identity. The Lie algebra constructed in this paper can be used to generate the nonlinear integrable couplings of other hierarchies. We will study these problems in the future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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