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Research Article

A Simplification for Exp-Function Method When the Balanced Nonlinear Term Is a Certain Product

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The Exp-function method plays an important role in searching for analytic solutions of many nonlinear differential equations. In this paper, we prove that the balancing procedure in the method is unnecessary when the balanced nonlinear term is a product of the dependent variable under consideration and its derivatives. And in this case, the ansatz of the method can be simplified to be with less parameters so as to be easy to calculate.

1. Introduction

In 2006, He and Wu firstly proposed the so-called Exp-function method to search for solitary solutions and periodic solutions of nonlinear partial differential equations (PDE) [1]. This method soon drew the attention of many researchers and was successfully applied to many nonlinear problems [2–31]. Among them, it is worth mentioning that Zhu firstly applied this method to difference-differential equations, which shows that the method is also effective in this case [8, 9]. After Zhu, Dai et al. generalized the Exp-function method to find new exact traveling wave solutions of nonlinear PDE and nonlinear differential-difference equations [19]. Recently, Ma et al. extended the Exp-function method to multiple Exp-function method for constructing multiple wave solutions [32, 33]. He elucidated how to solve fractional differential equations with local fractional derivatives via the fractional complex transformation and the Exp-function method [34].

For convenience, we first introduce the Exp-function method in brief.

1.1. Outline of the Exp-Function Method. Suppose that we consider a (1+1)-dimensional nonlinear PDE in the form

$$E\left(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots\right) = 0. \tag{1}$$

Using traveling wave transformation

$$u = u\left(\eta\right), \quad \eta = kx + \omega t, \tag{2}$$

we get a nonlinear ordinary differential equation (ODE)

$$P\left(u, u', u'', \ldots\right) = 0, \tag{3}$$

where the prime, as it is in the following, denotes the derivation with respect to η.

The Exp-function method is based on the assumption that the solutions of (3) can be expressed in the following form:

$$\begin{aligned} u\left(\eta\right) &= \frac{\sum_{n=-c}^{d} a_n \exp\left(n\eta\right)}{\sum_{n=-p}^{q} b_n \exp\left(n\eta\right)} \\ &= \frac{a_{-c}\exp\left(-c\eta\right) + \cdots + a_d \exp\left(d\eta\right)}{b_{-p}\exp\left(-p\eta\right) + \cdots + b_q \exp\left(q\eta\right)}, \end{aligned} \tag{4}$$

where c, d, p, and q are positive integers to be determined and a_n and b_n are constants to be specified.

Then we can express the highest order nonlinear and linear terms in (3) in terms of (4). In the resulting terms, determine d and q through balancing the highest order Exp-function and c and p by balancing the lowest order one.

Substituting (4) along with the determined c, d, p, and q into (3), we can obtain an equation for $\exp(\eta)$. Setting all the coefficients of the different powers of $\exp(\eta)$ to zero leads to a set of algebraic equations for a_n, b_n, k, and ω. Determine values of a_n, b_n, k, and ω by solving this algebraic equations and put these values into (4). Thus we may obtain nontrivial exact traveling wave solutions of (1).

1.2. An Open Problem. Among the Exp-function method, the balancing computation is laborious but prior. However, we observe that the balancing procedure of the Exp-function method in studied examples always leads to the same case $c = p$ and $d = q$ [21]. This fact has partly been proved in [16, 22]. In [16], Ali has proved it by assuming the highest order linear and nonlinear terms as $u^{(n)}$ and $u^r u^{(s)}$ $(s < n)$, respectively. In [22], making use of the same approach as was done by Ebaid proved the fact for nonlinear terms in the form u^γ $(\gamma \geqslant 2)$, $u^{(s)}u^k$ $(s, k \geqslant 1)$, $[u^{(s)}]^\Omega$ $(s \geqslant 1, \Omega \geqslant 2)$, and $[u^{(s)}]^\Omega u^\lambda$ $(s, \Omega, \lambda \geqslant 1)$, respectively, along with linear term $u^{(r)}(r \geqslant 1)$. Ebaid claimed in the abstract and section "Conclusions" of his article that the case $c = p$ and $d = q$ is the only relation that "can be obtained through applying the Exp-function ansatz for all possible cases of nonlinear ODEs."

