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Research Article

Existence and Uniqueness of Limit Cycles in a Class of Planar Differential Systems

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This paper is an extension to the recent results presented by M. Sabatini about the existence and uniqueness of limit cycles of a certain class of planar differential systems in order to include other new classes. A concrete example exhibiting the applicability of the result is introduced.

1. Introduction

We study planar differential systems defined in the real plane,

$$\begin{aligned} \dot{x} &= p(x, y), \\ \dot{y} &= Q(x, y), \end{aligned} \tag{1.1}$$

where, $P, Q \in C^1(R^2, R)$. These differential systems are mathematical models that arise in many fields of application like biology, physics, engineering, and so forth [1]. The study of the dynamics of (1.1) strongly depends on the existence stability properties, number, and location of special solution such as singular points and nonconstant isolated periodic solutions. In particular, if an attracting nonconstant isolated periodic solution exists, then it dominates the dynamics of the system (1.1) in an open connected subset of the plane, its region of attraction, such periodic solution called *limit cycles*. Studying the number and location of limit cycles is, by no means, a question of Hilbert 16th problem, see [2]. In some cases such a region of attraction can extend to cover the whole plane, with the exception of a singular point. In such a case the limit cycle is unique and dominates the system's dynamics. Uniqueness of limit cycles have been extensively studied in many books and articles, see, for example, [3–7] and references therein. Most of the results obtained for dynamical systems in the plane are concerned with the systems equivalent to the classical Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.2}$$

and its generalization such as Lotka-Volterra systems and systems equivalent to Rayleigh equation as special cases, see [1]. This system with some extent of generalization involves Van der Pol system and other systems can be involved as a special case. A recent result on more general class of systems,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x) - y\phi(x, y), \end{aligned} \tag{1.3}$$

which is equivalent to the equation

$$\ddot{x} + \dot{x}\phi(x, \dot{x}) + g(x) = 0, \tag{1.4}$$

and special cases are concerned and studied, see, for instance, [3, 8].

The main tools used in this paper are the recent results introduced by Sabatini in [4, 5]. We make use of one-to-one transformation in order to transform system (1.1) to system equivalent to (1.3) in the meaning that the phase portrait of the original and the transformed system are equivalent. To that end, we impose some conditions on $P(x, y)$,$Q(x, y)$ in system (1.1) under which it becomes equivalent to system (1.3). Then, consequently, we study the uniqueness and then the existences of the limit cycles which attract every nonconstant solution under two cases, when $g(x)$ is assumed to be linear and when it is assumed to be nonlinear. We apply the uniqueness results of [5] for limit cycles of (1.3) to find out conditions of uniqueness and then under suitable additional assumptions we obtain conditions of uniqueness and existence of limit cycles that attract every nonconstant solution for systems (1.1).

For the convenience of the reader, we mention the statements of Sabatini's Theorems that we apply in our results and some used definitions.

Let $\Omega \subseteq R^2$ be a star-shaped set. We say that a function $\phi \in C^1(\Omega, R)$ is star shaped if $(x, y) \cdot \nabla\phi = x\phi_x + y\phi_y$ does not change sign. We say that ϕ is strictly star shaped if $(x, y) \cdot \nabla\phi \neq 0$, except at the origin $(0, 0)$. The following are the statements of the theorems of M. Sabatini [5] that we apply.

Theorem 1.1 (see, [5]). *Let $\phi \in C^1(R^2, R)$ be a strictly star-shaped function. Then* (1.3) *has at most one limit cycle.*

Let us denote by D_r the disk $\{(x, y) : x^2 + y^2 \leq r^2\}$ and by ∂D_r its boundary $\{(x, y) : x^2 + y^2 = r^2\}$.

Theorem 1.2 (see, [5]). *Let $\phi \in C^1(R^2, R)$ be a strictly star-shaped function with $\phi(0, 0) < 0$, $\phi(x, y) \geq 0$ out of some bounded set U, and $\phi(x, y)$ does not vanish identically on any ∂D_r for $r > \sup\{\sqrt{x^2 + y^2}, (x, y) \in U\}$, then the system* (1.3) *has exactly one limit cycle, which attracts every nonconstant solution.*

Theorem 1.3 (see, [5]). *Assume* $g \in C^1(R,R)$, *with* $g'(x) > 0$ *and* $xg(x) > 0$ *for* $x \neq 0$. *Let* $\Phi \in C^1(R^2,R)$, *with* $\Phi(0,0) < 0$, $\Phi(x,y) \geq 0$ *out of some bounded set* U *containing* $(0,0)$, $\Phi(x,y)$ *does not vanish identically on any circumference including* U *and satisfy* $(\sigma(x)\sqrt{2G(x)}/g(x))[2G(x)((\Phi_x(x,y)g(x) - \Phi(x,y)g'(x))/g^2(x)) + \Phi(x,y)] + y\Phi_y(x,y) \neq 0$. *Then the system* (1.3) *has exactly one limit cycle, which attracts every nonconstant solution.*

