Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 257971, 12 pages
http://dx.doi.org/10.1155/2014/257971

Research Article

H_∞ Filtering for Discrete-Time Genetic Regulatory Networks with Random Delay Described by a Markovian Chain

Yantao Wang, Xingming Zhou, and Xian Zhang

School of Mathematical Science, Heilongjiang University, Harbin 150080, China

Correspondence should be addressed to Xian Zhang; zhangx663@126.com

Received 14 December 2013; Accepted 22 January 2014; Published 2 March 2014

Academic Editor: Xiaojie Su

This paper is concerned with the H_∞ filtering problem for a class of discretetime genetic regulatory networks with random delay and external disturbance. The aim is to design H_∞ filter to estimate the true concentrations of mRNAs and proteins based on available measurement data. By introducing an appropriate Lyapunov function, a sufficient condition is derived in terms of linear matrix inequalities (LMIs) which makes the filtering error system stochastically stable with a prescribed H_∞ disturbance attenuation level. The filter gains are given by solving the LMIs. Finally, an illustrative example is given to demonstrate the effectiveness of the proposed approach; that is, our approach is available for a smaller H_∞ disturbance attenuation level than one in (Liu et al., 2012).

1. Introduction

Genetic regulatory networks (GRNs) are collections of DNA segments in a cell which interact with each other indirectly through their mRNAs, protein expression products, and other substances. Understanding the nature and functions of various GRNs is very interesting and crucially important for the treatment of many diseases such as cancers [1, 2]. Therefore, in the past decade, the study on GRNs has been put more emphasis by the researchers at interdisciplinary field. Mathematical modeling of GRNs provides a powerful tool for studying gene regulation processes. In general, genetic network models can be classified into two types, that is, the discrete model [3, 4] and the continuous model [5–8]. Usually, a continuous model is described by a (functional) differential equation. Due to slow biochemical reactions such as gene transcription and translation, time delays can play an important role in GRNs, which results that the (functional) differential equation model has been one of the most fashionable GRN models, and a lot of research on analysis and synthesis of GRNs have been recently done based on (functional) differential equation models (see, e.g., [9–15]).

The concentrations of gene products, such as mRNAs and proteins, are described as system states in a (functional) differential equation model. In practice, biologists hope to gain actual concentrations of gene products in GRNs. However, due to model errors, external perturbation, time delays, and parameters jump, the steady-state values of GRNs can hardly be obtained. In order to obtain the steady-state values through available measurement data, the design of filter and estimator for (functional) differential equation models of GRNs has been investigated by some scholars (see, e.g., [16–23]). However, due to the requirement for implementing and application of GRNs for computer-based simulation, it is of vital importance to design filter or estimator for delayed discrete-time GRNs (i.e., discretized (functional) differential equation models of GRNs) in today's digital world, although there are, to the best author's knowledge, only three results reported at present [24–26]. Zhang et al. [25] is concerned with the set-values filtering for a class of discrete-time GRNs with time-varying parameters, constant time-delay, and bounded external noise. For a class of discrete-time GRNs with random delays described by a Markov chain, Liu et al. [26] designed a filter ensuring that the filtering error system is stochastically stable and has a prescribed H_∞ performance. By utilizing the Lyapunov stability theory and stochastic analysis technique, Wang et al. [24] investigated the existing conditions and explicit expressions of H_∞ state estimators for a class of stochastic discrete-time GRNs with probabilistic measurement delays described by Bernoulli distributed white sequences. These conditions are given in terms of LMIs and are dependent on the lower and upper bounds of the time-varying delays.

It should also be emphasized that for delayed discrete-time GRNs, the stability problem (as the most important properties for any dynamics systems) [27–29], H_∞ stabilization problem [30], and passivity problem [31] have been exploited. On the other hand, researchers have been paying attention to the problems of analysis and synthesis for Markovian jump system [32–36] and the filtering problems for some nonlinear systems [37–41].

Motivated by the above discussion, in this paper, we will deal with the H_∞ filtering problem for a class of discrete-time GRNs with random delay which is described by a Markovian chain. By constructing a novel Lyapunov function different from one in [26], a sufficient LMI condition is first established to ensure the existence of the desired filter. The condition is dependent on the transition probability matrix of the random delay. Then, the explicit expression of the desired filter is shown to ensure the resulting filtering error system to be stochastically stable and have a prescribed H_∞ disturbance attenuation level. Moreover, an optimization problem with LMIs constraints is established to design an H_∞ filter which ensures an optimal H_∞ disturbance attenuation level. Finally, a numerical example is given to show the effectiveness of the proposed approach.

2. Problem Formulation

Consider the following discrete-time GRN with random delays, n mRNAs, and n proteins [27, 28]:

$$\begin{aligned} M_i(k+1) &= e^{-a_i h} M_i(k) + \phi_i(h) \\ &\quad \times \left[\sum_{j=1}^{n} b_{ij} f_j\left(P_j(k-d(k))\right) + V_i \right], \\ P_i(k+1) &= e^{-c_i h} P_i(k) + \varphi_i(h) d_i M_i(k-d(k)), \\ &\qquad i = 1,2,\ldots,n, \end{aligned} \tag{1}$$

where $M_i(k)$ and $P_i(k)$, respectively, are the concentrations of mRNA and protein of the ith gene; $\phi_i(h) = (1-e^{-a_i h})/a_i > 0$ and $\varphi_i(h) = (1-e^{-c_i h})/c_i > 0$, where h is a given positive real number standing for the uniform discretionary step size; $d(k)$ denotes the random time delay of mRNAs and proteins, and is assumed to be a Markovian chain with state space $\mathcal{N} := \{1,2,\ldots,d\}$, and d is a fixed positive integer; $a_i > 0$ and $c_i > 0$ are the degradation rates of mRNA and protein, respectively; d_i is the translation rate; $V_i = \sum_{j\in I_i} v_{ij}$, where v_{ij} is a bounded constant denoting the dimensionless transcriptional rate of gene j to i, and I_i is the set of all the repressors of ith gene; b_{ij} $(i,j = 1,2,\ldots,n)$ are the coupling coefficients satisfying

$$b_{ij} = \begin{cases} v_{ij}, & \text{if transcription factor } j \text{ is} \\ & \text{an activator of gene } i, \\ 0, & \text{if there is no link from} \\ & \text{link node } j \text{ to } i, \\ -v_{ij}, & \text{if transcription factor } j \text{ is} \\ & \text{a repressor of gene } i; \end{cases} \tag{2}$$

the nonlinear function f_j $(j = 1,2,\ldots,n)$ denotes the feedback regulation of protein in process of transcription. In general, f_j is a monotonic function in Hill form; namely, $f_j(s) = s^{h_j}/(1+s^{h_j})$ $(j = 1,2,\ldots,n)$, where h_j is the Hill coefficient. Denote by $\pi := [\pi_{ij}]_{n\times n}$ the transition probability matrix of $d(k)$, where $\pi_{ij} = \text{Prob}\{d(k+1) = j \mid d(k) = i\}$.

Let us rewrite GRN (1) as the following compact matrix form:

$$\begin{aligned} M(k+1) &= AM(k) + Bf(P(k-d(k))) + V, \\ P(k+1) &= CP(k) + DM(k-d(k)), \end{aligned} \tag{3}$$

where

$$\begin{aligned} M(k) &= \left[M_1(k) \;\; M_2(k) \;\; \cdots \;\; M_n(k)\right]^T, \\ P(k) &= \left[P_1(k) \;\; P_2(k) \;\; \cdots \;\; P_n(k)\right]^T, \\ & f(P(k-d(k))) \\ &= \left[f_1(P_1(k-d(k))) \;\; f_2(P_2(k-d(k))) \;\; \cdots \;\; f_n(P_n(k-d(k)))\right]^T, \\ V &= \left[\phi_1(h)V_1 \;\; \phi_2(h)V_2 \;\; \cdots \;\; \phi_n(h)V_n\right]^T, \\ A &= \operatorname{diag}\left(e^{-a_1 h}, e^{-a_2 h}, \ldots, e^{-a_n h}\right), \\ C &= \operatorname{diag}\left(e^{-c_1 h}, e^{-c_2 h}, \ldots, e^{-c_n h}\right), \\ D &= \operatorname{diag}\left(\varphi_1(h) d_1, \varphi_2(h) d_2, \ldots, \varphi_n(h) d_n\right), \\ B &= \left[\phi_i(h) b_{ij}\right]_{n\times n} \quad (i = 1,2,\ldots,n). \end{aligned} \tag{4}$$

Let (M^*, P^*) be an equilibrium point of GRN (3), where $M^* = \left[M_1^* \;\; \cdots \;\; M_n^*\right]^T$ and $P^* = \left[P_1^* \;\; \cdots \;\; P_n^*\right]^T$; that is,

$$M^* = AM^* + Bf(P^*) + V, \qquad P^* = CP^* + DM^*. \tag{5}$$

To simplify the analysis, one can transform the equilibrium point to the origin by the relation $x_m(k) = M(k) - M^*$ and $x_p(k) = P(k) - P^*$. Then the transformed system is changed as follows:

$$\begin{aligned} x_m(k+1) &= Ax_m(k) + Bg\left(x_p(k-d(k))\right), \\ x_p(k+1) &= Cx_p(k) + Dx_m(k-d(k)), \end{aligned} \tag{6}$$

where $g(x_p(k)) = f(x_p(k) + P^*) - f(P^*)$. For every $i = 1,2,\ldots,n$, since f_i is a monotonic function in Hill form, one can easily obtain that g_i is a monotonically increasing function with saturation and satisfies the following inequality:

$$g_i(0) = 0, \quad 0 \le \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le l_i, \quad \forall s_1, s_2 \in R,\ s_1 \ne s_2, \tag{7}$$

where l_i is a given constant.

