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Research Article

Pullback $\mathscr{D}$-Attractor of Nonautonomous Three-Component Reversible Gray-Scott System on Unbounded Domains

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The long time behavior of solutions of the nonautonomous three-components reversible Gray-Scott system defined on the entire space $\mathbb{R}^n$ is studied when the external forcing terms are unbounded in a phase space. The existence of a pullback global attractor for the equation is established in $[L^2(\mathbb{R}^n)]^3$ and $[H^1(\mathbb{R}^n)]^3$, respectively. The pullback asymptotic compactness of solutions is proved by using uniform estimates on the tails of solutions on unbounded domains.

1. Introduction

In this paper, we consider the dynamical behavior of the nonautonomous three-components reversible Gray-Scott system

$$\begin{aligned}\frac{\partial u}{\partial t} &= d_1\Delta u - (F+k)\,u + u^2 v - Gu^3 + Nw + f_1(t,x),\\ \frac{\partial v}{\partial t} &= d_2\Delta v - Fv - u^2 v + Gu^3 + f_2(t,x),\\ \frac{\partial w}{\partial t} &= d_3\Delta w - (F+N)\,w + ku + f_3(t,x),\end{aligned} \tag{1}$$

on $[\tau,\infty)\times\mathcal{O}$ with initial data

$$\begin{aligned}u(\tau,x) = u_0(x),\qquad v(\tau,x) = v_0(x),\\ w(\tau,x) = w_0(x)\quad \text{on } \mathcal{O},\end{aligned} \tag{2}$$

where $\mathcal{O} = \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$; all the parameters are arbitrarily given positive constants; $f_i(x,t)$ for $t\in\mathbb{R}$ and $x\in\mathbb{R}^n$ is an external forcing term which is locally square integrable in time. That is, $f_i\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\mathbb{R}^n))$, $i=1,2,3$.

Historically, when $w = 0$, $G = 0$, and the external forces $f_1 = f_3 = 0$, $f_2 = F$, system (1) reduces to the two-component Gray-Scott system which signified one of the Brussell school led by the renowned physical chemist and Nobel Prize laureate (1977), Ilya Prigogine, which originated from describing an isothermal, cubic autocatalytic, continuously fed, and diffusive reactions of two chemicals (see, e.g., [1, 2]). The three-component reversible Gray-Scott model was firstly introduced by Mahara et al., which is based on the scheme of two reversible chemical or biochemical reactions [3]. Then in [4], You takes some nondimensional transformations, and the three-component reversible system is reduced to the system (1) without external forces. In [4], You considers the existence of global attractor for the system with Neumann boundary condition on a bounded domain of space dimension $n\le 3$ by the method of the rescaling and grouping estimation. For more recent dynamical behaviors of the nonautonomous three-component reversible Gray-Scott system, we can refer to [5, 6] for the existence of random attractors of the stochastic cases on bounded and unbounded domains and [7] for the existence of uniform attractor of the deterministic case on a bounded domain.

As pointed in [8], we can discuss the same or similar coupled reaction-diffusion systems on a higher dimensional domain with the space dimension $n > 3$ and on an unbounded domain, to work with various different phase spaces. Here, we intend to investigate the dynamical behavior of the nonautonomous three-component reversible Gray-Scott system on unbounded domains. It is worth mentioning that the Sobolev embeddings are not compact on domains of infinite volume. This introduces a major obstacle for proving the existence of attractors for PDEs on unbounded domains. For some deterministic equations, the difficulty caused by the unboundedness of domains can be overcome by the energy equation approach which is developed by Ball in [9] and used by many authors (see, e.g., [10, 11]). In this paper, we will use the uniform estimates on the tails of solutions to circumvent the difficulty caused by the unboundedness of the domain. This idea was developed in [12] to prove the asymptotic compactness of solutions for autonomous parabolic equations on $\mathbb{R}^n$ and later extended to stochastic equations in, for example, [6, 13–15]. Here, we will use the method of tail estimates to investigate the asymptotic behavior of system (1) with initial data (2). We first establish the pullback asymptotic compactness of solutions of system (1) and prove the existence of a pullback global attractor in $\mathbb{H}$. Then we extend this result and show the existence of a pullback global attractor in $\mathbb{E}$.

The paper is organized as follows. In the next section, we recall the fundamental concepts and results for pullback attractors for nonautonomous dynamical systems. In Section 3, we define a cocycle for the nonautonomous three-component reversible Gray-Scott system on $\mathbb{R}^n$. Section 4 is devoted to deriving the uniform estimates of solutions for large space and time variables. In the last section, we prove the existence of a pullback global attractor for the equation in $\mathbb{H}$ and $\mathbb{E}$.

The following notations will be used throughout the paper. We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and inner product in $L^2(\mathbb{R}^n)$ or $\mathbb{H} = [L^2(\mathbb{R}^n)]^3$ and use $\|\cdot\|_6$, $\|\cdot\|_4$ to denote the norm in $L^6(\mathbb{R}^n)$ or $\mathbb{V} = [L^6(\mathbb{R}^n)]^3$, $L^4(\mathbb{R}^n)$ or $\mathbb{U} = [L^4(\mathbb{R}^n)]^3$; $\mathbb{V}'$, $\mathbb{U}$ the dual of $\mathbb{V}$, $\mathbb{U}$; $\mathbb{E} = [H^1(\mathbb{R}^n)]^3$. The letters M is a generic positive constant which may change its value from line to line or even in the same line.

2. Preliminaries

In this section, we recall some basic concepts related to pullback attractors for nonautonomous dynamical systems. It is worth noticing that these concepts are quite similar to those of random attractors for stochastic systems. We can refer to [16–19] for more details.

Let Ω be a nonempty set and X a metric space with distance $d(\cdot,\cdot)$.

Definition 1. A family of mappings $\{\theta_t\}_{t\in\mathbb{R}}$ from Ω to itself is called a family of shift operators on Ω if $\{\theta_t\}_{t\in\mathbb{R}}$ satisfies the group properties:

(i) $\theta_0\omega = \omega$, for all $\omega \in \Omega$,

(ii) $\theta_t(\theta_\tau\omega) = \theta_{t+\tau}\omega$, for all $\omega \in \Omega$ and $t, \tau \in \mathbb{R}$.

Definition 2. Let $\{\theta_t\}_{t\in\mathbb{R}}$ be a family of shift operators on Ω. Then a continuous θ-cocycle ϕ on X is a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \longrightarrow X, \qquad (t,\omega,x) \longmapsto \phi(t,\omega,x), \quad (3)$$

which satisfies, for all $\omega \in \Omega$ and $t, \tau \in \mathbb{R}$, the following:

(i) $\phi(0,\omega,\cdot)$ is the identity on X,

(ii) $\phi(t+\tau,\omega,\cdot) = \phi(t,\theta_\tau\omega,\cdot)\circ\phi(\tau,\omega,\cdot)$,

(iii) $\phi(t,\omega,\cdot) : X \to X$ is continuous.

Hereafter, we always assume that ϕ is a continuous θ-cocycle on X and $\mathscr{D}$ is a collection of families of subsets of X:

$$\mathscr{D} = \left\{D = \{D(\omega)\}_{\omega\in\Omega} : D(\omega) \subseteq X \text{ for every } \omega \in \Omega\right\}. \quad (4)$$

Definition 3. Let $\mathscr{D}$ be a collection of families of subsets of X. Then $\mathscr{D}$ is called inclusion closed if $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $\widetilde{D} = \{\widetilde{D}(\omega) \subseteq X : \omega \in \Omega\}$ with $\widetilde{D}(\omega) \subseteq D(\omega)$ for all $\omega \in \Omega$ imply that $\widetilde{D} \in \mathscr{D}$.

Definition 4. Let $\mathscr{D}$ be a collection of families of subsets of X, and $\{K(\omega)\}_{\omega\in\Omega}$ is called a pullback absorbing set for ϕ in $\mathscr{D}$ if, for every $B \in \mathscr{D}$ and $\omega \in \Omega$, there exists $t(\omega, B) > 0$ such that

$$\phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega), \quad \forall t \geq t(\omega, B). \quad (5)$$

Definition 5. Let $\mathscr{D}$ be a collection of families of subsets of X. Then ϕ is said to be $\mathscr{D}$-pullback asymptotically compact in X if, for every $\omega \in \Omega$, $\{\phi(t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$ and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$.

Definition 6. Let $\mathscr{D}$ be a collection of families of subsets of X and $\{\mathscr{A}(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$. Then $\{\mathscr{A}(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ is called a $\mathscr{D}$-pullback global attractor for ϕ if the following conditions are satisfied, for every $\omega \in \Omega$:

(i) $\mathscr{A}(\omega)$ is compact,

(ii) $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ is invariant, that is,

$$\phi(t,\omega,\mathscr{A}(\omega)) = \mathscr{A}(\theta_t\omega), \quad \forall t \geq 0, \quad (6)$$

(iii) $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathscr{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$,

$$\lim_{t\to\infty} d(\phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), \mathscr{A}(\omega)) = 0, \quad (7)$$

where d is the Hausdorff semimetric given by $d(Y,Z) = \sup_{y\in Y}\inf_{z\in Z}\|y-z\|_X$ for any $Y \subseteq X$ and $Z \subseteq X$.