"However, one cannot construct a general form for the highest order nonlinear term because there are many possibilities other than the ones considered" [21]; Aslan and Marinakis concluded that these authors just took some special cases of the nonlinear term into account and hence the problem still remained open. In this paper, we will construct a special case to show that Ebaid's claim is not true; namely, the case $c = p$ and $d = q$ is not the only relation for some special differential equations and hence the problem is still open.

In what follows, we will discuss the relations of c, d, p, and q in a more effective and concise way.

2. Main Result

2.1. Some Terminology. To begin with, we recall some terminology in [35].

A monomial in a collection of variables $x_1, \ldots, x_n$ is a product

$$x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}, \tag{5}$$

where the α_i are nonnegative integers.

The total degree of (5) is the sum of the exponents: $\alpha_1 + \cdots + \alpha_n$.

A polynomial is said to be homogeneous if all the monomials appearing in it with nonzero coefficients have the same total degree.

For instance, $u^2 u' u''^3$ is a product of u and its derivatives u' and u'', and its total degree is 6. $u^2 + u'^2 + uu''$ is homogeneous.

2.2. Two Introducing Ansatz Function. For convenience, we assume $v(\eta)$ is expressed in the form

$$\begin{aligned} v(\eta) &= \frac{\sum_{n=-c_1}^{d_1} k_n \exp(n\eta)}{\sum_{n=-p_1}^{q_1} l_n \exp(n\eta)} \\ &= \frac{k_{-c_1} \exp(-c_1\eta) + \cdots + k_{d_1} \exp(d_1\eta)}{l_{-p_1} \exp(-p_1\eta) + \cdots + l_{q_1} \exp(q_1\eta)}, \end{aligned} \tag{6}$$

where c_1, d_1, p_1, and q_1 are positive integers to be determined and k_n and l_n are constants to be specified.

The following three formulas will be used in this section:

$$\begin{aligned} &u'(\eta) \\ &= \frac{(p-c)\, a_{-c} b_{-p} \exp(-(c+p)\eta) + \cdots + (d-q)\, a_d b_q \exp((d+q)\eta)}{b_{-p}^2 \exp(-2p\eta) + \cdots + b_q^2 \exp(2q\eta)}, \end{aligned} \tag{7}$$

$$\begin{aligned} &u(\eta) \cdot v(\eta) \\ &= \frac{a_{-c} k_{-c_1} \exp(-(c+c_1)\eta) + \cdots + a_d k_{d_1} \exp((d+d_1)\eta)}{b_{-p} l_{-p_1} \exp(-(p+p_1)\eta) + \cdots + b_q l_{q_1} \exp((q+q_1)\eta)}, \end{aligned} \tag{8}$$

$$\begin{aligned} &\frac{u(\eta)}{v(\eta)} \\ &= \frac{a_{-c} l_{-p_1} \exp(-(c+p_1)\eta) + \cdots + a_d l_{q_1} \exp((d+q_1)\eta)}{b_{-p} k_{-c_1} \exp(-(p+c_1)\eta) + \cdots + b_q k_{d_1} \exp((q+d_1)\eta)}. \end{aligned} \tag{9}$$

Before presenting our definitions, we recall an important fact that the constants a_{-c}, a_d, b_{-p}, and b_q in the ansatz of (4) can be assumed to be nonzero during the process of balancing the linear term of highest order with the highest order nonlinear term of certain ODE, and so do the constants k_{-c_1}, k_d, l_{-p_1}, and l_q in ansatz (6). Hence, in this paper, we assume that all above eight constants are nonzero.

Therefore, for ansatz (4), we can define the ansatz functions $L(\cdot)$ and $R(\cdot)$ as follows:

$$\begin{aligned} L(u) &= L\left(\frac{a_{-c} \exp(-c\eta) + \cdots + a_d \exp(d\eta)}{b_{-p} \exp(-p\eta) + \cdots + b_q \exp(q\eta)}\right) \\ &= -c - (-p) = p - c, \\ R(u) &= R\left(\frac{a_{-c} \exp(-c\eta) + \cdots + a_d \exp(d\eta)}{b_{-p} \exp(-p\eta) + \cdots + b_q \exp(q\eta)}\right) = d - q. \end{aligned} \tag{10}$$

In particular, we define $L(C) = R(C) = 0$ for arbitrary constant C.