When we take extracellular perturbations into account, a class of stochastic discrete-time GRN model with random delays is represented as follows:

$$\begin{aligned}
x_m(k+1) &= Ax_m(k) + Bg\left(x_p(k-d(k))\right) + E_1 w(k), \\
x_p(k+1) &= Cx_p(k) + Dx_m(k-d(k)) + F_1 v(k), \\
y_m(k) &= C_1 x_m(k) + E_2 w(k), \\
y_p(k) &= C_2 x_p(k) + F_2 v(k), \\
z_m(k) &= G_1 x_m(k), \\
z_p(k) &= G_2 x_p(k), \\
x_m(k) &= \theta_m(k), \quad x_p(k) = \theta_p(k), \\
k &= -d, -d+1, \ldots, 0,
\end{aligned} \tag{8}$$

where A, B, C, D, C_1, C_2, E_1, E_2, F_1, F_2, G_1, and G_2 are constant matrices of appropriate dimension; $y_m(k) := [y_{m1}(k) \ \cdots \ y_{mn}(k)]^T$ and $y_p(k) := [y_{p1}(k) \ \cdots \ y_{pn}(k)]^T$ denote the expression levels of mRNA and protein, respectively; $z_m(k) := [z_{m1}(k) \ \cdots \ z_{ml}(k)]^T$ and $z_p(k) := [z_{p1}(k) \ \cdots \ z_{pl}(k)]^T$ are the estimated signals; both $w(k)$ and $v(k)$ are exogenous disturbance signals; and $\theta_m(k)$ and $\theta_p(k)$ are the initial conditions of $x_m(k)$ and $x_p(k)$, respectively.

In complex GRNs, only the partial information of the network components can be usually obtained. Therefore, in order to obtain the states of GRNs, we need to estimate them via available measurements [42]. The full order linear filter which need to be designed as the following form:

$$\begin{aligned}
\hat{x}_m(k+1) &= A_f \hat{x}_m(k) + B_f y_m(k), \\
\hat{x}_p(k+1) &= C_f \hat{x}_p(k) + D_f y_p(k), \\
\hat{z}_m(k) &= G_{1f} \hat{x}_m(k) + H_{1f} y_m(k), \\
\hat{z}_p(k) &= G_{2f} \hat{x}_p(k) + H_{2f} y_p(k),
\end{aligned} \tag{9}$$

where $\hat{x}_m(k)$, $\hat{x}_p(k)$, $\hat{z}_m(k)$, and $\hat{z}_p(k)$ are the estimates of $x_m(k), x_p(k), z_m(k)$, and $z_p(k)$, respectively; $A_f, B_f, C_f, D_f \in R^{n\times n}$ and G_{1f}, G_{2f}, H_{1f}, $H_{2f} \in R^{l\times n}$ are filter parametric matrices to be determined.

Set

$$\begin{aligned}
\tilde{x}_m(k) &= \begin{bmatrix} x_m(k) \\ \hat{x}_m(k) \end{bmatrix}, \qquad \tilde{x}_p(k) = \begin{bmatrix} x_p(k) \\ \hat{x}_p(k) \end{bmatrix}, \\
e_m(k) &= z_m(k) - \hat{z}_m(k), \qquad e_p(k) = z_p(k) - \hat{z}_p(k).
\end{aligned} \tag{10}$$

Then the filtering error system can be expressed as

$$\begin{aligned}
\tilde{x}_m(k+1) &= \overline{A}\tilde{x}_m(k) + \overline{B}g\left(Z_1 \tilde{x}_p(k-d(k))\right) + \overline{E}w(k), \\
\tilde{x}_p(k+1) &= \overline{C}\tilde{x}_p(k) + \overline{D}Z_1 \tilde{x}_m(k-d(k)) + \overline{F}v(k), \\
e_m(k) &= \overline{G}_{1f}\tilde{x}_m(k) + \overline{H}_{1f}w(k), \\
e_p(k) &= \overline{G}_{2f}\tilde{x}_p(k) + \overline{H}_{2f}v(k), \\
\tilde{x}_m(k) &= \tilde{\theta}_m(k), \qquad \tilde{x}_p(k) = \tilde{\theta}_p(k), \\
k &= -d, -d+1, \ldots, 0,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
&\tilde{\theta}_m(k) = \begin{bmatrix} \theta_m(k) \\ 0 \end{bmatrix}, \qquad \tilde{\theta}_p(k) = \begin{bmatrix} \theta_p(k) \\ 0 \end{bmatrix}, \\
&\overline{A} = \begin{bmatrix} A & 0 \\ B_f C_1 & A_f \end{bmatrix}, \qquad \overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad \overline{C} = \begin{bmatrix} C & 0 \\ D_f C_2 & C_f \end{bmatrix}, \\
&\overline{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}, \qquad \overline{E} = \begin{bmatrix} E_1 \\ B_f E_2 \end{bmatrix}, \qquad \overline{F} = \begin{bmatrix} F_1 \\ D_f F_2 \end{bmatrix}, \\
&\overline{G}_{1f} = \left[G_1 - H_{1f}C_1 \ \ -G_{1f}\right], \\
&\overline{G}_{2f} = \left[G_2 - H_{2f}C_2 \ \ -G_{2f}\right], \qquad \overline{H}_{1f} = -H_{1f}E_2, \\
&\overline{H}_{2f} = -H_{2f}F_2, \qquad Z_1 = [I \ \ 0].
\end{aligned} \tag{12}$$

For convenience, for a nonnegative integer k we define

$$\begin{aligned}
\Theta_k = \ &\{\tilde{x}_m(k), \tilde{x}_m(k-1), \ldots, \tilde{x}_m(k-d), \\
&\tilde{x}_p(k), \tilde{x}_p(k-1), \ldots, \tilde{x}_p(k-d)\}.
\end{aligned} \tag{13}$$

Definition 1 (see [26]). The delay $d(k)$ is said to be the random delay described by a Markovian chain if it is bound by $1 \le d(k) \le d$, and $\{d(k) \in \mathcal{N}, k = 0, 1, 2, \ldots\}$ is a Markovian chain with state space $\mathcal{N}$ and transition probability matrix π.

Definition 2 (see [26]). When $w(k) = 0$ and $v(k) = 0$, the filtering error system (11) is said to be stochastically stable, if

$$\sum_{k=0}^{\infty} E\left\{\left\|\tilde{x}_m(k)\right\|^2 + \left\|\tilde{x}_p(k)\right\|^2 \mid \Theta_0, d(0)\right\} < \infty \tag{14}$$

for every initial condition Θ_0 and initial mode $d(0)$, where $E\{\cdot\}$ represents the mathematical expectation operator.

Definition 3. For a given constant $\gamma > 0$, the filtering error system (11) is said to be stochastically stable with H_∞ disturbance attenuation level γ if it is stochastically stable with

$w(k) = 0$ and $v(k) = 0$, and under the zero initial conditions it satisfies the following inequality:

$$\sum_{k=0}^{\infty} E\left\{ \begin{bmatrix} e_m(k) \\ e_p(k) \end{bmatrix}^T \begin{bmatrix} e_m(k) \\ e_p(k) \end{bmatrix} \mid \Theta_0, d(0) \right\} < \gamma^2 \sum_{k=0}^{\infty} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix}^T \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \tag{15}$$

for all nonzero $w(k), v(k) \in l_2[0, +\infty)$, and initial mode $d(0)$.

The objective of this paper is to design a filter of form (9) such that the filtering error system (11) is stochastically stable with H_∞ disturbance attenuation level γ. In order to realize the aim, we first introduce the following lemma.

Lemma 4 (see [43]). *For symmetric matrices $P > 0$ and $Q > 0$, the matrix inequality*

$$\begin{bmatrix} -P^{-1} & A \\ * & -Q \end{bmatrix} < 0 \tag{16}$$

holds, if and only if there is a matrix R such that

$$\begin{bmatrix} P - R - R^T & R^T A \\ * & -Q \end{bmatrix} < 0. \tag{17}$$

3. Stability Analysis and H_∞ Filter Design

The stability analysis for the filtering error system (11) with $w(k) = 0$ and $v(k) = 0$ is presented by the following theorem.