Theorem 7. *Let $\mathscr{D}$ be an inclusion-closed collection of families of subsets of X and ϕ a continuous θ-cocycle on X. Suppose that $\{K(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ is a closed absorbing set for ϕ in $\mathscr{D}$ and ϕ is $\mathscr{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathscr{D}$-pullback global attractor $\{\mathscr{A}(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ which is given by*

$$\mathscr{A}(\omega) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}\phi(t,\theta_{-t}\omega, K(\theta_{-t}\omega))}. \quad (8)$$

3. Cocycle Related to Three-Component Reversible Gray-Scott System

In this section, we constructed a θ-cocycle for the nonautonomous three-component reversible Gray-Scott system defined on $\mathbb{R}^n$. For every $\tau \in \mathbb{R}$ and $t > \tau$, system (1) with initial data (2) can be rewritten as

$$\frac{\partial g}{\partial t} - Ag + H(g) = f(x,t), \quad x \in \mathcal{O}, \tag{9}$$

with initial condition

$$g(x,\tau) = g_\tau(x), \quad x \in \mathbb{R}^n, \tag{10}$$

where

$$g = (u, v, w)^T,$$

$$f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t))^T, \quad x \in \mathcal{O},$$

$$A = \begin{pmatrix} d_1\Delta & 0 & 0 \\ 0 & d_2\Delta & 0 \\ 0 & 0 & d_3\Delta \end{pmatrix}, \tag{11}$$

$$H(g) = \begin{pmatrix} -(F+k)u + u^2v - Gu^3 + Nw \\ -Fv - u^2v + Gu^3 \\ ku - (F+N)w \end{pmatrix},$$

and here T denotes the transposition.

As in the case of bounded domains, by conducting a priori estimate on the Galerkin approximate solutions of system (9)-(10) similar to the autonomous system studied in [7], we can prove that if $f \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{H}))$, then problem (9)-(10) is well defined in $\mathbb{H}$. That is, for any $g_\tau \in \mathbb{H}$ and $\tau \in \mathbb{R}$, (9) possesses a unique solution g satisfying

$$g \in C([\tau,\infty);\mathbb{H}) \cap L^2_{\text{loc}}([\tau,\infty);\mathbb{E}), \tag{12}$$

which continuously depends on the initial data $g_\tau \in \mathbb{H}$. To construct a cocycle ϕ for problem (9)-(10), we denote by $\Omega = \mathbb{R}$ and define a shift operator θ_t on Ω for every $t \in \mathbb{R}$ by

$$\theta_t(\tau) = t + \tau, \quad \forall \tau \in \mathbb{R}. \tag{13}$$

Let ϕ be a mapping from $\mathbb{R}^+ \times \Omega \times \mathbb{H}$ to $\mathbb{H}$ given by

$$\phi(t,\tau,g_\tau) = g(t+\tau,\tau,g_\tau), \tag{14}$$

where $t \geq 0$, $\tau \in \mathbb{R}$, $g_\tau \in \mathbb{H}$, and g is the solution of problem (9)-(10). By the uniqueness of solutions, we find that, for every $t, s \geq 0$, $\tau \in \mathbb{R}$ and $g_\tau \in \mathbb{H}$,

$$\phi(t+s,\tau,g_\tau) = g(t, s+\tau, \phi(s,\tau,g_\tau)). \tag{15}$$

Then it follows that ϕ is a continuous θ-cocycle on $\mathbb{H}$. The purpose of this paper is to study the existence of pullback attractors for ϕ in an appropriate phase space.

Let $\overline{\mathbb{H}}$ be a subset of $\mathbb{H}$ and denote

$$\left\|\overline{\mathbb{H}}\right\| = \sup_{x\in\overline{\mathbb{H}}} \|x\|_{\mathbb{H}}. \tag{16}$$

Let $D = \{D(t)\}_{t\in\mathbb{R}}$ be a family of subsets of $\mathbb{H}$. That is, $D(t) \in \mathbb{H}$ for every $t \in \mathbb{R}$ and satisfying

$$\lim_{t\to-\infty} e^{Ft}\|D(t)\|^2 = 0, \tag{17}$$

where F is a positive number given in (1). Hereafter, we use $\mathcal{D}_F$ to denote the collection of all families of subsets of $\mathbb{H}$ satisfying (17), that is,

$$\mathcal{D}_F = \{D = \{D(t)\}_{t\in\mathbb{R}} : D \text{ satisfies (17)}\}. \tag{18}$$

Throughout this paper, we assume the following conditions for the external term:

$$\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|^2 d\xi < \infty, \quad \forall \tau \in \mathbb{R}, \tag{19}$$

$$\begin{gathered} \int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|^6_{\mathbb{V}'} d\xi < \infty, \\ \int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|^4_{\mathbb{U}'} d\xi < \infty, \quad \forall \tau \in \mathbb{R}, \end{gathered} \tag{20}$$

$$\limsup_{K\to\infty} \int_{-\infty}^{\tau}\int_{|x|\geq K} e^{F\xi}|f(x,\xi)|^2 dx\, d\xi = 0, \quad \forall \tau \in \mathbb{R}. \tag{21}$$

It is useful to note that condition (21) implies that, for every $\tau \in \mathbb{R}$ and $\beta > 0$, there is $K = K(\tau,\beta) > 0$ such that

$$\int_{-\infty}^{\tau}\int_{|x|\geq K} e^{F\xi}|f_i(x,\xi)|^2 dx\, d\xi \leq \beta e^{F\tau}, \quad i = 1,2,3. \tag{22}$$

As we will see later, inequality (22) is crucial for deriving uniform estimates on the tails of solutions, and these estimates are necessary for proving the asymptotic compactness of solutions.

4. Uniform Estimates of Solutions

In this section, we derive uniform estimates of solutions of problem (9)-(10) defined on $\mathbb{R}^n$ when $t \to \infty$. We start with the estimates in $\mathbb{H}$.

Lemma 8. *Suppose that* (19) *holds. Then for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_F$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$,*

$$\begin{gathered} \|g(\tau, \tau-t, g_0(\tau-t))\|^2 \leq Me^{-F\tau}\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|^2 d\xi, \\ \int_{\tau-t}^{\tau} e^{F\xi}\|\nabla g(\xi,\tau-t,g_0(\tau-t))\|^2 ds \leq M\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|^2 d\xi, \end{gathered} \tag{23}$$

where $g_0(\tau-t) \in D(\tau-t)$ and M depends on G, F, N, and k.

Proof. Define

$$W(t,x) = \frac{N}{k}w(t,x), \qquad \mu = \frac{k}{N}, \tag{24}$$

then the system (1) becomes

$$\frac{\partial u}{\partial t} = d_1\Delta u - (F+k)u + u^2v - Gu^3 + kW + f_1(t,x), \tag{25}$$

$$\frac{\partial v}{\partial t} = d_2\Delta v - Fv - u^2v + Gu^3 + f_2(t,x), \tag{26}$$

$$\mu\frac{\partial W}{\partial t} = \mu d_3\Delta W + ku - (\mu F + k)W + f_3(t,x). \tag{27}$$

Taking the inner products $(\partial u/\partial t, Gu(t))$, $(\partial v/\partial t, v(t))$, and $(\mu(\partial W/\partial t), GW(t))$ and then suming up the resulting equalities, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(G\|u\|^2+\|v\|^2+\mu G\|W\|^2\right)+d_1G\|\nabla u\|^2 \\
&\qquad +d_2\|\nabla v\|^2+\mu G d_3\|\nabla W\|^2 \\
&\quad = 2kG\int_{\mathbb{R}^n} uW\,dx \\
&\qquad -\left(G(F+k)\|u\|^2-F\|v\|^2-G(\mu F+k)\|W\|^2\right) \\
&\qquad -\int_{\mathbb{R}^n}\left(Gu^2-uv\right)^2dx+G\int_{\mathbb{R}^n}uf_1dx \\
&\qquad +\int_{\mathbb{R}^n}vf_2dx+G\int_{\mathbb{R}^n}Wf_3dx \\
&\quad \le G\int_{\mathbb{R}^n}uf_1dx+\int_{\mathbb{R}^n}vf_2dx+G\int_{\mathbb{R}^n}Wf_3dx \\
&\qquad -F\left(G\|u\|^2+\|v\|^2+\mu G\|W\|^2\right) \\
&\quad \le -\frac{1}{2}F\left(G\|u\|^2+\|v\|^2+\mu G\|W\|^2\right) \\
&\qquad +\frac{G}{2F}|f_1|^2+\frac{1}{2F}|f_2|^2+\frac{G}{2\mu F}|f_3|^2.
\end{aligned} \tag{28}$$

Setting

$$d=\min\{d_1,d_2,d_3\},\qquad C_1=\frac{\max\{1,G,G/\mu\}}{F\min\{1,G,G/\mu\}}, \tag{29}$$

then (28) becomes

$$\frac{d}{dt}\|g(t)\|^2+2d\|\nabla g(t)\|^2+F\|g(t)\|^2\le C_1\|f\|^2. \tag{30}$$

Multiplying (30) by e^{Ft} and then integrating the resulting inequality on $(\tau-t,\tau)$ with $t\ge 0$, we find that

$$\begin{aligned}
&\|g(\tau,\tau-t,g_0(\tau-t))\|^2+2de^{-F\tau}\int_{\tau-t}^{\tau}e^{F\xi}\|\nabla g(\xi)\|^2d\xi \\
&\quad \le e^{-F\tau}e^{F(\tau-t)}\|g_0(\tau-t)\|^2+C_1e^{-F\tau}\int_{\tau-t}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi \\
&\quad \le e^{-F\tau}e^{F(\tau-t)}\|g_0(\tau-t)\|^2+C_1e^{-F\tau}\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi.
\end{aligned} \tag{31}$$

Notice that $g_0(\tau-t)\in D(\tau-t)$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathscr{D}_F$. We find that for every $\tau\in\mathbb{R}$, there exists $T=T(\tau,D)$ such that for all $t\ge T$,

$$e^{F(\tau-t)}\|g_0(\tau-t)\|^2\le C_1\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi. \tag{32}$$

By (31) and (32), we get that, for all $t\ge T$,

$$\begin{aligned}
&\|g(\tau,\tau-t,g_0(\tau-t))\|^2+2de^{-F\tau}\int_{\tau-t}^{\tau}e^{F\xi}\|\nabla g(\xi)\|^2d\xi \\
&\quad \le 2C_1e^{-F\tau}\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi,
\end{aligned} \tag{33}$$

which completes the proof. □

The following lemma is useful for deriving uniform estimates of solutions in $\mathbb{E}$.