For example, we have

$$\begin{aligned} L\left(\frac{\exp(-\eta) + \exp(2\eta)}{\exp(-2\eta) + \exp(2\eta)}\right) &= 1, \\ R\left(\frac{\exp(-\eta) + \exp(2\eta)}{\exp(-2\eta) + \exp(2\eta)}\right) &= 0. \end{aligned} \tag{11}$$

According to the definitions, we can find that in ansatz (4) $c = p$ equals $L(u) = 0$ and $d = q$ equals $R(u) = 0$. Therefore, the open problem for ansatz (4) is equal to whether the relations $L(u) = 0$ and $L(u) = 0$ hold.

2.3. Properties of the Ansatz Functions. Assuming $c \neq p$, $d \neq q$, $c_1 \neq p_1$, and $d_1 \neq q_1$ in ansatz (4) and (6), from (7), we obtain

$$\begin{aligned} L\left(u'\right) &= 2p - (c+p) = p - c = L(u), \\ R\left(u'\right) &= (d+q) - 2q = d - q = R(u). \end{aligned} \tag{12}$$

So we have

$$\begin{aligned} L(u) &= L\left(u'\right) = L\left(u''\right) = \cdots = L\left(u^{(n)}\right), \\ R(u) &= R\left(u'\right) = \cdots = R\left(u^{(n)}\right), \end{aligned} \tag{13}$$

for arbitrary nonnegative integer n.

And from (8) and (9), we have

$$\begin{aligned} L(u \cdot v) &= (p + p_1) - (c + c_1) \\ &= (p - c) + (p_1 - c_1) \\ &= L(u) + L(v), \\ R(u \cdot v) &= (d + d_1) - (q + q_1) \\ &= (d - q) + (d_1 - q_1) \\ &= R(u) + R(v), \\ L\left(\frac{u}{v}\right) &= (p + c_1) - (c + p_1) \\ &= (p - c) - (p_1 - c_1) \\ &= L(u) - L(v), \\ R\left(\frac{u}{v}\right) &= (d + q_1) - (q + d_1) \\ &= (d - q) - (d_1 - q_1) \\ &= R(u) - R(v). \end{aligned} \tag{14}$$

Hence, we have

$$L\left(u^{\kappa}\right) = \kappa \cdot L(u), \qquad R\left(u^{\kappa}\right) = \kappa \cdot R(u), \tag{15}$$

for any integer κ.

2.4. Theorem and Proof. In this section, we assume the balanced nonlinear term is a product of dependent variable u and its derivatives; namely,

$$u^{i_1}\left(u'\right)^{i_2} \cdots \left(u^{(m)}\right)^{i_m}, \tag{16}$$

where i_j $(j = 1, \ldots, m)$ are nonnegative integers. The fact that the product (16) is a nonlinear term implies $i_1 + i_2 + \cdots + i_m \geqslant 2$. In other words, the total degree of (16) is at least 2.

Theorem 1. *Suppose that the balanced nonlinear term in* (3) *is a product of u and its derivatives in the form of* (16) *and the balancing linear term is $u^{(s)}$, where s is a nonnegative integer; then the Exp-function ansatz* (4) *admits $L(u) = 0$ and $R(u) = 0$.*

Proof. By contradiction, suppose that $L(u) \neq 0$. Then we have

$$\begin{aligned} & L\left(u^{i_1}\left(u'\right)^{i_2} \cdots \left(u^{(m)}\right)^{i_m}\right) \\ &\quad = L\left(u^{i_1}\right) + L\left(\left(u'\right)^{i_2}\right) + \cdots + L\left(\left(u^{(m)}\right)^{i_m}\right) \\ &\quad = i_1 \cdot L(u) + i_2 \cdot L\left(u'\right) + \cdots + i_m \cdot L\left(u^{(m)}\right) \\ &\quad = i_1 \cdot L(u) + i_2 \cdot L(u) + \cdots + i_m \cdot L(u) \\ &\quad = \left(i_1 + i_2 + \cdots + i_m\right) \cdot L(u), \\ & L\left(u^{(s)}\right) = L(u). \end{aligned} \tag{17}$$