This paper is organized as follows. In Section 2, we give the main results with their proofs. The main tools applied in this article are the theorems presented by Sabatini in [5] which are mentioned above for completeness. In Section 3, we present a concrete example to illustrate the applicability of the results. As far as we know the approach which has been used in the example to study the limit cycles of the systems is new and not presented before.

2. The Main Results

We study system (1.1) under the assumption that the functions $P(x,y)$ and $Q(x,y)$ satisfy the hypotheses

$H1$: *for any given x, z=P(x,y) is 1-1 correspondence between y and z;*

$H2$: *there exists a function* $\phi(x,y)$ *such that for* $k \in R,\ k > 0$,

$$: \nabla P(x,y) \cdot (P(x,y), Q(x,y)) + P(x,y)\phi(x,P(x,y)) + kx = 0,\ \forall (x,y).$$

Hypothesis $H1$ means that $P(x,y)$ is invertible in y for every x, that is, there is a function $y = R(x,z)$ as the inverse of $P(x,y)$, such that, for every x, $P(x,R(x,z)) = z$ and $R(x,P(x,y)) = y$.

Theorem 2.1. *Suppose that* $P(x,y)$ *in system* (1.1) *satisfies Hypothesis* $H1$, *if there exists a strictly star-shaped function* $\phi \in C^1(R^2,R)$ *such that Hypothesis* $H2$ *is satisfied, then system* (1.1) *has at most one limit cycle.*

Proof. Consider the transformation $z = P(x,y)$. This transformation is, from Hypothesis $H1$, one-to-one correspondence between y and z, for any given x.

System (1.1) will be transformed to the system

$$\begin{aligned} \dot{x} &= z, \\ \dot{z} &= \dot{P}(x, R(x,z)). \end{aligned} \tag{2.1}$$

But

$$\begin{aligned} \dot{P}(x,R(x,z)) &= \dot{P}(x,y) \\ &= \nabla P(x,y) \cdot (P(x,y), Q(x,y)) \\ &= -P(x,y)\phi(x,P(x,y)) - kx \quad \text{from Hypothesis } H2. \\ &= -z\phi(x,z) - kx. \end{aligned} \tag{2.2}$$

Then system (2.1) will be in the form

$$\dot{x} = z,$$
$$\dot{z} = -kx - z\phi(x,z), \quad k \in R, \ k > 0. \tag{2.3}$$

Without loss of generality, possibly performing a time rescaling, we may restrict to the system (2.3), for $k = 1$,

$$\dot{x} = z,$$
$$\dot{z} = -x - z\phi(x,z). \tag{2.4}$$

Because of the invertibility of the transformation, the phase portrait of system (1.1) is equivalent to the phase portrait of system (2.4) and consequently they have the same number of limit cycles.

Applying Theorem 1 [5], we conclude that system (2.4), and consequently system (1.1), has at most one limit cycle. □

Now we focus on the existence of limit cycles for system (1.1). Let U be a bounded set, with $\sigma := \sup\{\mathrm{dist}((x,y),(0,0)), (x,y) \in U\}$.

Theorem 2.2. *Suppose that $P(x,y)$ in System* (1.1) *satisfies Hypothesis H1. If there exists $\phi \in C^1(R^2,R)$ a strictly star-shaped function satisfying Hypothesis H2 with $\phi(0,0) < 0$, $\phi(x,y) \geq 0$ out of some bounded set U containing* $(0,0)$*, and $\phi(x,y)$ does not vanish identically on any ∂D_r for $r > \sigma$, then the system* (1.1) *has exactly one limit cycle, which attracts every nonconstant solution.*

Proof. Making use of the same transformation applied in the proof of Theorem 2.1 and applying Theorem 2 of [5], we conclude that the system (2.4) and, consequently, the system (1.1), has exactly one limit cycle, which attracts every nonconstant solution. □

On the other hand, there exist classes of differential system models of more general type which are not covered by previous results, that one which is equivalent to (1.4) assuming $xg(x) > 0$ for $x \neq 0$. We investigate the uniqueness of limit cycles, and under suitable additional assumptions, its existence which attracts every nonconstant solution. The function $g(x)$ in (1.4) is considered now nonlinear, more general type, so what is given above is a special case of the following result.