Lemma 9. *Suppose that* (19) *holds. Then for every $\tau\in\mathbb{R}$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathscr{D}_F$, there exists $T=T(\tau,D)>2$ such that for all $t\ge T$,*

$$\begin{aligned}
&\int_{\tau-2}^{\tau}e^{F\xi}\|g(\xi,\tau-t,g_0(\tau-t))\|^2d\xi\le M\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi, \\
&\int_{\tau-2}^{\tau}e^{F\xi}\|\nabla g(\xi,\tau-t,g_0(\tau-t))\|^2d\xi\le M\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi,
\end{aligned} \tag{34}$$

where $g_0(\tau-t)\in D(\tau-t)$ and M relies on G*,* F*,* N*, and* k*.*

Proof. By (30) we find that

$$\frac{d}{dt}\|g(t)\|^2+F\|g(t)\|^2\le C_1\|f\|^2. \tag{35}$$

Let $s\in[\tau-2,\tau]$ and $\tau>2$. Multiplying the above by e^{Ft} and integrating over $(\tau-t,s)$, we get

$$\begin{aligned}
&e^{Fs}\|g(s,\tau-t,g_0(\tau-t))\|^2 \\
&\quad \le e^{F(\tau-t)}\|g_0(\tau-t)\|^2+C_1\int_{\tau-t}^{s}e^{F\xi}\|f(\xi)\|^2d\xi \\
&\quad \le e^{F(\tau-t)}\|g_0(\tau-t)\|^2+C_1\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi.
\end{aligned} \tag{36}$$

Therefore, there exists $T=T(\tau,D)>2$ such that for all $t\ge T$ and $s\in[\tau-2,\tau]$,

$$e^{Fs}\|g(s,\tau-t,g_0(\tau-t))\|^2\le 2C_1\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi. \tag{37}$$

Integrate the above with respect to s on $(\tau-2,\tau)$ to obtain

$$\begin{aligned}
&\int_{\tau-2}^{\tau}e^{Fs}\|g(s,\tau-t,g_0(\tau-t))\|^2ds \\
&\quad \le 4C_1\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi.
\end{aligned} \tag{38}$$

On the other hand, for $s=\tau-2$, (37) implies that

$$e^{F(\tau-2)}\|g(\tau-2,\tau-t,g_0(\tau-t))\|^2\le 2C_1\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi. \tag{39}$$

Multiplying (30) by $e^{F\tau}$ and then integrating over $(\tau-2,\tau)$, by (38) we get that, for all $t\ge T$,

$$\begin{aligned}
&e^{F\tau}\|g(\tau,\tau-t,g_0(\tau-t))\|^2+2d\int_{\tau-2}^{\tau}e^{F\xi}\|\nabla g(\xi)\|^2d\xi \\
&\quad \le e^{F(\tau-2)}\|g(\tau-2,\tau-t,g_0(\tau-t))\|^2 \\
&\qquad +C_1\int_{\tau-t}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi \\
&\quad \le 3C_1\int_{-\infty}^{\tau}e^{F\xi}\|f(\xi)\|^2d\xi.
\end{aligned} \tag{40}$$

which along with (39) completes the proof. □

Note that $e^{\lambda\xi} \geq e^{\lambda\tau-2\lambda}$ for any $\xi \geq \tau-2$. So as an immediate consequence of Lemma 9 we have the following estimates.

Corollary 10. *Suppose that* (19) *holds. Then for every* $\tau \in \mathbb{R}$ *and* $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_F$, *there exists* $T = T(\tau, D) > 2$ *such that for all* $t \geq T$,

$$\begin{aligned} &\int_{\tau-2}^{\tau} \|g(\xi, \tau-t, g_0(\tau-t))\|^2 ds \\ &\quad \leq Me^{-F\tau} \int_{-\infty}^{\tau} e^{F\xi} \|f(\xi)\|^2 d\xi, \\ &\int_{\tau-2}^{\tau} \|\nabla g(\xi, \tau-t, g_0(\tau-t))\|^2 ds \\ &\quad \leq Me^{-F\tau} \int_{-\infty}^{\tau} e^{F\xi} \|f(\xi)\|^2 d\xi, \end{aligned} \tag{41}$$

where $g_0(\tau-t) \in D(\tau-t)$ *and* M *depends on* G, F, N, *and* k.

Before the derivation of uniform estimates of solutions in $\mathbb{E}$, we firstly give two propositions which will frequently be used in the next results.

Proposition 11. *Suppose that* (20) *holds. Then for every* $\tau \in \mathbb{R}$ *and* $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_F$, *there exists* $T = T(\tau, D) > 0$ *such that for all* $t \geq T$,

$$\|g(\tau, \tau-t, g_0(\tau-t))\|_6^6 \leq Me^{-F\tau} \int_{-\infty}^{\tau} e^{F\xi} \|f(\xi)\|_{\mathbb{V}'}^6 d\xi, \tag{42}$$

$$\int_{\tau-2}^{\tau} e^{F\xi} \|g(\xi, \tau-t, g_0(\tau-t))\|_6^6 d\xi \leq M \int_{-\infty}^{\tau} e^{F\xi} \|f(\xi)\|_{\mathbb{V}'}^6 d\xi, \tag{43}$$

where $g_0(\tau-t) \in D(\tau-t)$ *and* M *depends on* G, F, N, *and* k.

Proof. Let $V(t,x) = (v(t,x))/G$, then (25)–(27) become

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u - (F+k)u + Gu^2V - Gu^3 + kW + h_1(t,x), \\ \frac{\partial V}{\partial t} &= d_2 \Delta V - FV - u^2V + u^3 + h_2(t,x), \\ \mu \frac{\partial W}{\partial t} &= \mu d_3 \Delta W + ku - (\mu F + k)W + h_3(t,x). \end{aligned} \tag{44}$$

Taking the inner products $(\partial u/\partial t, u^5(t))$, $(\partial V/\partial t, GV^5(t))$ and $(\mu(\partial W/\partial t), W^5(t))$ and then suming up the resulting equalities, we get

$$\begin{aligned} &\frac{1}{6}\frac{d}{dt}\left(\|u\|_6^6 + G\|V\|_6^6 + \mu\|W\|_6^6\right) \\ &\qquad + 5\left(d_1\left\|u^2\nabla u\right\|^2 + d_2 G\left\|V^2\nabla V\right\|^2 + \mu d_3 G\left\|W^2\nabla W\right\|^2\right) \\ &\quad \leq -(F+k)\int_{\mathbb{R}^n} u^6 ds + F\int_{\mathbb{R}^n} V^5 dx - GF\int_{\mathbb{R}^n} V^6 dx \\ &\qquad - (\mu F + k)\int_{\mathbb{R}^n} W^6 dx + k\int_{\mathbb{R}^n} u^5 W\, dx + k\int_{\mathbb{R}^n} uW^5 dx \\ &\qquad - G\int_{\mathbb{R}^n}\left(u^8 - u^7 V - u^3V^5 + u^2V^6\right) dx \\ &\qquad + \int_{\mathbb{R}^n} u^5 f_1 dx + G\int_{\mathbb{R}^n} V^5 f_2 dx + \int_{\mathbb{R}^n} W^5 f_3 dx. \end{aligned} \tag{45}$$

By Young's inequality,

$$\begin{aligned} -G\int_{\mathbb{R}^n}\left(u^8 - u^7V - u^3V^5 + u^2V^6\right) dx &\leq 0, \\ G\int_{\mathbb{R}^n} V^5 f_2 dx &\leq \frac{5FG}{6}\|V\|_6^6 + \frac{G\|f_2\|_{\mathbb{V}'}^6}{6F^5}, \\ \int_{\mathbb{R}^n} u^5 f_1 dx &\leq \frac{5F}{6}\|u\|_6^6 + \frac{\|f_1\|_{\mathbb{V}'}^6}{6F^5}, \\ \int_{\mathbb{R}^n} W^5 h_3 dx &\leq \frac{5\mu F}{6}\|W\|_6^6 + \frac{\|f_3\|_{\mathbb{V}'}^6}{6(\mu F)^5}. \end{aligned} \tag{46}$$

From (46) then (45) yields

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|_6^6 + G\|V\|_6^6 + \mu\|W\|_6^6\right) + F\left(\|u\|_6^6 + G\|V\|_6^6 + \mu\|W\|_6^6\right) \\ &\quad \leq \frac{\|f_1\|_{\mathbb{V}'}^6}{F^5} + \frac{G\|f_2\|_{\mathbb{V}'}^6}{F^5} + \frac{\|f_3\|_{\mathbb{V}'}^6}{(\mu F)^5}. \end{aligned} \tag{47}$$

Denote

$$C_4 = \frac{\max\{1, G, 1/\mu^5\}}{F^5 \min\{1, 1/G^5, 1/\mu^5\}}, \tag{48}$$

then from (47) implies that

$$\frac{d}{dt}\|g(t)\|_6^6 + F\|g(t)\|_6^6 \leq C_4\|f(t)\|_{\mathbb{V}'}^6. \tag{49}$$

Multiplying (49) by e^{Ft} and then integrating the resulting inequality on $(\tau-t, \tau)$ with $t \geq 0$, we find that

$$\begin{aligned} &\|g(\tau, \tau-t, g_0(\tau-t))\|_6^6 \\ &\quad \leq e^{-F\tau}e^{F(\tau-t)}\|g_0(\tau-t)\|_6^6 + C_4 e^{-F\tau}\int_{\tau-t}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6 d\xi \end{aligned}$$

$$\leq e^{-F\tau}e^{F(\tau-t)}\|g_0(\tau-t)\|_6^6 + C_4 e^{-F\tau}\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6 d\xi. \tag{50}$$

Notice that $g_0(\tau-t) \in D(\tau-t)$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_F$. We find that, for every $\tau \in \mathbb{R}$, there exists $T = T(\tau, D)$ such that for all $t \geq T$,

$$e^{F(\tau-t)}\|g_0(\tau-t)\|_6^6 \leq C_4\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6 d\xi. \tag{51}$$

By (50) and (51), we get that, for all $t \geq T$,

$$\|g(\tau,\tau-t,g_0(\tau-t))\|_6^6 \leq 2C_4 e^{-F\tau}\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6 d\xi, \tag{52}$$

which completes the proof. □

Similarly, we have the following.