Theorem 5. *The filtering error system* (11) *with $w(k) = 0$ and $v(k) = 0$ is stochastically stable, if there exist matrices $\varsigma := \mathrm{diag}(\varsigma_1, \varsigma_2, \ldots, \varsigma_n) > 0$, $\mu := \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n) > 0$, $P_i^T(r) = P_i(r) > 0 (i = 1, 2, \ldots, 6; r = 1, 2, \ldots, d)$, and $P_j^T = P_j > 0$ ($j = 2, 3, 5, 6$) such that the following matrix inequalities* (18) *and* (19) *hold for all $r \in \mathcal{N}$:*

$$\Omega := \widetilde{\Omega} + \widehat{\Omega} < 0, \tag{18}$$

$$\overline{P}_j(r) < P_j, \qquad j = 2, 3, 5, 6, \tag{19}$$

where

$$\begin{aligned}
\widehat{\Omega} &= \Lambda_1^T \overline{P}_1(r) \Lambda_1 + \Lambda_2^T \left(d\overline{P}_3(r) + \frac{d^2 + d}{2} P_3 \right) \Lambda_2 + \Lambda_3^T \overline{P}_4(r) \Lambda_3, \\
\Lambda_1 &= \begin{bmatrix} \overline{A} & 0 & 0 & \overline{B} & 0 & 0 \end{bmatrix}, \\
\Lambda_2 &= \begin{bmatrix} \overline{A} - I & 0 & 0 & \overline{B} & 0 & 0 \end{bmatrix}, \\
\Lambda_3 &= \begin{bmatrix} 0 & \overline{C} & \overline{D} Z_1 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\Omega} &= \begin{bmatrix} \Omega_{11} & 0 & \Omega_{13} & 0 & 0 & 0 \\ * & -P_4(r) & 0 & 0 & -Z_1^T \varsigma L & -Z_1^T C^T \mu L \\ * & * & \Omega_{33} & 0 & 0 & -Z_1^T D^T \mu L \\ * & * & * & \Omega_{44} & \Omega_{45} & 0 \\ * & * & * & * & \Omega_{55} & \Omega_{56} \\ * & * & * & * & * & \Omega_{66} \end{bmatrix}, \\
\Omega_{11} &= (d-1) P_2 + \overline{P}_2(r) - P_1(r) - \Omega_{13}, \\
\Omega_{13} &= \frac{1}{r} P_3(r) + \frac{1}{r} P_3, \qquad \Omega_{33} = -P_2(r) - \Omega_{13}, \\
\Omega_{44} &= -P_5(r) - \Omega_{45}, \qquad \Omega_{45} = \frac{1}{r} P_6(r) + \frac{1}{r} P_6, \\
\Omega_{55} &= (d-1) P_5 + \overline{P}_5(r) - \Omega_{56} - \Omega_{45} - \varsigma, \\
\Omega_{56} &= -d\overline{P}_6(r) - \frac{(d^2 + d) P_6}{2}, \qquad \Omega_{66} = -\Omega_{56} - \mu, \\
L &= \mathrm{diag}\left(-\frac{l_1}{2}, -\frac{l_2}{2}, \ldots, -\frac{l_n}{2} \right), \\
\overline{P}_i(r) &= \sum_{s=1}^{d} \pi_{rs} P_i(s), \quad i = 1, 2, \ldots, 6.
\end{aligned} \tag{20}$$

Proof. Choose an appropriate Lyapunov function $V(\Theta_k, k, d(k))$ for the filtering error system (11) with $w(k) = 0$ and $v(k) = 0$ as follows:

$$V(\Theta_k, k, d(k)) = \sum_{i=1}^{3} \left(V_{m,i}(\Theta_k, k, d(k)) + V_{p,i}(\Theta_k, k, d(k)) \right) \tag{21}$$

with

$$\begin{aligned}
V_{m,1}(\Theta_k, k, d(k)) &= \tilde{x}_m^T(k) P_1(d(k)) \tilde{x}_m(k), \\
V_{p,1}(\Theta_k, k, d(k)) &= \tilde{x}_p^T(k) P_4(d(k)) \tilde{x}_p(k), \\
V_{m,2}(\Theta_k, k, d(k)) &= \sum_{i=k-d(k)}^{k-1} \tilde{x}_m^T(i) P_2(d(k)) \tilde{x}_m(i) + \sum_{j=-d+1}^{-1} \sum_{i=k+j}^{k-1} \tilde{x}_m^T(i) P_2 \tilde{x}_m(i), \\
V_{p,2}(\Theta_k, k, d(k)) &= \sum_{i=k-d(k)}^{k-1} g^T(Z_1 \tilde{x}_p(i)) P_5(d(k)) g(Z_1 \tilde{x}_p(i)) + \sum_{j=-d+1}^{-1} \sum_{i=k+j}^{k-1} g^T(Z_1 \tilde{x}_p(i)) P_5 g(Z_1 \tilde{x}_p(i)), \\
V_{m,3}(\Theta_k, k, d(k)) &= \sum_{j=-d(k)}^{-1} \sum_{i=k+j}^{k-1} \eta^T(i) P_3(d(k)) \eta(i) + \sum_{j=-d}^{-1} \sum_{l=j}^{-1} \sum_{i=k+l}^{k-1} \eta^T(i) P_3 \eta(i),
\end{aligned}$$

$$V_{p,3}(\Theta_k,k,d(k)) = \sum_{j=-d(k)}^{-1}\sum_{i=k+j}^{k-1}\zeta^T(i)P_6(d(k))\zeta(i) + \sum_{j=-d}^{-1}\sum_{l=j}^{-1}\sum_{i=k+l}^{k-1}\zeta^T(i)P_6\zeta(i), \quad (22)$$

where $\eta(k) = \tilde{x}_m(k+1) - \tilde{x}_m(k)$ and $\zeta(k) = g(Z_1\tilde{x}_p(k+1)) - g(Z_1\tilde{x}_p(k))$. By taking the forward difference of the function $V_{m,1}(\Theta_k,k,d(k))$ along with the solution of system (11), one can obtain that

$$\begin{aligned}
&E\{V_{m,1}(\Theta_{k+1},k+1,d(k+1)) \mid \Theta_k, d(k)=r\} \\
&\quad - V_{m,1}(\Theta_k,k,r) \\
&\quad = \sum_{s=1}^{d}\pi_{rs}\tilde{x}_m^T(k+1)P_1(s)\tilde{x}_m(k+1) - \tilde{x}_m^T(k)P_1(r)\tilde{x}_m(k) \\
&\quad = \tilde{x}_m^T(k+1)\overline{P}_1(r)\tilde{x}_m(k+1) - \tilde{x}_m^T(k)P_1(r)\tilde{x}_m(k).
\end{aligned} \quad (23)$$

Additionally, it can be verified that

$$\begin{aligned}
&E\{V_{m,2}(\Theta_{k+1},k+1,d(k+1)) \mid \Theta_k, d(k)=r\} \\
&\quad - V_{m,2}(\Theta_k,k,r) \\
&\quad = \sum_{s=1}^{d}\pi_{rs}\sum_{i=k+1-s}^{k}\tilde{x}_m^T(i)P_2(s)\tilde{x}_m(i) \\
&\qquad - \sum_{i=k-r}^{k-1}\tilde{x}_m^T(i)P_2(r)\tilde{x}_m(i) \\
&\qquad + \sum_{j=-d+1}^{-1}\sum_{i=k+1+j}^{k}\tilde{x}_m^T(i)P_2\tilde{x}_m(i) \\
&\qquad - \sum_{j=-d+1}^{-1}\sum_{i=k+j}^{k-1}\tilde{x}_m^T(i)P_2\tilde{x}_m(i) \\
&\quad = \tilde{x}_m^T(k)\overline{P}_2(r)\tilde{x}_m(k) - \tilde{x}_m^T(k-r)P_2(r)\tilde{x}_m(k-r) \\
&\qquad + \sum_{i=k+1-s}^{k-1}\tilde{x}_m^T(i)\overline{P}_2(r)\tilde{x}_m(i) - \sum_{i=k+1-r}^{k-1}\tilde{x}_m^T(i)P_2(r)\tilde{x}_m(i) \\
&\qquad + \sum_{j=-d+1}^{-1}\tilde{x}_m^T(k)P_2\tilde{x}_m(k) - \sum_{j=k+1-d}^{k-1}\tilde{x}_m^T(j)P_2\tilde{x}_m(j) \\
&\quad \le \tilde{x}_m^T(k)\overline{P}_2(r)\tilde{x}_m(k) - \tilde{x}_m^T(k-r)P_2(r)\tilde{x}_m(k-r) \\
&\qquad + (d-1)\tilde{x}_m^T(k)P_2\tilde{x}_m(k) \\
&\qquad + \sum_{i=k+1-d}^{k-1}\tilde{x}_m^T(i)\overline{P}_2(r)\tilde{x}_m(i) - \sum_{i=k+1-d}^{k-1}\tilde{x}_m^T(i)P_2\tilde{x}_m(i) \\
&\quad \le \tilde{x}_m^T(k)\left[(d-1)P_2 + \overline{P}_2(r)\right]\tilde{x}_m^T(k) \\
&\qquad - \tilde{x}_m^T(k-r)P_2(r)\tilde{x}_m^T(k-r),
\end{aligned}$$