Balancing linear and nonlinear terms requires

$$L\left(u^{i_1}\left(u'\right)^{i_2} \ldots \left(u^{(m)}\right)^{i_m}\right) = L\left(u^{(s)}\right). \tag{18}$$

So we obtain

$$\left(i_1 + i_2 + \cdots + i_m - 1\right) \cdot L(u) = 0. \tag{19}$$

Since $i_1 + i_2 + \cdots + i_m - 1 \neq 0$, we arrive at the result $L(u) = 0$. This is a contradiction.

The result $R(u) = 0$ can be obtained in a similar way; here we omit the details. □

Remark. Since our forms of the linear and nonlinear terms are in a more general setting, Theorem 1 covers the results presented by Ali and Ebaid, respectively.

2.5. A Simplification for the Exp-Function Method. According to Theorem 1, if the balanced nonlinear term is a product, ansatz (4) can be reduced to the following equivalent form:

$$u = \frac{\alpha_0 + \alpha_1 \exp(\eta) + \cdots + \alpha_\tau \exp(\tau\eta)}{1 + \beta_1 \exp(\eta) + \cdots + \beta_\tau \exp(\tau\eta)}, \tag{20}$$

where τ is a free positive integer with $\tau \geqslant 2$ and α_i and β_i are constants to be specified. Ansatz (20) is concise and easy to calculate and makes the Exp-function method more straightforward.

For example, Naher et al. applied the Exp-function method to constructing the traveling wave solutions of the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation in the form [25]

$$u_t + \alpha u^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0. \tag{21}$$

The traveling wave transformation

$$u(x, y, z, t) = u(\eta), \quad \eta = x + y + z - Vt \tag{22}$$

carries (21) into an ODE

$$-Vu' + \alpha u^2 u' + 3u''' = 0. \tag{23}$$

Since the nonlinear term $u^2 u'$ is a product of u and u', we can immediately assume ansatz (4) in the form

$$u = \frac{\alpha_0 + \alpha_1 \exp(\eta) + \alpha_2 \exp(2\eta)}{1 + \beta_1 \exp(\eta) + \beta_2 \exp(2\eta)}, \tag{24}$$

which is equivalent to the case $c = p = 1$ and $d = q = 1$ and in accord with Naher's (namely, (3.8) in [25]).

3. A Counter-Example

In [22], Ebaid claimed in his abstract that the case $c = p$ and $d = q$ was the only relation that could be obtained by applying this method to any nonlinear ODE. In this section, we construct a counter-example to show that the case $c = p$ and $d = q$ is not the only relation. In fact, the claim does not applied to each nonlinear homogeneous ODE.

For example, we can create the following ODE:

$$u^2 + u'^2 + uu'' = 0, \tag{25}$$

which can be rewritten as

$$u + \frac{u'^2}{u} + u'' = 0. \tag{26}$$

We have

$$\begin{aligned} L\left(\frac{u'^2}{u}\right) &= L(u), \qquad L\left(u''\right) = L(u), \\ R\left(\frac{u'^2}{u}\right) &= R(u), \qquad R\left(u''\right) = R(u). \end{aligned} \tag{27}$$

From (27), the relations $L(u'^2/u) = L(u'')$ and $R(u'^2/u) = R(u'')$ hold automatically. Hence by balancing nonlinear term u'^2/u and linear term u'', we cannot determine the relations of c, d, p, and q. That is to say, all of them are free constants. Hence it is possible that the relation $c \neq p$ and $d \neq q$ holds. In other words, the relation $c = p$ and $d = q$ is not all-inclusive.

4. Conclusion

In summary, we present an entire novel approach to prove that the balancing procedure in Exp-function method is unnecessary when the balanced nonlinear term is a product of the dependent variable under consideration and its derivatives. Our results cover the results presented by Ali and Ebaid. We believe that our work can serve as an answer to the open problem proposed by Aslan et al.

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