Proposition 12. *Suppose that* (20) *holds. Then for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_F$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$,*

$$\begin{aligned}
&\|g(\tau,\tau-t,g_0(\tau-t))\|_4^4 \leq Me^{-F\tau}\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{U}'}^4 d\xi,\\
&\int_{\tau-2}^{\tau} e^{F\xi}\|g(\xi,\tau-t,g_0(\tau-t))\|_4^4 d\xi \leq M\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{U}'}^4 d\xi,
\end{aligned} \tag{53}$$

where $g_0(\tau-t) \in D(\tau-t)$ and M depends on G, F, N, and k.

Proof. The proof is similar to Proposition 11 except for few trivial details, and thus we omit it here. □

Lemma 13. *Suppose that* (19) *and* (20) *hold. Then for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_F$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,*

$$\begin{aligned}
&\|\nabla g(\tau,\tau-t,g_0(\tau-t))\|^2\\
&\quad\leq Me^{-F\tau}\left(\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|^2 d\xi + \int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6 d\xi\right),\\
&\int_{\tau-1}^{t}\|g_\xi(\xi,\tau-t,g_0(\tau-t))\|^2 d\xi\\
&\quad\leq Me^{-F\tau}\left(\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|^2 d\xi + \int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6 d\xi\right),
\end{aligned} \tag{54}$$

where $g_0(\tau-t) \in D(\tau-t)$ and M depends on d, G, F, N, and k.

Proof. Taking the inner product of the first equation of system (1) with u_t, the second equation with v_t, and the third equation with w_t, respectively, in $L^2(\mathbb{R}^n)$ and then replacing t by ξ, we obtain

$$\begin{aligned}
&\|u_\xi(\xi,\tau-t,u_0(\tau-t))\|^2\\
&\quad+\frac{d}{d\xi}\left(\frac{d_1}{2}\|\nabla u(\xi,\tau-t,u_0(\tau-t))\|^2\right.\\
&\qquad\left.+\frac{(F+k)}{2}\|u(\xi,\tau-t,u_0(\tau-t))\|^2\right)\\
&\leq\frac{1}{2}\|u_\xi(\xi,\tau-t,u_0(\tau-t))\|^2\\
&\quad+\left(\frac{4}{3}+2G^2\right)\|u(\xi,\tau-t,u_0(\tau-t))\|_6^6\\
&\quad+\frac{2}{3}\|v(\xi,\tau-t,v_0(\tau-t))\|_6^6\\
&\quad+2N^2\|w(\xi,\tau-t,w_0(\tau-t))\|^2+2\|f_1(\xi)\|^2,
\end{aligned} \tag{55}$$

$$\begin{aligned}
&\|v_\xi(\xi,\tau-t,v_0(\tau-t))\|^2\\
&\quad+\frac{d}{d\xi}\left(\frac{d_2}{2}\|\nabla v(\xi,\tau-t,u_0(\tau-t))\|^2\right.\\
&\qquad\left.+\frac{F}{2}\|v(\xi,\tau-t,v_0(\tau-t))\|^2\right)\\
&\leq\frac{1}{2}\|v_\xi(\xi,\tau-t,v_0(\tau-t))\|^2\\
&\quad+\left(1+\frac{3}{2}G^2\right)\|u(\xi,\tau-t,u_0(\tau-t))\|_6^6\\
&\quad+\frac{1}{2}\|v(\xi,\tau-t,v_0(\tau-t))\|_6^6+\frac{3}{2}\|f_2(\xi)\|^2,
\end{aligned} \tag{56}$$

$$\begin{aligned}
&\|w_\xi(\xi,\tau-t,w_0(\tau-t))\|^2\\
&\quad+\frac{d}{d\xi}\left(\frac{d_3}{2}\|\nabla w(\xi,\tau-t,w_0(\tau-t))\|^2\right.\\
&\qquad\left.+\frac{(F+N)}{2}\|w(\xi,\tau-t,w_0(\tau-t))\|^2\right)\\
&\leq\frac{1}{2}\|w_\xi(\xi,\tau-t,w_0(\tau-t))\|^2\\
&\quad+k^2\|u(\xi,\tau-t,u_0(\tau-t))\|^2+\|f_3(\xi)\|^2,
\end{aligned} \tag{57}$$

where (55) and (56) are partly due to the Young inequality $\|u^4v^2\| \leq (2/3)\|u\|_6^6 + (1/3)\|v\|_6^6$. Denote $u_0(\tau-t)$, $v_0(\tau-$

$t)$, $w_0(\tau-t)$, and $g_0(\tau-t)$ with u_0, v_0, w_0, and g_0, respectively. Adding the three inequalities (55)–(57) together, we have

$$\begin{aligned}
&\left\|u_\xi(\xi,\tau-t,u_0)\right\|^2+\left\|v_\xi(\xi,\tau-t,v_0)\right\|^2+\left\|w_\xi(\xi,\tau-t,w_0)\right\|^2\\
&\quad+\frac{d}{d\xi}\Big(d_1\|\nabla u(\xi,\tau-t,u_0)\|^2+d_2\|\nabla v(\xi,\tau-t,v_0)\|^2\\
&\qquad+d_3\|\nabla w(\xi,\tau-t,w_0)\|^2\\
&\qquad+(F+k)\|u(\xi,\tau-t,u_0)\|^2\\
&\qquad+F\|v(\xi,\tau-t,v_0)\|^2\\
&\qquad+(F+N)\|w(\xi,\tau-t,w_0)\|^2\Big)\\
&\le 4\left(1+\frac{3}{2}G^2\right)\|u(\xi,\tau-t,u_0)\|_6^6+2\|v(\xi,\tau-t,v_0)\|_6^6\\
&\quad+4N^2\|w(\xi,\tau-t,w_0)\|^2+2k^2\|u(\xi,\tau-t,u_0)\|^2\\
&\quad+4\|f_1(\xi)\|^2+3\|f_2(\xi)\|^2+2\|f_3(\xi)\|^2.
\end{aligned}\tag{58}$$

Let

$$C_2=4+6G^2,\qquad C_3=2\max\left\{2N^2,k^2\right\}.\tag{59}$$

Then (58) yields

$$\begin{aligned}
&\left\|g_\xi(\xi,\tau-t,g_0)\right\|^2+\frac{d}{d\xi}\Big(d\|\nabla g(\xi,\tau-t,g_0)\|^2\\
&\qquad+F\|g(\xi,\tau-t,g_0)\|^2\Big)\\
&\quad\le C_2\|g(\xi,\tau-t,g_0)\|_6^6+C_3\|g(\xi,\tau-t,g_0)\|^2\\
&\qquad+4\|f(\xi)\|^2.
\end{aligned}\tag{60}$$

That is,

$$\begin{aligned}
&\frac{d}{d\xi}\left(d\|\nabla g(\xi,\tau-t,g_0)\|^2+F\|g(\xi,\tau-t,g_0)\|^2\right)\\
&\quad\le C_2\|g(\xi,\tau-t,g_0)\|_6^6+C_3\|g(\xi,\tau-t,g_0)\|^2+4\|f(\xi)\|^2.
\end{aligned}\tag{61}$$

Let $s\le\tau$ and $t\ge2$. By integrating (61) over (s,τ), we get

$$\begin{aligned}
&d\|\nabla g(\tau,\tau-t,g_0)\|^2+F\|g(\tau,\tau-t,g_0)\|^2\\
&\quad\le d\|\nabla g(s,\tau-t,g_0)\|^2+F\|g(s,\tau-t,g_0)\|^2\\
&\qquad+C_2\int_s^\tau\|g(\xi,\tau-t,g_0)\|_6^6d\xi\\
&\qquad+C_3\int_s^\tau\|g(\xi,\tau-t,g_0)\|^2d\xi+4\int_s^\tau\|f(\xi)\|^2d\xi.
\end{aligned}\tag{62}$$

Now integrating the above with respect to s on $(\tau-1,\tau)$, we find

$$\begin{aligned}
&d\|\nabla g(\tau,\tau-t,g_0)\|^2+F\|g(\tau,\tau-t,g_0)\|^2\\
&\quad\le d\int_{\tau-1}^\tau\|\nabla g(s,\tau-t,g_0)\|^2ds\\
&\qquad+F\int_{\tau-1}^\tau\|g(s,\tau-t,g_0)\|^2ds\\
&\qquad+C_2\int_{\tau-1}^\tau\|g(\xi,\tau-t,g_0)\|_6^6d\xi\\
&\qquad+C_3\int_{\tau-1}^\tau\|g(\xi,\tau-t,g_0)\|^2d\xi\\
&\qquad+4\int_{\tau-1}^\tau\|f(\xi)\|^2d\xi,
\end{aligned}\tag{63}$$

which along with Corollary 10 and Proposition 11 implies that there exists $T=T(\tau,D)>2$ such that for all $t\ge T$,

$$\begin{aligned}
&d\|\nabla g(\tau,\tau-t,g_0)\|^2+F\|g(\tau,\tau-t,g_0)\|^2\\
&\quad\le\left(d+F+C_3+4e^F\right)Me^{-F\tau}\int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|^2d\xi\\
&\qquad+C_2e^FMe^{-F\tau}\int_{-\infty}^\tau\|f(\xi)\|_{\mathbb{V}'}^6d\xi\\
&\quad\le Me^{-F\tau}\left(\int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|^2d\xi+\int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6d\xi\right).
\end{aligned}\tag{64}$$