$$\begin{aligned}
&E\{V_{m,3}(\Theta_{k+1},k+1,d(k+1)) \mid \Theta_k, d(k)=r\} \\
&\quad - V_{m,3}(\Theta_k,k,r) \\
&\quad = \sum_{s=1}^{d}\pi_{rs}\sum_{j=-s}^{-1}\sum_{i=k+1+j}^{k}\eta^T(i)P_3(s)\eta(i) \\
&\qquad - \sum_{j=-r}^{-1}\sum_{i=k+j}^{k-1}\eta^T(i)P_3(r)\eta(i) \\
&\qquad + \sum_{j=-d}^{-1}\sum_{l=j}^{-1}\left[\sum_{i=k+1+l}^{k}\eta^T(i)P_3\eta(i) - \sum_{i=k+l}^{k-1}\eta^T(i)P_3\eta(i)\right] \\
&\quad \le \sum_{j=-d}^{-1}\eta^T(k)\overline{P}_3(r)\eta(k) - \sum_{j=-r}^{-1}\eta^T(k+j)P_3(r)\eta(k+j) \\
&\qquad + \sum_{j=-d}^{-1}\sum_{i=k+1+j}^{k-1}\eta^T(i)\overline{P}_3(r)\eta(i) + \frac{d^2+d}{2}\eta^T(k)P_3\eta(k) \\
&\qquad - \sum_{j=-d}^{-1}\sum_{i=k+1+j}^{k-1}\eta^T(i)P_3\eta(i) - \sum_{j=-d}^{-1}\eta^T(k+j)P_3\eta(k+j) \\
&\quad \le \eta^T(k)\left[d\overline{P}_3(r) + \frac{d^2+d}{2}P_3\right]\eta(k) \\
&\qquad - \sum_{j=-r}^{-1}\eta^T(k+j)\frac{1}{r}P_3(r)\sum_{j=-r}^{-1}\eta(k+j) \\
&\qquad + \sum_{j=-d}^{-1}\sum_{i=k+1+j}^{k-1}\eta^T(i)\overline{P}_3(r)\eta(i) - \sum_{j=-d}^{-1}\sum_{i=k+1+j}^{k-1}\eta^T(i)P_3\eta(i) \\
&\qquad - \sum_{j=-r}^{-1}\eta^T(k+j)\frac{1}{r}P_3\sum_{j=-r}^{-1}\eta(k+j) \\
&\quad \le \eta^T(k)\left[d\overline{P}_3(r) + \frac{d^2+d}{2}P_3\right]\eta(k) \\
&\qquad - \sum_{j=-r}^{-1}\eta^T(k+j)\frac{1}{r}(P_3(r)+P_3)\sum_{j=-r}^{-1}\eta(k+j).
\end{aligned} \quad (24)$$

Similarly, the following inequalities (25) can be derived:

$$\begin{aligned}
&E\left\{V_{p,1}(\Theta_{k+1},k+1,d(k+1)) \mid \Theta_k, d(k)=r\right\} \\
&\quad - V_{p,1}(\Theta_k,k,r) \\
&\quad = \sum_{s=1}^{d}\pi_{rs}\tilde{x}_p^T(k+1)P_4(s)\tilde{x}_p(k+1) - \tilde{x}_p^T(k)P_4(r)\tilde{x}_p(k) \\
&\quad = \tilde{x}_p^T(k+1)\overline{P}_4(r)\tilde{x}_p(k+1) - \tilde{x}_p^T(k)P_4(r)\tilde{x}_p(k),
\end{aligned}$$

$$\begin{aligned}
&E\left\{V_{p,2}\left(\Theta_{k+1},k+1,d\left(k+1\right)\right)\mid\Theta_k,d\left(k\right)=r\right\}\\
&\quad-V_{p,2}\left(\Theta_k,k,r\right)\\
&\quad=\sum_{s=1}^{d}\pi_{rs}\sum_{i=k+1-s}^{k}g^T\left(Z_1\tilde{x}_p\left(i\right)\right)P_5\left(s\right)g\left(Z_1\tilde{x}_p\left(i\right)\right)\\
&\qquad-\sum_{i=k-r}^{k-1}g^T\left(Z_1\tilde{x}_p\left(i\right)\right)P_5\left(r\right)g\left(Z_1\tilde{x}_p\left(i\right)\right)\\
&\qquad+\sum_{j=-d+1}^{-1}\sum_{i=k+1+j}^{k}g^T\left(Z_1\tilde{x}_p\left(i\right)\right)P_5g\left(Z_1\tilde{x}_p\left(i\right)\right)\\
&\qquad-\sum_{j=-d+1}^{-1}\sum_{i=k+j}^{k-1}g^T\left(Z_1\tilde{x}_p\left(i\right)\right)P_5g\left(Z_1\tilde{x}_p\left(i\right)\right)\\
&\quad=g^T\left(Z_1\tilde{x}_p\left(k\right)\right)\overline{P}_5\left(r\right)g\left(Z_1\tilde{x}_p\left(k\right)\right)\\
&\qquad-g^T\left(Z_1\tilde{x}_p\left(k-r\right)\right)P_5\left(r\right)g\left(Z_1\tilde{x}_p\left(k-r\right)\right)\\
&\qquad+\sum_{s=1}^{d}\pi_{rs}\sum_{i=k+1-s}^{k-1}g^T\left(Z_1\tilde{x}_p\left(i\right)\right)P_5\left(s\right)g\left(Z_1\tilde{x}_p\left(i\right)\right)\\
&\qquad-\sum_{i=k+1-r}^{k-1}g^T\left(Z_1\tilde{x}_p\left(i\right)\right)P_5\left(r\right)g\left(Z_1\tilde{x}_p\left(i\right)\right)\\
&\qquad+\sum_{j=-d+1}^{-1}\left[g^T\left(Z_1\tilde{x}_p\left(k\right)\right)P_5g\left(Z_1\tilde{x}_p\left(k\right)\right)\right.\\
&\qquad\qquad\left.-g^T\left(Z_1\tilde{x}_p\left(k+j\right)\right)P_5g\left(Z_1\tilde{x}_p\left(k+j\right)\right)\right]\\
&\quad\le g^T\left(Z_1\tilde{x}_p\left(k\right)\right)\left[\left(d-1\right)P_5+\overline{P}_5\left(r\right)\right]g\left(Z_1\tilde{x}_p\left(k\right)\right)\\
&\qquad-g^T\left(Z_1\tilde{x}_p\left(k-r\right)\right)P_5\left(r\right)g\left(Z_1\tilde{x}_p\left(k-r\right)\right)\\
&\qquad+\sum_{i=k+1-d}^{k-1}g^T\left(Z_1\tilde{x}_p\left(i\right)\right)\overline{P}_5\left(r\right)g\left(Z_1\tilde{x}_p\left(i\right)\right)\\
&\qquad-\sum_{i=k+1-d}^{k-1}g^T\left(Z_1\tilde{x}_p\left(i\right)\right)P_5g\left(Z_1\tilde{x}_p\left(i\right)\right)\\
&\quad\le g^T\left(Z_1\tilde{x}_p\left(k\right)\right)\left[\left(d-1\right)P_5+\overline{P}_5\left(r\right)\right]g\left(Z_1\tilde{x}_p\left(k\right)\right)\\
&\qquad-g^T\left(Z_1\tilde{x}_p\left(k-r\right)\right)P_5\left(r\right)g\left(Z_1\tilde{x}_p\left(k-r\right)\right),\\
&E\left\{V_{p,3}\left(\Theta_{k+1},k+1,d\left(k+1\right)\right)\mid\Theta_k,d\left(k\right)=r\right\}\\
&\quad-V_{p,3}\left(\Theta_k,k,r\right)\\
&\quad=\sum_{s=1}^{d}\pi_{rs}\sum_{j=-s}^{-1}\sum_{i=k+1+j}^{k}\zeta^T\left(i\right)P_6\left(s\right)\zeta\left(i\right)\\
&\qquad-\sum_{j=-r}^{-1}\sum_{i=k+j}^{k-1}\zeta^T\left(i\right)P_6\left(r\right)\zeta\left(i\right)\\
&\qquad+\sum_{j=-d}^{-1}\sum_{l=j}^{-1}\left[\sum_{i=k+1+l}^{k}\zeta^T\left(i\right)P_6\zeta\left(i\right)-\sum_{i=k+l}^{k-1}\zeta^T\left(i\right)P_6\zeta\left(i\right)\right]\\
&\quad\le\sum_{s=1}^{d}\pi_{rs}\sum_{j=-d}^{-1}\zeta^T\left(k\right)P_6\left(s\right)\zeta\left(k\right)\\
&\qquad-\sum_{j=-r}^{-1}\zeta^T\left(k+j\right)P_6\left(r\right)\zeta\left(k+j\right)\\
&\qquad+\sum_{s=1}^{d}\pi_{rs}\sum_{j=-d}^{-1}\sum_{i=k+1+j}^{k-1}\zeta^T\left(i\right)P_6\left(s\right)\zeta\left(i\right)\\
&\qquad-\sum_{j=-r}^{-1}\sum_{i=k+1+j}^{k-1}\zeta^T\left(i\right)P_6\left(r\right)\zeta\left(i\right)+\frac{d^2+d}{2}\zeta^T\left(k\right)P_6\zeta\left(k\right)\\
&\qquad-\sum_{j=-d}^{-1}\sum_{i=k+1+j}^{k-1}\zeta^T\left(i\right)P_6\zeta\left(i\right)-\sum_{j=-d}^{-1}\zeta^T\left(k+j\right)P_6\zeta\left(k+j\right)\\
&\quad\le\zeta^T\left(k\right)\Omega_{56}\zeta\left(k\right)-\sum_{j=-r}^{-1}\zeta^T\left(k+j\right)\Omega_{45}\sum_{j=-r}^{-1}\zeta\left(k+j\right).
\end{aligned}\tag{25}$$

In view of (7), we can conclude that

$$g_i\left(s\right)\left[g_i\left(s\right)-l_is\right]\le 0,\quad\forall s\in R,\ i=1,2,\dots,n.\tag{26}$$

Then, it follows from (26) that

$$\begin{aligned}
&-g^T\left(Z_1\tilde{x}_p\left(k\right)\right)\varsigma g\left(Z_1\tilde{x}_p\left(k\right)\right)-2\tilde{x}_p^T\left(k\right)Z_1^T\varsigma Lg\left(Z_1\tilde{x}_p\left(k\right)\right)\\
&\quad\ge 0,\\
&-g^T\left(Z_1\tilde{x}_p\left(k+1\right)\right)\mu g\left(Z_1\tilde{x}_p\left(k+1\right)\right)\\
&\quad-2\tilde{x}_p^T\left(k+1\right)Z_1^T\mu Lg\left(Z_1\tilde{x}_p\left(k+1\right)\right)\ge 0.
\end{aligned}\tag{27}$$

Now, combining (23)–(25) and (27) results in

$$\begin{aligned}
&E\left\{V\left(\Theta_{k+1},k+1,d\left(k+1\right)\right)\mid\Theta_k,d\left(k\right)=r\right\}\\
&\quad-V\left(\Theta_k,k,r\right)\le\xi^T\left(k\right)\Omega\xi\left(k\right),
\end{aligned}\tag{28}$$

where $\xi^T(k) = [\tilde{x}_m^T(k)\ \tilde{x}_p^T(k)\ \tilde{x}_m^T(k-r)\ g^T(Z_1\tilde{x}_p(k-r))\ g^T(Z_1\tilde{x}_p(k))\ g^T(Z_1\tilde{x}_p(k+1))]$, and Ω is defined as in (18).