Now, we consider System (1.3) which is equivalent to (1.4). Assume that $xg(x) > 0$ for $x \neq 0$, $g \in C^1(R,R)$, $g'(0) \neq 0$.

Let us set

$$G(x) = \int_0^x g(s)\,ds,$$

$$\sigma(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

$$\alpha : \quad R \longrightarrow R \quad \text{such that } \alpha(x) = \sigma(x)\sqrt{2G(x)},$$
$$E(x,y) = G(x) + \frac{y^2}{2}. \tag{2.2}$$

Theorem 2.3. *Suppose that $P(x,y)$ in System* (1.1) *satisfies Hypothesis $H1$, if there is a function $g \in C^1(R,R)$ with $g'(x) > 0$, $xg(x) > 0$ for $x \neq 0$, and a function $\Phi \in C^1(R^2,R)$ satisfying the following conditions:*

(1) $\Phi(0,0) < 0$,

(2) $\Phi(x,y) \geqslant 0$ *out of some bounded set U, containing* $(0,0)$, *and $\Phi(x,y)$ does not vanish identically on* $\{(x,y) : 2E(x,y) = r\}$ *for $r > \sigma$, where,* $\sigma := \sup\{\sqrt{2E(x,y)},\ (x,y) \in U\}$,

(3) $(\sigma(x)\sqrt{2G(x)}/g(x))[2G(x)((\Phi_x(x,y)g(x) - \Phi(x,y)g'(x))/g^2(x)) + \Phi(x,y)] + y\Phi_y(x,y) \neq 0$,

then the system (1.1) *has exactly one limit cycle, which attracts every nonconstant solution.*

Proof. The proof completes by applying Theorem 3 [5]. □

Hypothesis $H1$ implies that the function $P(x,y)$ may be in the form

$$P(x,y) = a_0(x) + y + \sum_{i=1}^{N} a_{2i+1}(x)y^{2i+1}, \tag{2.3}$$

where, $a_0(0) = 0$, and all $a_k(x)$ must be positive for all x.

3. Examples

The following is a concrete example illustrating the applicability of Theorem 2.2.

3.1. Example

Consider the system

$$\begin{aligned}
\dot{x} &= x^2 + y + x^2y^3 \\
\dot{y} &= \Big[-x + y + x^2 - 2xy - 2x^3 - x^2y - y^3 - x^4 - x^3y - 4x^2y^2 - x^5 - 4x^4y \\
&\quad + x^2y^3 - 2xy^4 - x^6 - 4x^3y^3 - x^3y^4 - 3x^2y^5 - x^4y^3 - 2x^5y^3 - 7x^4y^4 \\
&\quad -3x^6y^3 - 2x^3y^6 - x^5y^6 - 3x^4y^7 - 3x^6y^6 - x^6y^9\Big] / \left(1 + 3x^2y^2\right).
\end{aligned} \tag{3.1}$$

The function $P(x, y)$ satisfies Hypothesis $H1$. The second equation of the system can be rewritten in the form

$$\begin{aligned} &P_x(x, y)\dot{x} + P_y(x, y)\dot{y} \\ &\quad = -x - P(x, y)\left(-1 + 3x^2 + 2x^2y + y^2 + x^4 + 2x^2y^3 + 2x^2y^4 + 2x^4y^3 + x^4y^6\right). \end{aligned} \tag{3.2}$$

Choose $\phi(x, y)$ such that $\phi(x, P(x, y))$ equals to the multiplier of $P(x, y)$ in the right-hand side, that is,

$$\phi(x, P(x, y)) = -1 + 3x^2 + 2x^2y + y^2 + x^4 + 2x^2y^3 + 2x^2y^4 + 2x^4y^3 + x^4y^6. \tag{3.3}$$

In order to find an expression of the desired function $\phi(x, y)$, we may assume a form of quadratic polynomial in x, y with undefined coefficients then impose it in the above equation, from which we identify the coefficients. Consequently, an expression of the function ϕ may be in the following form:

$$\phi(x, y) = -1 + x^2 + xy + y^2. \tag{3.4}$$

Clearly, Hypothesis $H2$ is satisfied (with $k = 1$). Moreover,

$$(x, y).\nabla\phi(x, y) = 2\left(x^2 + xy + y^2\right) > 0, \quad \forall(x, y) \neq (0, 0). \tag{3.5}$$

This means that ϕ is, strictly star-shaped function with $\phi(0,0) < 0$, $\phi(x, y) \geq 0$ out of some bounded set U containing $(0, 0)$, and $\phi(x, y)$ does not vanish identically on any circumferences, hence the system has a unique limit cycle.

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