Similarly, first integrating (61) with respect to ξ on $(s,\tau-1)$ and then integrating with respect to s on $(\tau-2,\tau-1)$, by using Corollary 10 and Proposition 11, we can get that for all $t\ge T$,

$$\begin{aligned}
&d\|\nabla g(\tau-1,\tau-t,g_0)\|^2+F\|g(\tau-1,\tau-t,g_0)\|^2\\
&\quad\le\left(d+F+C_3+4e^F\right)Me^{-F\tau}\int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|^2d\xi\\
&\qquad+C_2e^FMe^{-F\tau}\int_{-\infty}^\tau\|f(\xi)\|_{\mathbb{V}'}^6d\xi\\
&\quad\le Me^{-F\tau}\left(\int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|^2d\xi+\int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6d\xi\right).
\end{aligned}\tag{65}$$

Now integrating (60) over $(\tau-1,\tau)$, we obtain

$$\begin{aligned}
&\int_{\tau-1}^{\tau}\left\|g_\xi(\xi,\tau-t,g_0)\right\|^2 d\xi \\
&\quad + d\left\|\nabla g(\tau,\tau-t,g_0)\right\|^2 + F\left\|g(\tau,\tau-t,g_0)\right\|^2 \\
&\le d\left\|\nabla g(\tau-1,\tau-t,g_0)\right\|^2 + F\left\|g(\tau-1,\tau-t,g_0)\right\|^2 \\
&\quad + C_2\int_{\tau-1}^{\tau}\|g(\xi,\tau-t,g_0)\|_6^6 d\xi \\
&\quad + C_3\int_{\tau-1}^{\tau}\left\|g(\xi,\tau-t,g_0)\right\|^2 d\xi \\
&\quad + 4\int_{\tau-1}^{\tau}\|f(\xi)\|^2 d\xi,
\end{aligned} \tag{66}$$

which along with (65) shows that, for all $t \ge T$,

$$\begin{aligned}
&\int_{\tau-1}^{\tau}\left\|g_\xi(\xi,\tau-t,g_0)\right\|^2 d\xi \\
&\quad \le Me^{-F\tau}\left(\int_{-\infty}^{\tau} e^{F\xi}\left\|f(\xi)\right\|^2 d\xi + \int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6 d\xi\right).
\end{aligned} \tag{67}$$

Then Lemma 13 follows from (64) and (67) which completes the proof. □

Lemma 14. *Suppose that* (19) *and* (20) *hold, and let*

$$\frac{df}{dt} \in \left[L^2_{\mathrm{loc}}\left(\mathbb{R}, L^2\left(\mathbb{R}^n\right)\right)\right]^3. \tag{68}$$

Then for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_F$, there exists $T = T(\tau, D) > 2$ such that for all $t \ge T$,

$$\begin{aligned}
&\left\|g_\tau(\tau,\tau-t,g_0(\tau-t))\right\|^2 \\
&\quad \le Me^{-F\tau}e^{Me^{-F\tau}\int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{U}'}^4 d\xi} \\
&\qquad \times\left(\int_{-\infty}^{\tau} e^{F\xi}\left\|f(\xi)\right\|^2 d\xi \right.\\
&\qquad\quad \left. + \int_{-\infty}^{\tau} e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6 d\xi\right) \\
&\qquad + M\int_{\tau-1}^{\tau}\left\|f_\xi(\xi)\right\|^2 d\xi,
\end{aligned} \tag{69}$$

where $g_0(\tau-t) \in D(\tau-t)$ and M depends on d, G, F, N, k, and ϱ (a positive constant in the Gagliardo-Nirenberg inequality).

Proof. Let $u_t = \widetilde{u}$, $v_t = \widetilde{v}$, and $W_t = \widetilde{w}$ and differentiate system (1) with respect to t to get that

$$\begin{aligned}
\frac{\partial\widetilde{u}}{\partial t} &= d_1\Delta\widetilde{u} - (F+k)\widetilde{u} + 2uv\widetilde{u} + u^2\widetilde{v} \\
&\quad - 3Gu^2\widetilde{u} + k\widetilde{w} + \frac{\partial f_1}{\partial t}(t,x), \\
\frac{\partial\widetilde{v}}{\partial t} &= d_2\Delta\widetilde{v} - F\widetilde{v} - 2uv\widetilde{u} - u^2\widetilde{v} + 3Gu^2\widetilde{u} + \frac{\partial f_2}{\partial t}(t,x), \\
\frac{\partial\widetilde{w}}{\partial t} &= d_3\Delta\widetilde{w} + k\widetilde{u} - (F+N)\widetilde{w} + \frac{\partial f_3}{\partial t}(t,x).
\end{aligned} \tag{70}$$

Taking the inner products $(\partial\widetilde{u}/\partial t, \widetilde{u})$, $(\partial\widetilde{v}/\partial t, \widetilde{v})$, and $(\partial\widetilde{w}/\partial t, \widetilde{w})$ in $L^2(\mathbb{R}^n)$ and then putting the three equalities together, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2 + \|\widetilde{w}\|^2\right) + d_1\|\nabla\widetilde{u}\|^2 \\
&\quad + d_2\|\nabla\widetilde{v}\|^2 + d_3\|\nabla\widetilde{w}\|^2 \\
&\quad + (F+k)\|\widetilde{u}\|^2 + F\|\widetilde{v}\|^2 + (N+k)\|\widetilde{w}\|^2 \\
&\le \left(2 + \frac{3}{2}G\right)\left(\|u\|_4^2 + \|v\|_4^2\right)\left(\|\widetilde{u}\|_4^2 + \|\widetilde{v}\|_4^2\right) \\
&\quad + N\|\widetilde{w}\|_4^2 + k\|\widetilde{u}\|_4^2 + \frac{N}{4}\|\widetilde{u}\|_4^2 + \frac{k}{4}\|\widetilde{w}\|_4^2 \\
&\quad + \frac{F}{2}\left(\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2 + \|\widetilde{w}\|^2\right) \\
&\quad + \frac{1}{2F}\left(\left\|f_{1t}(t)\right\|^2 + \left\|f_{2t}(t)\right\|^2 + \left\|f_{3t}(t)\right\|^2\right).
\end{aligned} \tag{71}$$

That is,

$$\begin{aligned}
&\frac{d}{dt}\left\|\widetilde{g}\right\|^2 + 2d\left\|\nabla\widetilde{g}\right\|^2 + F\left\|\widetilde{g}\right\|^2 \\
&\quad \le (4+3G)\left\|g\right\|_4^2\left\|\widetilde{g}\right\|_4^2 + \frac{1}{2}\max\{N,k\}\left\|\widetilde{g}\right\|^2 + \frac{1}{F}\left\|f_t(t)\right\|^2.
\end{aligned} \tag{72}$$

Due to the Hölder inequality and Gagliardo-Nirenberg inequality,

$$\begin{aligned}
&\frac{d}{dt}\left\|\widetilde{g}\right\|^2 + 2d\left\|\nabla\widetilde{g}\right\|^2 + F\left\|\widetilde{g}\right\|^2 \\
&\quad \le (4+3G)\left\|g\right\|_4^2\left\|\widetilde{g}\right\|_4^2 + \frac{1}{2}\max\{N,k\}\left\|\widetilde{g}\right\|^2 + \frac{1}{F}\left\|f_t(t)\right\|^2 \\
&\quad \le (4+3G)\varrho\left\|g\right\|_4^2\left\|\widetilde{g}\right\|\left\|\nabla\widetilde{g}\right\| \\
&\qquad + \frac{1}{2}\max\{N,k\}\left\|\widetilde{g}\right\|^2 + \frac{1}{F}\left\|f_t(t)\right\|^2 \\
&\quad \le 2d\left\|\nabla\widetilde{g}\right\|^2 + \frac{(4+3G)^2\varrho^2}{8d}\left\|g\right\|_4^4\left\|\widetilde{g}\right\|^2 + \delta\left\|\widetilde{g}\right\|^2 + \frac{1}{F}\left\|f_t(t)\right\|^2,
\end{aligned} \tag{73}$$

where $\delta = (1/2)\max\{N, k\}$. Then (73) implies that

$$\frac{d}{dt}\|\tilde{g}\|^2 \le \left(\frac{(4+3G)^2\varrho^2}{8d}\|g\|_4^4 + \delta\right)\|\tilde{g}\|^2 + \frac{1}{F}\|f_t(t)\|^2. \quad (74)$$

By the Gronwall lemma, letting $s \in [\tau-1, \tau]$ and $t \ge 1$ and integrating on (s, τ), by $\tilde{g} = g_t$ we get

$$\begin{aligned}
&\|g_\tau(\tau, \tau-t, g_0(\tau-t))\|^2 \\
&\quad \le \|g_\tau(s, \tau-t, g_0(\tau-t))\|^2 \\
&\qquad \times e^{((4+3G)^2\varrho^2/8d)\int_s^\tau \|g(\xi,\tau-t,g_0(\tau-t))\|_4^4 d\xi + \delta(\tau-s)} \\
&\qquad + \frac{1}{F}\int_s^\tau \|f_\xi(\xi)\|^2 d\xi \\
&\quad \le \|g_\tau(\tau, \tau-t, g_0(\tau-t))\|^2 \\
&\qquad \times e^{((4+3G)^2\varrho^2/8d)\int_s^\tau \|g(\xi,\tau-t,g_0(\tau-t))\|_4^4 d\xi + \delta(\tau-s)} \\
&\qquad + \frac{1}{F}\int_s^\tau \|f_\xi(\xi)\|^2 d\xi.
\end{aligned} \quad (75)$$