Due to (18), formula (28) results in

$$\begin{aligned}
&E\left\{V\left(\Theta_{k+1},k+1,d\left(k+1\right)\right)\mid\Theta_k,d\left(k\right)=r\right\}\\
&\quad\le V\left(\Theta_k,k,r\right)-\lambda_{\min}\left\{\tilde{x}_m^T\left(k\right)\tilde{x}_m\left(k\right)+\tilde{x}_p^T\left(k\right)\tilde{x}_p\left(k\right)\right\},
\end{aligned}\tag{29}$$

where $\lambda_{\min}$ denotes the minimal eigenvalue of $-\Omega$. Since

$$\begin{aligned}
&E\left\{E\left\{V\left(\Theta_{k+1},k+1,d\left(k+1\right)\right)\mid\Theta_k,d\left(k\right)\right\}\mid\Theta_0,d\left(0\right)\right\}\\
&\quad=E\left\{V\left(\Theta_{k+1},k+1,d\left(k+1\right)\right)\mid\Theta_0,d\left(0\right)\right\},
\end{aligned}\tag{30}$$

we obtain

$$\begin{aligned}
&E\left\{\left\|\tilde{x}_m\left(k\right)\right\|^2+\left\|\tilde{x}_p\left(k\right)\right\|^2\mid\Theta_0,d\left(0\right)\right\}\le\lambda_{\min}^{-1}V\left(\Theta_0,0,d\left(0\right)\right)\\
&\quad<\infty.
\end{aligned}\tag{31}$$

by taking the conditional expectation $E\{\cdot \mid \Theta_0, d(0)\}$ and summing from $k = 0$ to $+\infty$ on both sides of (29). Consequently, by Definition 2, one can conclude from the above inequality that the filtering error system (11) is stochastically stable, and the proof is thus completed. □

Remark 6. It is worth noting that the H_∞ filtering problem for (8) has been studied in [26], but the obtained results in [26] are not dependent on the transition probability matrix of the random delay described by a Markovian chain. In order to reduce the conservatism and give the explicit expression of the desired filter, in the above theorem we have constituted intensive studying of the H_∞ filtering problem for (8) and have investigated a result dependent on the transition probability matrix of the random delay described by a Markovian chain.

Remark 7. The novel Lyapunov functional in this paper is selected to be of (21). Since in (21) we have not only chosen the triple summation term but also considered sufficiently the information of the random delay described by a Markovian chain, the conservatism might be reduced than one in [26], which will be illustrated through a numerical example in Section 4.

Theorem 5 does not give a design procedure for the desired filter. Based on Theorem 5, the following theorem offers an approach to design a H_∞ filter for GRN (8) such that the filtering error system (11) is stochastically stable with H_∞ disturbance attenuation level γ.

Theorem 8. *For given a scalar $\gamma > 0$ and a positive integer d, if for each $r \in \mathcal{N}$, there exist matrices $P_i^T(r) = P_i(r) > 0$ ($i = 1, 2, \ldots, 6$), $P_j^T = P_j > 0$ ($j = 2, 3, 5, 6$),*

$$R_k := \begin{bmatrix} R_{k1} & R_{k2} \\ R_{k3} & R_{k2} \end{bmatrix}^T, \qquad \det R_{k2} \neq 0, \quad k = 1, 2, \tag{32}$$

$\varsigma := \mathrm{diag}(\varsigma_1, \varsigma_2, \ldots, \varsigma_n) > 0$, $\mu := \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n) > 0$, $\overline{A}_f$, $\overline{B}_f$, $\overline{C}_f$, $\overline{D}_f$, G_{1f}, H_{1f}, G_{2f}, and H_{2f}, such that the following LMIs (34) and (35) hold, then the filtering error system (11) is stochastically stable with H_∞ disturbance attenuation level γ. Moreover, the required filter is given by (9) with

$$\begin{aligned} A_f &= R_{12}^{-1}\overline{A}_f, & B_f &= R_{12}^{-1}\overline{B}_f, \\ C_f &= R_{22}^{-1}\overline{C}_f, & D_f &= R_{22}^{-1}\overline{D}_f, \end{aligned} \tag{33}$$

$$\Upsilon := \begin{bmatrix} \Upsilon_{11} & 0 & 0 & 0 & 0 & \Upsilon_{16} \\ * & \Upsilon_{22} & 0 & 0 & 0 & \Upsilon_{26} \\ * & * & \Upsilon_{33} & 0 & 0 & \Upsilon_{36} \\ * & * & * & -I & 0 & \Upsilon_{46} \\ * & * & * & * & -I & \Upsilon_{56} \\ * & * & * & * & * & \Upsilon_{66} \end{bmatrix} < 0, \tag{34}$$

$$\overline{P}_j(r) < P_j, \quad j = 2, 3, 5, 6, \tag{35}$$

where

$$\begin{aligned}
\Upsilon_{11} &= \overline{P}_1(r) - R_1 - R_1^T, \\
\Upsilon_{22} &= d\overline{P}_3(r) + \frac{d^2 + d}{2}P_3 - R_1 - R_1^T, \\
\Upsilon_{33} &= \overline{P}_4(r) - R_2 - R_2^T, \\
\overline{P}_i(r) &= \sum_{s=1}^{d} \pi_{rs} P_i(s), \qquad i = 1, 2, \ldots, 6, \\
\Upsilon_{16} &= R_1^T \Psi_1 + (Z_1 + Z_2)^T \left(\overline{B}_f \Psi_2 + \overline{A}_f \Psi_3\right), \\
\Upsilon_{26} &= R_1^T \Psi_4 + (Z_1 + Z_2)^T \left(\overline{B}_f \Psi_2 + \overline{A}_f \Psi_3\right), \\
\Upsilon_{36} &= R_2^T \Psi_5 + (Z_1 + Z_2)^T \left(\overline{D}_f \Psi_6 + \overline{C}_f \Psi_7\right), \\
Z_1 &= \begin{bmatrix} I & 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} 0 & I \end{bmatrix}, \\
\Psi_1 &= \begin{bmatrix} AZ_1 & 0 & 0 & B & 0 & 0 & E_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Psi_2 &= \begin{bmatrix} C_1 Z_1 & 0 & 0 & 0 & 0 & 0 & E_2 & 0 \end{bmatrix}, \\
\Psi_3 &= \begin{bmatrix} Z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Psi_4 &= \begin{bmatrix} (A - I) Z_1 & 0 & 0 & B & 0 & 0 & E_1 & 0 \\ -Z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Psi_5 &= \begin{bmatrix} 0 & CZ_1 & DZ_1 & 0 & 0 & 0 & 0 & F_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Psi_6 &= \begin{bmatrix} 0 & C_2 Z_1 & 0 & 0 & 0 & 0 & 0 & F_2 \end{bmatrix}, \\
\Psi_7 &= \begin{bmatrix} 0 & Z_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Upsilon_{46} &= \begin{bmatrix} \overline{G}_{1f} & 0 & 0 & 0 & 0 & 0 & \overline{H}_{1f} & 0 \end{bmatrix}, \\
\Upsilon_{56} &= \begin{bmatrix} 0 & \overline{G}_{2f} & 0 & 0 & 0 & 0 & 0 & \overline{H}_{2f} \end{bmatrix}, \\
\Upsilon_{66} &= \begin{bmatrix} \widetilde{\Omega} & 0 & \Phi_2 \\ * & -\gamma^2 I & 0 \\ * & * & -\gamma^2 I \end{bmatrix}, \\
\Phi_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -F_1^T \mu L \end{bmatrix}^T,
\end{aligned} \tag{36}$$

and L, $\overline{G}_{1f}$, $\overline{G}_{2f}$, $\overline{H}_{1f}$, and $\overline{H}_{2f}$ are defined as previously.

Proof. A_f, B_f, C_f, and D_f are defined as in (33). Then it is easy to verify that $\Upsilon_{16} = R_1^T \overline{\Lambda}_1$, $\Upsilon_{26} = R_1^T \overline{\Lambda}_2$, and $\Upsilon_{36} = R_2^T \overline{\Lambda}_3$, where

$$\begin{aligned} \overline{\Lambda}_1 &= \begin{bmatrix} \Lambda_1 & \overline{E} & 0 \end{bmatrix}, \qquad \overline{\Lambda}_2 = \begin{bmatrix} \Lambda_2 & \overline{E} & 0 \end{bmatrix}, \\ \overline{\Lambda}_3 &= \begin{bmatrix} \Lambda_3 & 0 & \overline{F} \end{bmatrix}, \end{aligned} \tag{37}$$

and Λ_1, Λ_2, Λ_3, $\overline{E}$, and $\overline{F}$ are defined as previously. This, together with (34) and Lemma 4, implies that

$$\begin{bmatrix} -\overline{P}_1^{-1}(r) & 0 & 0 & 0 & 0 & \overline{\Lambda}_1 \\ * & -\left(d\overline{P}_3(r) + \dfrac{d^2+d}{2}P_3\right)^{-1} & 0 & 0 & 0 & \overline{\Lambda}_2 \\ * & * & -\overline{P}_4^{-1}(r) & 0 & 0 & \overline{\Lambda}_3 \\ * & * & * & -I & 0 & \Upsilon_{46} \\ * & * & * & * & -I & \Upsilon_{56} \\ * & * & * & * & * & \Upsilon_{66} \end{bmatrix} < 0. \tag{38}$$

Due to the Schur complement lemma, inequality (38) is equal to

$$\Phi + \overline{\Phi} < 0, \tag{39}$$

where

$$\overline{\Phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} + \Upsilon_{46}^T\Upsilon_{46} + \Upsilon_{56}^T\Upsilon_{56},$$
$$\Phi = \begin{bmatrix} \widetilde{\Omega} & 0 & \Phi_2 \\ 0 & 0 & 0 \\ \Phi_2^T & 0 & 0 \end{bmatrix} + \overline{\Lambda}_1^T\overline{P}_1(r)\overline{\Lambda}_1 \tag{40}$$
$$+ \overline{\Lambda}_2^T\left(d\overline{P}_3(r) + \frac{d^2+d}{2}P_3\right)\overline{\Lambda}_2 + \overline{\Lambda}_3^T\overline{P}_4(r)\overline{\Lambda}_3.$$

Thus

$$\Lambda := \Upsilon_{66} + \overline{\Lambda}_1^T\overline{P}_1(r)\overline{\Lambda}_1 + \overline{\Lambda}_2^T\left(d\overline{P}_3(r) + \frac{d^2+d}{2}P_3\right)\overline{\Lambda}_2 + \overline{\Lambda}_3^T\overline{P}_4(r)\overline{\Lambda}_3 < 0. \tag{41}$$

Noting that Ω is a submatrix of Λ, we can conclude that $\Omega < 0$. By Theorem 5, the filtering error system (11) with $w(k) = 0$ and $v(k) = 0$ is stochastically stable.

Choose the same Lyapunov function as in (21) for the filtering error system (11) and employ the similar approach in the proof of Theorem 5, one has

$$\begin{aligned} \Delta V_k &:= E\left\{V\left(\Theta_{k+1}, k+1, d(k+1)\right) \mid \Theta_k, d(k) = r\right\} \\ &\quad - V\left(\Theta_k, k, r\right) \\ &\leq E\left\{\delta^T(k)\Phi\delta(k)\right\}, \end{aligned} \tag{42}$$

where $\delta(k) = \left[\xi^T(k) \ \ w^T(k) \ \ v^T(k)\right]^T$, and $\xi(k)$ is defined as previously. To deal with the H_∞ performance, the following performance function is considered

$$J_K := \sum_{k=0}^{K} E\left\{\begin{bmatrix} e_m(k) \\ e_p(k)\end{bmatrix}^T \begin{bmatrix} e_m(k) \\ e_p(k)\end{bmatrix} - \gamma^2 \begin{bmatrix} w(k) \\ v(k)\end{bmatrix}^T \begin{bmatrix} w(k) \\ v(k)\end{bmatrix} \mid \Theta_0, d(0)\right\}. \tag{43}$$

Due to the zero initial condition and

$$\begin{aligned} &E\left\{E\left\{V\left(\Theta_{k+1}, k+1, d(k+1)\right) \mid \Theta_k, d(k)\right\} \mid \Theta_0, d(0)\right\} \\ &\quad = E\left\{V\left(\Theta_{k+1}, k+1, d(k+1)\right) \mid \Theta_0, d(0)\right\}, \end{aligned} \tag{44}$$

it is easy to see from (39) and (42) that

$$\begin{aligned} J_K &= \sum_{k=0}^{K} E\left\{\begin{bmatrix} e_m(k) \\ e_p(k)\end{bmatrix}^T \begin{bmatrix} e_m(k) \\ e_p(k)\end{bmatrix} - \gamma^2 \begin{bmatrix} w(k) \\ v(k)\end{bmatrix}^T \begin{bmatrix} w(k) \\ v(k)\end{bmatrix} + \Delta V_k \mid \Theta_0, d(0)\right\} \\ &\quad - \sum_{k=0}^{K} E\left\{\Delta V_k \mid \Theta_0, d(0)\right\} \\ &= \sum_{k=0}^{K} E\left\{\begin{bmatrix} e_m(k) \\ e_p(k)\end{bmatrix}^T \begin{bmatrix} e_m(k) \\ e_p(k)\end{bmatrix} - \gamma^2 \begin{bmatrix} w(k) \\ v(k)\end{bmatrix}^T \begin{bmatrix} w(k) \\ v(k)\end{bmatrix} + \Delta V_k \mid \Theta_0, d(0)\right\} \\ &\quad - E\left\{V\left(\Theta_{K+1}, K+1, d(K+1)\right) \mid \Theta_0, d(0)\right\} \\ &\quad + V\left(\Theta_0, 0, d(0)\right) \\ &\leq \sum_{k=0}^{K} E\left\{e_m^T(k)e_m(k) + e_p^T(k)e_p(k) - \gamma^2 w^T(k)w(k) - \gamma^2 v^T(k)v(k) + \Delta V_k \mid \Theta_0, d(0)\right\} \\ &\leq \sum_{k=0}^{K} E\left\{\delta^T(k)\left(\Phi + \overline{\Phi}\right)\delta(k) \mid \Theta_0, d(0)\right\} < 0. \end{aligned} \tag{45}$$

Let $k \to \infty$; it is concluded from Definition 3 that the filtering error system (11) is stochastically stable with H_∞ disturbance attenuation level γ.

The proof is thus completed. □

Remark 9. What can be seen from Theorem 8 is that the scalar γ can be calculated as an optimization variable to obtain the minimum H_∞ disturbance attenuation level. To be more specific, the minimal H_∞ disturbance attenuation level can be obtained by solving the following convex optimization problem:

$$\min_{\text{s.t. (34)-(35)}} \beta, \quad \beta = \gamma^2. \tag{46}$$

Note that if there exists a solution β^* to the problem (46), then the minimal H_∞ disturbance attenuation level is $\sqrt{\beta^*}$.

4. Illustrative Example

In this section we illustrate the effectiveness of the proposed approach by testing the following numerical example which has been used in [26].

Consider GRN (8) with the following parameters:

$$\begin{gathered} A = \begin{bmatrix} 0.3679 & 0 & 0 \\ 0 & 0.3679 & 0 \\ 0 & 0 & 0.3679 \end{bmatrix}, \\ B = \begin{bmatrix} 0 & 0 & -0.126 \\ -0.126 & 0 & 0 \\ 0 & -0.126 & 0 \end{bmatrix}, \\ E_1 = \begin{bmatrix} 0.3 \\ 0.5 \\ 0 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 0.6 \\ 0.4 \\ 0.2 \end{bmatrix}, \\ C = \begin{bmatrix} 0.3679 & 0 & 0 \\ 0 & 0.6065 & 0 \\ 0 & 0 & 0.3679 \end{bmatrix}, \\ D = \begin{bmatrix} 0.6321 & 0 & 0 \\ 0 & 0.3935 & 0 \\ 0 & 0 & 0.6321 \end{bmatrix}, \\ E_2 = \begin{bmatrix} 0.5 \\ 0.4 \\ 0.2 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0.2 \\ 0.6 \\ 0.3 \end{bmatrix}, \\ G_2 = G_1 = C_2 = C_1 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}. \end{gathered} \tag{47}$$