Now integrating (75) with respect to s on $(\tau-1, \tau)$, we find

$$\begin{aligned}
&\|g_\tau(\tau, \tau-t, g_0(\tau-t))\|^2 \\
&\quad \le e^{((4+3G)^2\varrho^2/8d)e^{-F\tau}\int_{\tau-1}^\tau e^{-F\xi}\|g(\xi,\tau-t,g_0(\tau-t))\|_4^4 d\xi + \delta} \\
&\qquad \times \int_{\tau-1}^\tau \|g_\tau(s, \tau-t, g_0(\tau-t))\|^2 ds \\
&\qquad + \frac{1}{F}\int_{\tau-1}^\tau \|f_\xi(\xi)\|^2 d\xi \\
&\quad \le e^{Me^{-F\tau}\int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|_{\mathbb{U}'}^4 d\xi + \delta} \\
&\qquad \times \int_{\tau-1}^\tau \|g_\tau(s, \tau-t, g_0(\tau-t))\|^2 ds \\
&\qquad + \frac{1}{F}\int_{\tau-1}^\tau \|f_\xi(\xi)\|^2 d\xi,
\end{aligned} \quad (76)$$

which along with Lemma 13 and Proposition 12 shows that there exists $T = (\tau, D) > 2$ such that for all $t \ge T$,

$$\begin{aligned}
&\|g_\tau(\tau, \tau-t, g_0(\tau-t))\|^2 \\
&\quad \le Me^{-F\tau}e^{Me^{-F\tau}\int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|_{\mathbb{U}'}^4 d\xi} \\
&\qquad \times \left(\int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|^2 d\xi + \int_{-\infty}^\tau e^{F\xi}\|f(\xi)\|_{\mathbb{V}'}^6 d\xi\right) \\
&\qquad + M\int_{\tau-1}^\tau \|f_\xi(\xi)\|^2 d\xi,
\end{aligned} \quad (77)$$

which completes the proof. □

We now establish uniform estimates on the tails of solutions when $t \to \infty$. We show that the tails of solutions are uniformly small for large space and time variables. These uniform estimates are crucial for proving the pullback asymptotic compactness of the cocycle ϕ.

Lemma 15. *Suppose that* (19) *and* (20) *hold. Then for every* $\varepsilon > 0$, $\tau \in \mathbb{R}$, *and* $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_F$, *there exist* $T = T(\tau, D, \varepsilon) > 2$ *and* $K = K(\tau, \varepsilon) > 0$ *such that for all* $t \ge T$ *and* $N \ge K$,

$$\int_{|x|\ge K} |g(x, \tau, \tau-t, g_0(\tau-t))|^2 dx \le \varepsilon, \quad (78)$$

where $g_0(\tau-t) \in D(\tau-t)$, *K depends on* τ *and* ε, *and T depends on* τ, D, *and* ε.

Proof. Choose a smooth cut-off function satisfying $0 \le \rho(s) \le 1$ for $s \in \mathbb{R}^+$, $\rho(s) = 0$ for $0 \le s \le 1$, and $\rho(s) = 1$ for $s \ge 2$. Suppose that there exists a constant c such that $|\rho'(s)| \le c$ for $s \in \mathbb{R}^+$.

Taking the inner product of (25), (26), and (27) with $G\rho(|x|^2/K^2)u$, $\rho(|x|^2/K^2)v$, and $G\rho(|x|^2/K^2)W$ in $L^2(\mathbb{R}^n)$, respectively, we get

$$\begin{aligned}
&\frac{G}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|u|^2 - d_1 G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)u\Delta u \\
&\qquad + G(F+k)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|u|^2 \\
&\quad = G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)u^3 v\,dx - G^2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)u^4 dx \\
&\qquad + kG\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)uW\,dx + G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)f_1 u\,dx, \\
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|v|^2 - d_2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)v\Delta v\,dx \\
&\qquad + F\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|v|^2 dx \\
&\quad = -\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)u^2v^2 dx + G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)vu^3 dx \\
&\qquad + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)f_2 v\,dx, \\
&\frac{\mu G}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|W|^2 - \mu G d_3\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)W\Delta W \\
&\qquad + G(\mu F+k)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|W|^2 \\
&\quad = kG\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)Wu\,dx + G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)Wf_3 dx.
\end{aligned} \quad (79)$$

Add up the three equalities. Then we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{G}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|u|^2+\frac{1}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|v|^2\right.\\
&\qquad\left.+\frac{\mu G}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|W|^2\right)\\
&\qquad-\left(d_1G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)u\Delta u+d_2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)v\Delta v\right.\\
&\qquad\left.+\mu G d_3\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)W\Delta W\right)\\
&\qquad+G(F+k)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|u|^2+F\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|v|^2dx\\
&\qquad+G(\mu F+k)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|W|^2\\
&\leq-\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)\left(Gu^2-uv\right)^2dx\\
&\qquad+2kG\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)uW\,dx+G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)f_1u\,dx\\
&\qquad+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)f_2v\,dx+G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)Wf_3dx\\
&\leq Gk\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|u|^2+Gk\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|W|^2\\
&\qquad+\frac{GF}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|u|^2+\frac{F}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|v|^2dx\\
&\qquad+\frac{\mu GF}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|W|^2dx\\
&\qquad+\frac{G}{2F}\int_{|x|\geq K}\left|f_1(x,t)\right|^2dx+\frac{G}{2\mu F}\int_{|x|\geq K}\left|f_3(x,t)\right|^2dx\\
&\qquad+\frac{1}{2F}\int_{|x|\geq K}\left|f_2(x,t)\right|^2dx.
\end{aligned}\tag{80}$$

That is,

$$\begin{aligned}
&\frac{d}{dt}\left(G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|u|^2+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|v|^2\right.\\
&\qquad\left.+\mu G\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|W|^2\right)\\
&\quad+GF\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|u|^2+F\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|v|^2\\
&\quad+G\mu F\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|W|^2\\
&\leq\frac{2c}{K}\left(d_1G\left(\|u\|^2+\|\nabla u\|^2\right)+d_2\left(\|v\|^2+\|\nabla v\|^2\right)\right.\\
&\qquad\left.+d_3G\mu\left(\|W\|^2+\|\nabla W\|^2\right)\right)+\frac{G}{F}\int_{|x|\geq K}\left|f_1(x,t)\right|^2dx\\
&\quad+\frac{G}{\mu F}\int_{|x|\geq K}\left|f_3(x,t)\right|^2dx+\frac{1}{F}\int_{|x|\geq K}\left|f_2(x,t)\right|^2dx.
\end{aligned}\tag{81}$$

Here, we use the integration by parts. We have

$$\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)u\Delta u\,dx\leq\frac{c}{K}\left(\|u\|^2+\|\nabla u\|^2\right).\tag{82}$$

Denote $d^{\circ}=\max\{d_1,d_2,d_3\}$, then we can deduce from (81) that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|g|^2dx+F\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)|g|^2\\
&\quad\leq\frac{2cd^{\circ}}{K}\left(\|g\|^2+\|\nabla g\|^2\right)+C_1\int_{|x|\geq K}\left|f(x,t)\right|^2dx.
\end{aligned}\tag{83}$$

Multiplying (83) by e^{Ft} and then integrating over $(\tau-t,\tau)$ with $t\geq0$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)\left|g\left(x,\tau,\tau-t,g_0(\tau-t)\right)\right|^2dx\\
&\quad\leq e^{-F\tau}e^{F(\tau-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)\left|g_0(x,\tau-t)\right|^2dx\\
&\qquad+\frac{2cd^{\circ}}{K}e^{-F\tau}\int_{\tau-t}^{\tau}e^{F\xi}\left\|g\left(\xi,\tau-t,g_0(\tau-t)\right)\right\|^2\\
&\qquad+\left\|\nabla g\left(\xi,\tau-t,g_0(\tau-t)\right)\right\|^2d\xi\\
&\qquad+C_1e^{-F\tau}\int_{\tau-t}^{\tau}\int_{|x|\geq K}\left|f(x,\xi)\right|^2dx\,d\xi\\
&\quad\leq e^{-F\tau}e^{F(\tau-t)}\int_{\mathbb{R}^n}\left|g_0(x,\tau-t)\right|^2dx\\
&\qquad+C_1e^{-F\tau}\int_{-\infty}^{\tau}\int_{|x|\geq K}\left|f(x,\xi)\right|^2dx\,d\xi\\
&\qquad+\frac{2cd^{\circ}}{K}e^{-F\tau}\int_{\tau-t}^{\tau}e^{F\xi}\left\|g\left(\xi,\tau-t,g_0(\tau-t)\right)\right\|^2\\
&\qquad+\left\|\nabla g\left(\xi,\tau-t,g_0(\tau-t)\right)\right\|^2d\xi.
\end{aligned}\tag{84}$$

Note that for given $\varepsilon>0$, there is $T_1=T_1(\tau,D,\varepsilon)>0$ such that for all $t\geq T_1$,

$$e^{-F\tau}e^{F(\tau-t)}\int_{\mathbb{R}^n}\left|g_0(x,\tau-t)\right|^2dx\leq\varepsilon.\tag{85}$$

By (22), there is $K_1=K_1(\tau,\varepsilon)>0$ such that for all $N\geq K_1$,

$$C_1e^{-F\tau}\int_{-\infty}^{\tau}\int_{|x|\geq K}\left|f(x,\xi)\right|^2dx\,d\xi\leq\varepsilon.\tag{86}$$

For the last term on the right-hand side of (84), it follows from Lemma 8 that there is $T_2 = T_2(\tau, D)$ such that for all $t \geq T_2$,

$$\begin{aligned} &\frac{2cd^{o}}{K}e^{-F\tau}\int_{\tau-t}^{\tau}e^{F\xi}\left\|g\left(\xi,\tau-t,g_0\left(\tau-t\right)\right)\right\|^2 \\ &\quad+\left\|\nabla g\left(\xi,\tau-t,g_0\left(\tau-t\right)\right)\right\|^2 d\xi \\ &\leq \frac{2cd^{o}}{K}Me^{-F\tau}\int_{-\infty}^{\tau}e^{F\xi}\left\|f\left(\xi\right)\right\|^2 d\xi. \end{aligned} \tag{87}$$

Therefore, there is $K_2 = K_2(\tau, \varepsilon) > K_1$ such that for all $K \geq K_2$ and $t \geq T_2$,

$$\begin{aligned} &\frac{2cd^{o}}{K}e^{-F\tau}\int_{\tau-t}^{\tau}e^{F\xi}\left\|g\left(\xi,\tau-t,g_0\left(\tau-t\right)\right)\right\|^2 \\ &\quad+\left\|\nabla g\left(\xi,\tau-t,g_0\left(\tau-t\right)\right)\right\|^2 d\xi \leq \varepsilon. \end{aligned} \tag{88}$$

Let $T = \max\{T_1, T_2\}$ and then by (85)–(88), we find that, for all $K \geq K_2$ and $t \geq T$,

$$\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)\left|g\left(x,\tau,\tau-t,g_0\left(\tau-t\right)\right)\right|^2 dx \leq 3\varepsilon, \tag{89}$$

and hence for all $K \geq K_2$ and $t \geq T$,

$$\begin{aligned} &\int_{|x|\geq\sqrt{2}N}\rho\left(\frac{|x|^2}{K^2}\right)\left|g\left(x,\tau,\tau-t,g_0\left(\tau-t\right)\right)\right|^2 dx \\ &\leq \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{K^2}\right)\left|g\left(x,\tau,\tau-t,g_0\left(\tau-t\right)\right)\right|^2 dx \\ &\leq 3\varepsilon, \end{aligned} \tag{90}$$

which completes the proof. □

5. Existence of Pullback Attractors

In this section, we prove the existence of a $\mathscr{D}_F$-pullback global attractor for the nonautonomous three-component reversible Gray-Scott system on $\mathbb{R}^n$. We first establish the $\mathscr{D}_F$-pullback asymptotic compactness of solutions and prove the existence of a pullback attractor in $\mathbb{H}$. Then we show that this attractor is actually a $\mathscr{D}_F$-pullback attractor in $\mathbb{E}$.

Lemma 16. *Suppose that* (19) *and* (20) *hold. Then ϕ is $\mathscr{D}_F$-pullback asymptotically compact in $\mathbb{H}$. That is, for every $\tau \in \mathbb{R}$, $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_F$, $t_n \to \infty$, and $g_{0,n} \in D(\tau - t_n)$, the sequence $\phi(t_n, \tau - t_n, g_{0,n})$ has a convergent subsequence in $\mathbb{H}$.*

Proof. The proof is a slightly modification of Lemma 5.1 in [20] and thus is omitted here. □

Theorem 17. *Suppose that* (19) *and* (20) *hold. Then problem* (9)-(10) *has a unique $\mathscr{D}_F$-pullback global attractor $\{\mathscr{A}(\tau)\}_{\tau\in\mathbb{R}} \in \mathscr{D}_F$ in $\mathbb{H}$. That is, for every $\tau \in \mathbb{R}$,*

(i) *$\mathscr{A}(\tau)$ is compact in $\mathbb{H}$,*

(ii) *$\{\mathscr{A}(\tau)\}_{\tau\in\mathbb{R}}$ is invariant. That is,*

$$\phi\left(t,\tau,\mathscr{A}\left(\tau\right)\right) = \mathscr{A}\left(t+\tau\right), \quad \forall t \geq 0, \tag{91}$$

(iii) *$\{\mathscr{A}(\tau)\}_{\tau\in\mathbb{R}}$ attracts every set in $\mathscr{D}_F$ with respect to the norm of $\mathbb{H}$. That is, for every $B = \{B(\tau)\}_{\tau\in\mathbb{R}} \in \mathscr{D}_F$,*

$$\lim_{t\to\infty} d_H\left(\phi\left(t,\tau-t,B\left(\tau-t\right),\mathscr{A}\left(\tau\right)\right)\right) = 0, \tag{92}$$

where for any $X, Y \subseteq \mathbb{H}$, $d_{\mathbb{H}}(X, Y) = \sup_{x\in X}\inf_{y\in Y}\|x - y\|_{\mathbb{H}}$.

Proof. For $\tau \in \mathbb{R}$, denote

$$B\left(\tau\right) = \left\{g : \left\|g\right\|^2 \leq Me^{-F\tau}\int_{-\infty}^{\tau}e^{F\xi}\left\|f\left(\xi\right)\right\|^2 d\xi\right\}. \tag{93}$$

Note that $B = \{B(\tau)\}_{\tau\in\mathbb{R}} \in \mathscr{D}_F$ is a $\mathscr{D}_F$-pullback absorbing for ϕ in $\mathbb{H}$ by Lemma 8. In addition, ϕ is $\mathscr{D}_F$-pullback asymptotically compact by Lemma 16. Thus, the existence of a $\mathscr{D}_F$-pullback global attractor for ϕ in $\mathbb{H}$ follows from Theorem 7. □

In what follows, we strengthen Theorem 17 and show that the global attractor $\{\mathscr{A}(\tau)\}_{\tau\in\mathbb{R}}$ is actually a $\mathscr{D}_F$-pullback global attractor in $\mathbb{E}$. As a necessary step towards this goal, we first prove the asymptotic compactness of solutions in $\mathbb{E}$.

Lemma 18. *Suppose that* (19) *and* (20) *hold. Let*

$$\frac{df}{dt} \in \left[L^2_{\text{loc}}\left(\mathbb{R}, L^2\left(\mathbb{R}^n\right)\right)\right]^3. \tag{94}$$

Then ϕ is $\mathscr{D}_F$-pullback asymptotically compact in $\mathbb{E}$. That is, for every $\tau \in \mathbb{R}$, $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_F$, $t_n \to \infty$, and $g_{0,n} \in D(\tau - t_n)$, the sequence $\phi(t_n, \tau - t_n, g_{0,n})$ has a convergent subsequence in $\mathbb{E}$.

Proof. By Lemma 16, the sequence $\phi(t_n, \tau - t_n, g_{0,n}) = g(\tau, \tau - t, g_{0,n})$ has a convergent subsequence in $\mathbb{H}$, and hence there exists $\overline{g} \in \mathbb{H}$ such that, up to a subsequence,

$$g\left(\tau,\tau-t,g_{0,n}\right) \longrightarrow \overline{g} \quad \text{in } \mathbb{H}. \tag{95}$$

This shows that $\phi(t_n, \tau - t_n, g_{0,n})$ is a Cauchy sequence in $\mathbb{H}$. Next, we prove that the sequence is actually a Cauchy sequence in $\mathbb{E}$. For any $n, m \geq 1$, it follows from (9) that

$$\begin{aligned} &g_\tau\left(\tau,\tau-t_n,g_{0,n}\right) - g_\tau\left(\tau,\tau-t_m,g_{0,m}\right) \\ &\quad= A\left(g\left(\tau,\tau-t_n,g_{0,n}\right) - g\left(\tau,\tau-t_m,g_{0,m}\right)\right) \\ &\qquad- H\left(g\left(\tau,\tau-t_n,g_{0,n}\right) - g\left(\tau,\tau-t_m,g_{0,m}\right)\right). \end{aligned} \tag{96}$$

That is,

$$\begin{aligned}\frac{\partial(u_n-u_m)}{\partial\tau}&=d_1\Delta(u_n-u_m)-(F+k)(u_n-u_m)\\&\quad+(u_n^2v_n-u_m^2v_m)-G(u_n^3-u_m^3)\\&\quad+N(w_n-w_m),\\\frac{\partial(v_n-v_m)}{\partial\tau}&=d_2\Delta(v_n-v_m)-F(v_n-v_m)\\&\quad-(u_n^2v_n-u_m^2v_m)+G(u_n^3-u_m^3),\\\frac{\partial(w_n-w_m)}{\partial\tau}&=d_3\Delta(w_n-w_m)+k(u_n-u_m)\\&\quad-(F+N)(w_n-w_m),\end{aligned}\tag{97}$$

where u_n, u_m denote $u(\tau,\tau-t_n,u_{0,n})$, $u(\tau,\tau-t_m,u_{0,m})$ and so do as v_n, v_m, w_n, w_m, and g_n, g_m. Taking the inner products $(\partial(u_n-u_m)/\partial\tau, u_n-u_m)$, $(\partial(v_n-v_m)/\partial\tau, v_n-v_m)$ and $(\partial(w_n-w_m)/\partial\tau, w_n-w_m)$, respectively, and summing up the three equalities, we get

$$\begin{aligned}&d\left(\|\nabla(u_n-u_m)\|^2+\|\nabla(v_n-v_m)\|^2+\|\nabla(w_n-w_m)\|^2\right)\\&\quad+(F+k)\|u_n-u_m\|^2+F\|v_n-v_m\|^2\\&\quad+(N+k)\|w_n-w_m\|^2\\&\le(N+k)(w_n-w_m,u_n-u_m)\\&\quad+(u_n^2v_n-u_m^2v_m,u_n-u_m-v_n+v_m)\\&\quad-G(u_n^3-u_m^3,u_n-u_m-v_n+v_m)\\&\quad-\left\|\frac{\partial}{\partial\tau}(u_n-u_m+v_n-v_m+w_n-w_m)\right\|\\&\quad\times\|u_n-u_m+v_n-v_m+w_n-w_m\|.\end{aligned}\tag{98}$$