The regulation function is taken as $g_i(x) = x^2/(1+x^2)$ $(i = 1,2,3)$. It is easy to know that the derivative of $g_i(x)$ is less than $l = 0.65$, which shows $L = \mathrm{diag}(-0.325, -0.325, -0.325)$. Suppose the bound of the time delay is $d = 3$: then $d(k) \in \mathcal{N} = \{1,2,3\}$. The transition probability matrix Π is given by

$$\Pi = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}. \tag{48}$$

By solving the optimization problem (46), it can be obtained that the optimal disturbance attenuation level γ^* is 0.2289, which is better than one (i.e., 1.5046) in [26]. And the corresponding filter gain matrices are as follows:

$$\begin{gathered} A_f = \begin{bmatrix} 0.3033 & 0.0362 & -0.0085 \\ -0.1172 & 0.0675 & 0.0442 \\ -0.0196 & -0.0232 & 0.3032 \end{bmatrix}, \\ B_f = \begin{bmatrix} -1.3657 & 1.0166 & -0.1168 \\ 0.2405 & -1.8421 & -0.0969 \\ 0.0521 & 0.4326 & -1.7666 \end{bmatrix}, \\ C_f = \begin{bmatrix} 0.0604 & -0.0945 & -0.0015 \\ -0.2121 & 0.4325 & 0.0640 \\ -0.1054 & 0.1555 & 0.1184 \end{bmatrix}, \\ D_f = \begin{bmatrix} 0.4505 & -2.1899 & -0.2779 \\ 0.0936 & -1.8583 & 0.3270 \\ -0.4267 & 1.2589 & -2.8255 \end{bmatrix}, \\ G_{1f} = \begin{bmatrix} -0.0965 & -0.1809 & 0.0340 \\ -0.0772 & -0.1447 & 0.0272 \\ -0.0386 & -0.0724 & 0.0136 \end{bmatrix}, \\ G_{2f} = \begin{bmatrix} 0.0413 & -0.0725 & -0.0097 \\ 0.1240 & -0.2175 & -0.0292 \\ 0.0620 & -0.1087 & -0.0146 \end{bmatrix}, \\ H_{1f} = \begin{bmatrix} 0.6784 & -0.9047 & 0.1134 \\ -0.2573 & 0.2762 & 0.0907 \\ -0.1286 & -0.3619 & 1.0454 \end{bmatrix}, \\ H_{2f} = \begin{bmatrix} 1.1304 & -0.3575 & -0.0320 \\ 0.3912 & -0.0724 & -0.0960 \\ 0.1956 & -0.5362 & 0.9520 \end{bmatrix}. \end{gathered} \tag{49}$$

In the following simulation setup, the noise signal is chosen as

$$w(k) = v(k) = \begin{cases} \sin(0.3k), & k \le 20, \\ 0, & k > 20. \end{cases} \tag{50}$$

Let the filtering error system run by random sequence $d(k)$, the trajectories and their estimations of the mRNAs and proteins are shown in Figures 1 and 2, where the solid line and dotted line describe the state trajectories and estimations of mRNAs and proteins, respectively. The filtering errors are shown in Figures 3 and 4. It can be seen from Figures 3 and 4 that the filtering error converges to zero in the absence of disturbances.

Next, we illustrate the H_∞ performance of the filtering error system (11). By direct computation, we have

$$\sum_{k=0}^{60} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix}^T \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} = 20.9454. \tag{51}$$

For values of 1000 random sequences of $d(k)$, we obtain by MATLAB that the maximum of $\sum_{k=0}^{60} \begin{bmatrix} e_m(k) \\ e_p(k) \end{bmatrix}^T \begin{bmatrix} e_m(k) \\ e_p(k) \end{bmatrix}$ is

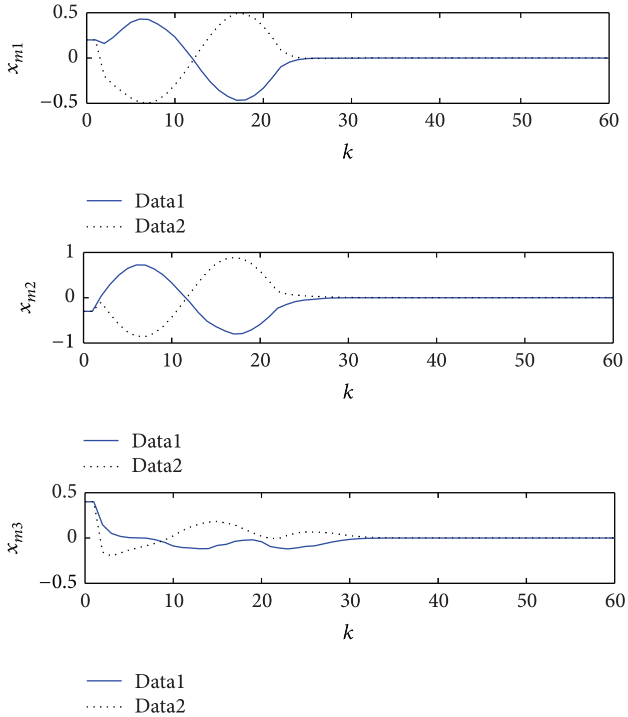

Figure 1: Trajectories and estimations of mRNAs.

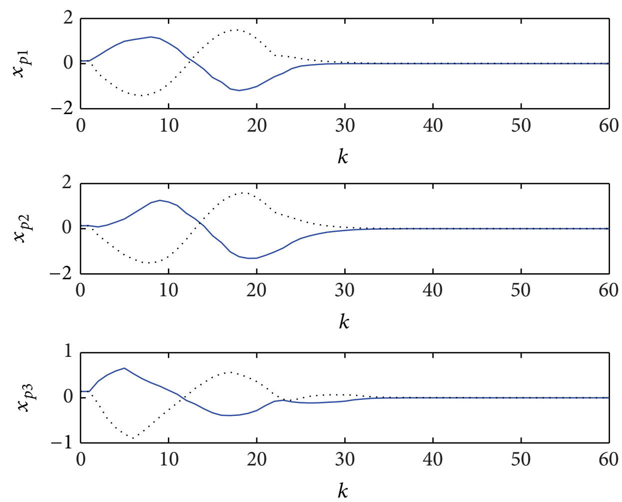

Figure 2: Trajectories and estimations of proteins.

0.1647, and hence the maximum disturbance attenuation level is

$$\sqrt{\frac{\sum_{k=0}^{60}\begin{bmatrix} e_m(k) \\ e_p(k) \end{bmatrix}^T \begin{bmatrix} e_m(k) \\ e_p(k) \end{bmatrix}}{\sum_{k=0}^{60}\begin{bmatrix} w(k) \\ v(k) \end{bmatrix}^T \begin{bmatrix} w(k) \\ v(k) \end{bmatrix}}} = \sqrt{\frac{0.1647}{20.9454}} = 0.0887 < \gamma^*. \tag{52}$$

This verifies that the H_∞ disturbance attenuation level is below the given upper bound.

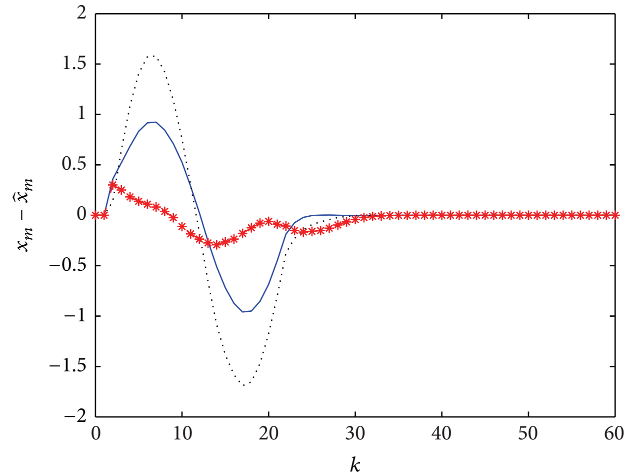

Figure 3: Estimation error of mRNAs.

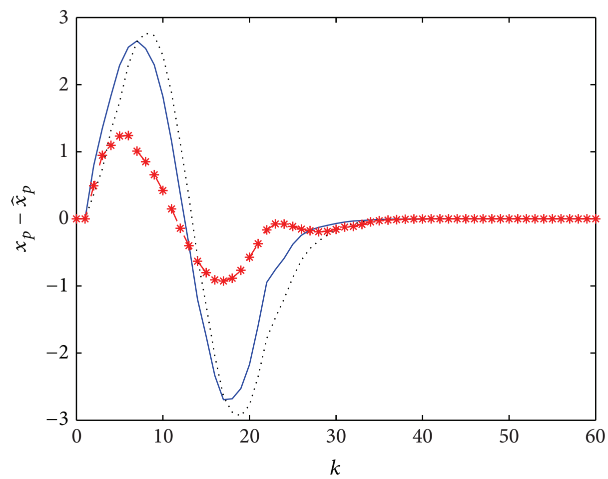

Figure 4: Estimation error of proteins.

5. Conclusion

In this paper, we investigate the filtering problem on a class of discrete-time GRNs with random delays. The filtering error system is established as a Markovian switched system and the random delay is described as a Markovian chain. By introducing an appropriate Lyapunov function, sufficient conditions for concerned problems are derived in terms of LMIs. The designed filter guarantees that the filtering error system is stochastically stable with H_∞ disturbance attenuation level. Finally, the effectiveness and performance of the obtained results are demonstrated by a numerical example.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant no. 11371006, the fund of Heilongjiang Province Innovation Team Support Plan under Grant no. 2012TD007, the fund of Heilongjiang University Innovation Team Support Plan under Grant no. Hdtd2010-03, the Fund of Key Laboratory of Electronics Engineering, College of Heilongjiang Province, (Heilongjiang University), China, the fund of Heilongjiang Education Committee, and the Heilongjiang University Innovation Fund for Graduates. The authors thank the anonymous referees for their helpful comments and suggestions which improve greatly this note.

References

[1] F. K. Ahmad, S. Deris, and N. H. Othman, "The inference of breast cancer metastasis through gene regulatory networks," *Journal of Biomedical Informatics*, vol. 45, no. 2, pp. 350–362, 2012.