Because of the Hölder inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}&(u_n^2v_n-u_m^2v_m,u_n-u_m-v_n+v_m)\\&\quad-G(u_n^3-u_m^3,u_n-u_m-v_n+v_m)\\&\le\frac{d}{2}\|\nabla(g_n-g_m)\|^2\\&\quad+\frac{C^2\varrho^2}{2d}\left(\|g_n\|^2+\|g_n\|_4^4+\|g_n\|_6^6\right)\|g_n-g_m\|^2.\end{aligned}\tag{99}$$

From (98) and (99), it yields

$$\begin{aligned}&d\|\nabla(u_n-u_m+v_n-v_m+w_n-w_m)\|^2\\&\quad+2F\|u_n-u_m+v_n-v_m+w_n-w_m\|^2\\&\le-2\left\|\frac{\partial}{\partial\tau}(u_n-u_m+v_n-v_m+w_n-w_m)\right\|\\&\quad\times\|u_n-u_m+v_n-v_m+w_n-w_m\|\\&\quad+\left(\frac{C^2\varrho^2}{d}\left(\|g_n\|^2+\|g_n\|_4^4+\|g_n\|_6^6\right)+2\delta\right)\\&\quad\times\|u_n-u_m+v_n-v_m+w_n-w_m\|^2.\end{aligned}\tag{100}$$

By Lemma 14 we find that, for every $\tau\in\mathbb{R}$, there exists $T=T(\tau,D)$ such that for all $t\ge T$,

$$\|g_\tau(\tau,\tau-t,g_0(\tau-t))\|\le M.\tag{101}$$

Since $t_n\to\infty$, there exists $\overline{N}=\overline{N}(\tau,D)$ such that $t_n\ge T$ for all $n\ge T$. Thus, we obtain that, for all $n\ge\overline{N}$,

$$\|g_\tau(\tau,\tau-t,g_{0,n})\|\le M,\tag{102}$$

which along with (100) shows that, for all $n,m\ge\overline{N}$,

$$\begin{aligned}&d\|\nabla(u_n-u_m+v_n-v_m+w_n-w_m)\|^2\\&\quad+2F\|u_n-u_m+v_n-v_m+w_n-w_m\|^2\\&\le2M\|u_n-u_m+v_n-v_m+w_n-w_m\|\\&\quad+\left(\frac{C^2\varrho^2}{d}\left(\|g_n\|^2+\|g_n\|_4^4+\|g_n\|_6^6\right)+2\delta\right)\\&\quad\times\|u_n-u_m+v_n-v_m+w_n-w_m\|^2.\end{aligned}\tag{103}$$

That is,

$$\begin{aligned}&\|\nabla(g(\tau,\tau-t,g_{0,n})-g(\tau,\tau-t,g_{0,m}))\|^2\\&\le2M\|g(\tau,\tau-t,g_{0,n})-g(\tau,\tau-t,g_{0,m})\|\\&\quad+\left(\frac{C^2\varrho^2}{d}\left(\|g_n\|^2+\|g_n\|_4^4+\|g_n\|_6^6\right)+2\delta\right)\\&\quad\times\|g(\tau,\tau-t,g_{0,n})-g(\tau,\tau-t,g_{0,m})\|^2.\end{aligned}\tag{104}$$

By Lemma 8 and Propositions 11 and 12, and combining with the fact that $g(\tau,\tau-t,g_{0,n})$ is a Cauchy sequence in $\mathbb{H}$, we complete the proof. □

We are now ready to prove the existence of a global attractor for problem (9)-(10) in $\mathbb{E}$.

Theorem 19. *Suppose that* (19) *and* (20) *hold. Let*

$$\frac{df}{dt}\in\left[L^2_{\mathrm{loc}}\left(\mathbb{R},L^2(\mathbb{R}^n)\right)\right]^3.\tag{105}$$

Then problem (9)-(10) *has a unique $\mathcal{D}_F$-pullback global attractor $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}\in\mathcal{D}_F$ in $\mathbb{E}$. That is, for every $\tau\in\mathbb{R}$,*

(i) $\mathscr{A}(\tau)$ *is compact in* $\mathbb{E}$,

(ii) $\{\mathscr{A}(\tau)\}_{\tau\in\mathbb{R}}$ *is invariant. That is,*

$$\phi(t,\tau,\mathscr{A}(\tau)) = \mathscr{A}(t+\tau), \quad \forall t \geq 0, \tag{106}$$

(iii) $\{\mathscr{A}(\tau)\}_{\tau\in\mathbb{R}}$ *attracts every set in* $\mathscr{D}_F$ *with respect to the norm of* $\mathbb{E}$. *That is, for every* $B = \{B(\tau)\}_{\tau\in\mathbb{R}} \in \mathscr{D}_F$,

$$\lim_{t\to\infty} d_{\mathbb{E}}(\phi(t,\tau-t,B(\tau-t)),\mathscr{A}(\tau)) = 0, \tag{107}$$

where, for any $X, Y \subseteq \mathbb{H}$, $d_{\mathbb{H}}(X,Y) = \sup_{x\in X}\inf_{y\in Y}\|x-y\|_{\mathbb{E}}$.

Proof. The proof is a slightly modification of Theorem 5.4 in [20] and thus is omitted here. □

Remark 20. In this paper, we study the nonautonomous three-component reversible Gray-Scott system, which is probably similar to the system considered in [20]. But, in fact, the asymptotically dissipative condition of the system cannot be satisfied (see, e.g., [4, 8] for more details), which is different from the system considered in [20]. Here, we have to use the method of the rescaling and grouping estimation to deduce the uniform estimates of solutions.

Remark 21. In the original three-component reversible Gray-Scott system (see [4]), the first constant F that appears in the second variable v-section does not depend on the space variable x. Here we have to affiliate the constant F to $f_2(x)$, or else, it will give an obstacle to establish the uniform estimates of solutions when $t \to \infty$ (see also [6, Remark 5.3]).

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References

[1] P. Gray and S. K. Scott, "Autocatalytic reactions in the isothermal, continuous stirred tank reactor: oscillations and instabilities in the system $a + 2b \to 3b, b \to c$," *Chemical Engineering Science*, vol. 39, pp. 1087–1097, 1984.

[2] S. K. Scott and K. Showalter, "Simple and complex reaction-diffusion fronts," in *Chemical Waves and Patterns: Understanding Chemical Reactivity*, R. Kapral and K. Showalter, Eds., vol. 10, pp. 485–516, Springer, New York, NY, USA, 1995.

[3] H. Mahara, N. J. Suematsu, T. Yamaguchi, K. Ohgane, Y. Nishiura, and M. Shimomura, "Three-variable reversible Gray-Scott model," *Journal of Chemical Physics*, vol. 121, no. 18, pp. 8968–8972, 2004.

[4] Y. You, "Dynamics of three-component reversible Gray-Scott model," *Discrete and Continuous Dynamical Systems B*, vol. 14, no. 4, pp. 1671–1688, 2010.

[5] A. Gu, "Random attractors for stochastic three-component reversible Gray-Scott system with multiplicative white noise," *Journal of Applied Mathematics*, vol. 2012, Article ID 810198, 15 pages, 2012.

[6] A. Gu, "Random attractors of stochastic three-component reversible Gray-Scott system on unbounded domains," *Abstract and Applied Analysis*, vol. 2012, Article ID 419472, 22 pages, 2012.

[7] A. Gu, S. Zhou, and Z. Wang, "Uniform attractor of non-autonomous three-component reversible Gray-Scott system," *Applied Mathematics and Computation*, vol. 219, no. 16, pp. 8718–8729, 2013.

[8] Y. You, "Dynamics of two-compartment Gray-Scott equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 5, pp. 1969–1986, 2011.

[9] J. M. Ball, "Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations," *Journal of Nonlinear Science*, vol. 7, no. 5, pp. 475–502, 1997.

[10] T. Caraballo, G. Łukaszewicz, and J. Real, "Pullback attractors for asymptotically compact non-autonomous dynamical systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 3, pp. 484–498, 2006.

[11] K. Lu and B. Wang, "Global attractors for the Klein-Gordon-Schrödinger equation in unbounded domains," *Journal of Differential Equations*, vol. 170, no. 2, pp. 281–316, 2001.

[12] B. Wang, "Attractors for reaction-diffusion equations in unbounded domains," *Physica D*, vol. 128, no. 1, pp. 41–52, 1999.

[13] P. W. Bates, K. Lu, and B. Wang, "Random attractors for stochastic reaction-diffusion equations on unbounded domains," *Journal of Differential Equations*, vol. 246, no. 2, pp. 845–869, 2009.

[14] B. Wang, "Random attractors for the stochastic Benjamin-Bona-Mahony equation on unbounded domains," *Journal of Differential Equations*, vol. 246, no. 6, pp. 2506–2537, 2009.

[15] X. Ding and J. Jiang, "Random attractors for stochastic retarded lattice dynamical systems," *Abstract and Applied Analysis*, vol. 2012, Article ID 409282, 27 pages, 2012.

[16] P. E. Kloeden and M. Rasmussen, *Nonautonomous Dynamical Systems*, vol. 176 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2011.

[17] A. Carvalho, J. Langa, and J. Robinson, *Attractors for infinite-Dimensional Nonautonomous Dynamical Systems*, Springer, New York, NY, USA, 2012.

[18] A. Miranville and S. Zelik, "Attractors for dissipative partial differential equations in bounded and unbounded domains," in *Handbook of Differential Equations: Evolutionary Equations*, vol. 4, pp. 103–200, Elsevier/North-Holland, Amsterdam, The Netherlands, 2008.

[19] P. W. Bates, H. Lisei, and K. Lu, "Attractors for stochastic lattice dynamical systems," *Stochastics and Dynamics*, vol. 6, no. 1, pp. 1–21, 2006.

[20] B. Wang, "Pullback attractors for non-autonomous reaction-diffusion equations on $\mathbb{R}^n$," *Frontiers of Mathematics in China*, vol. 4, no. 3, pp. 563–583, 2009.