[2] H.-Y. Yeh, S.-W. Cheng, Y.-C. Lin, C.-Y. Yeh, S.-F. Lin, and V.-W. Soo, "Identifying significant genetic regulatory networks in the prostate cancer from microarray data based on transcription factor analysis and conditional independency," *BMC Medical Genomics*, vol. 2, article 70, 2009.

[3] R. Somogyi and C. A. Sniegoski, "Modeling the complexity of genetic networks: understanding multigenic and pleiotropic regulation," *Complexity*, vol. 1, no. 6, pp. 45–63, 1995/96.

[4] D. C. Weaver, C. T. Workman, and G. D. Stormo, "Modeling regulatory networks with weight matrices," in *Proceedings of the Pacific Symposium on Biocomputing*, vol. 4, pp. 112–123, World Scientific, Maui, Hawaii, 1999.

[5] M. B. Elowitz and S. Leibier, "A synthetic oscillatory network of transcriptional regulators," *Nature*, vol. 403, no. 6767, pp. 335–338, 2000.

[6] T. Chen, H. L. He, and G. M. Church, "Modeling gene expression with differential equations," *Pacific Symposium on Biocomputing*, vol. 4, pp. 29–40, 1999.

[7] D. Thieffry and R. Thomas, "Qualitative analysis of gene networks," *Pacific Symposium on Biocomputing*, vol. 3, pp. 77–88, 1998.

[8] L. Chen and K. Aihara, "Stability of genetic regulatory networks with time delay," *IEEE Transactions on Circuits and Systems I*, vol. 49, no. 5, pp. 602–608, 2002.

[9] X. Zhang, A. H. Yu, and G. D. Zhang, "M-matrix-based delay-range-dependent global asymptotical stability criterion for genetic regulatory networks with time-varying delays," *Neurocomputing*, vol. 113, pp. 8–15, 2013.

[10] Y. He, J. Zeng, M. Wu, and C.-K. Zhang, "Robust stabilization and H_∞ controllers design for stochastic genetic regulatory networks with time-varying delays and structured uncertainties," *Mathematical Biosciences*, vol. 236, no. 1, pp. 53–63, 2012.

[11] F.-X. Wu, "Stability and bifurcation of ring-structured genetic regulatory networks with time delays," *IEEE Transactions on Circuits and Systems I*, vol. 59, no. 6, pp. 1312–1320, 2012.

[12] Y. T. Wang, A. H. Yu, and X. Zhang, "Robust stability of stochastic genetic regulatory networks with time-varying delays: a delay fractioning approach," *Neural Computing and Applications*, vol. 23, no. 5, pp. 1217–1227, 2013.

[13] J. H. Koo, D. H. Ji, S. C. Won, and J. H. Park, "An improved robust delay-dependent stability criterion for genetic regulatory networks with interval time delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 8, pp. 3399–3405, 2012.

[14] J. Liu and D. Yue, "Asymptotic and robust stability of T-S fuzzy genetic regulatory networks with time-varying delays," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 8, pp. 827–840, 2012.

[15] W. Wang and S. Zhong, "Delay-dependent stability criteria for genetic regulatory networks with time-varying delays and nonlinear disturbance," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 9, pp. 3597–3611, 2012.

[16] P. Li, J. Lam, and Z. Shu, "On the transient and steady-state estimates of interval genetic regulatory networks," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 40, no. 2, pp. 336–349, 2010.

[17] B. Lv, J. Liang, and J. Cao, "Robust distributed state estimation for genetic regulatory networks with Markovian jumping parameters," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 10, pp. 4060–4078, 2011.

[18] J. Liang, J. Lam, and Z. Wang, "State estimation for Markov-type genetic regulatory networks with delays and uncertain mode transition rates," *Physics Letters A*, vol. 373, no. 47, pp. 4328–4337, 2009.

[19] W. Yu, J. Lü, G. Chen, Z. Duan, and Q. Zhou, "Estimating uncertain delayed genetic regulatory networks: an adaptive filtering approach," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 892–897, 2009.

[20] M. Mohammadian, A. Hossein Abolmasoumi, and H. Reza Momeni, "H_∞ mode-independent filter design for Markovian jump genetic regulatory networks with time-varying delays," *Neurocomputing*, vol. 87, pp. 10–18, 2012.

[21] B. Chen, L. Yu, and W.-A. Zhang, "H_∞ filtering for Markovian switching genetic regulatory networks with time-delays and stochastic disturbances," *Circuits, Systems, and Signal Processing*, vol. 30, no. 6, pp. 1231–1252, 2011.

[22] W. Wang, S. Zhong, and F. Liu, "Robust filtering of uncertain stochastic genetic regulatory networks with time-varying delays," *Chaos, Solitons & Fractals*, vol. 45, no. 7, pp. 915–929, 2012.

[23] D. Zhang, L. Yu, and Q.-G. Wang, "Exponential H_∞ filtering for switched stochastic genetic regulatory networks with random sensor delays," *Asian Journal of Control*, vol. 13, no. 5, pp. 749–755, 2011.

[24] T. Wang, Y. S. Ding, L. Zhang, and K. R. Hao, "Robust state estimation for discrete-time stochastic genetic regulatory networks with probabilistic measurement delays," *Neurocomputing*, vol. 111, pp. 1–12, 2013.

[25] D. Zhang, H. Song, L. Yu, Q.-G. Wang, and C. Ong, "Set-values filtering for discrete time-delay genetic regulatory networks with time-varying parameters," *Nonlinear Dynamics*, vol. 69, no. 1-2, pp. 693–703, 2012.

[26] A. Liu, L. Yu, W.-A. Zhang, and B. Chen, "H_∞ filtering for discrete-time genetic regulatory networks with random delays," *Mathematical Biosciences*, vol. 239, no. 1, pp. 97–105, 2012.

[27] J. Cao and F. Ren, "Exponential stability of discrete-time genetic regulatory networks with delays," *IEEE Transactions on Neural Networks*, vol. 19, no. 3, pp. 520–523, 2008.

[28] Q. Ye and B. Cui, "Mean square exponential and robust stability of stochastic discrete-time genetic regulatory networks with uncertainties," *Cognitive Neurodynamics*, vol. 4, no. 2, pp. 165–176, 2010.

[29] Q. Ma, S. Xu, Y. Zou, and J. Lu, "Robust stability for discrete-time stochastic genetic regulatory networks," *Nonlinear Analysis. Real World Applications*, vol. 12, no. 5, pp. 2586–2595, 2011.

[30] K. Mathiyalagan and R. Sakthivel, "Robust stabilization and H_∞ control for discrete-time stochastic genetic regulatory networks with time delays," *Canadian Journal of Physics*, vol. 90, no. 10, pp. 939–953, 2012.

[31] K. Mathiyalagan, R. Sakthivel, and S. M. Anthoni, "New robust passivity criteria for discrete-time genetic regulatory networks with Markovian jumping parameters," *Canadian Journal of Physics*, vol. 90, no. 2, pp. 107–118, 2012.

[32] L. G. Wu, X. J. Su, and P. Shi, "Output feedback control of Markovian jump repeated scalar nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 1, pp. 199–204, 2014.

[33] L. Wu, P. Shi, and H. Gao, "State estimation and sliding-mode control of Markovian jump singular systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1213–1219, 2010.

[34] L. Wu, X. Su, and P. Shi, "Sliding mode control with bounded L_2 gain performance of Markovian jump singular time-delay systems," *Automatica*, vol. 48, no. 8, pp. 1929–1933, 2012.

[35] L. Wu, P. Shi, H. Gao, and C. Wang, "H_∞ filtering for 2D Markovian jump systems," *Automatica*, vol. 44, no. 7, pp. 1849–1858, 2008.

[36] L. Wu, X. Yao, and W. X. Zheng, "Generalized H_2 fault detection for two-dimensional Markovian jump systems," *Automatica*, vol. 48, no. 8, pp. 1741–1750, 2012.

[37] X. J. Su, L. G. Wu, and P. Shi, "Sensor networks with random link failures: distributed filtering for T-S fuzzy systems," *IEEE Transactions on Industrial Informatics*, vol. 9, no. 3, pp. 1739–1750, 2013.

[38] L. Wu and D. W. C. Ho, "Fuzzy filter design for Itô stochastic systems with application to sensor fault detection," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 1, pp. 233–242, 2009.

[39] X. Su, P. Shi, L. Wu, and S. K. Nguang, "Induced filtering of fuzzy stochastic systems with time-varying delays," *IEEE Transactions on Computers*, vol. 43, no. 4, pp. 1251–1264, 2012.

[40] A. Hmamed, C. E. Kasri, E. H. Tissir, T. Alvarez, and F. Tadeo, "Robust H_∞ filtering for uncertain 2-D continuous systems with delays," *International Journal of Innovative Computing Information and Control*, vol. 9, no. 5, pp. 2167–2183, 2013.

[41] F. B. Li and X. Zhang, "Delay-range-dependent robust H1 filtering for singular LPV systems with time variant delay," *International Journal of Innovative Computing Information and Control*, vol. 9, no. 1, pp. 339–353, 2013.

[42] J. Liang and J. Lam, "Robust state estimation for stochastic genetic regulatory networks," *International Journal of Systems Science*, vol. 41, no. 1, pp. 47–63, 2010.

[43] M. C. de Oliveira, J. Bernussou, and J. C. Geromel, "A new discrete-time robust stability condition," *Systems & Control Letters*, vol. 37, no. 4, pp. 261–265, 1999.