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Research Article

Existence Theory for Integrodifferential Equations and Henstock-Kurzweil Integral in Banach Spaces

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We prove existence theorems for the integrodifferential equation $x'(t) = f(t,x(t),\int_0^t k(t,s,x(s))ds)$, $x(0) = x_0$, $t \in I_a = [0,a]$, $a > 0$, where f, k, x are functions with values in a Banach space E and the integral is taken in the sense of HL. Additionally, the functions f and k satisfy certain boundary conditions expressed in terms of the measure of noncompactness.

1. Introduction

In this paper, we establish some existence principles for integrodifferential operator equations and present existence results for integrodifferential and integral equations.

Let $(E,\|\cdot\|)$ be a Banach space, $(E_1,\|\cdot\|)$ a separable Banach space, f, k, x are functions with values in a Banach space E (or in a separable Banach space E_1).

We prove two existence theorems for the problem

$$\begin{gathered} x'(t) = f\left(t,x(t),\int_0^t k(t,s,x(s))\,ds\right), \\ x(0) = x_0, \quad t \in I_a = [0,a],\ a > 0,\ x_0 \in E, \end{gathered} \tag{1.1}$$

where the integral is taken in the sense of HL [1].

The Henstock-Kurzweil integral encompasses the Newton, Riemann, and Lebesgue integrals [2–5]. A particular feature of this integral is that integrals of highly oscillating functions such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$ on $(0,1]$ and $F(0) = 0$, can be defined. This integral was introduced by Henstock and Kurzweil independently in 1957-1958 and has since proved useful in the study of ordinary differential equations [6–10].

It is well known that Henstock's lemma plays an important role in the theory of the Henstock-Kurzweil integral in the real-valued case. On the other hand, in connection with the Henstock-Kurzweil integral for Banach space-valued functions, Cao pointed out in [1] that Henstock's lemma holds for the case of finite dimension, but it does not always hold for the case of infinite dimension.

In this paper, we will use the definition of HL integral which satisfies Henstock's lemma.

As each Bochner integrable function is HL integrable function, so our results extend corresponding theorems for differential, integral, and integrodifferential equations where the right-hand side of (1.1) was Bochner integrable. We obtain this result even using Carathéodory concept of a solution and not the approximate derivative, which comes from the properties of primitives of Henstock-Kurzweil-Denjoy concept of the integral.

We should mention that an extensive work has been done in the study of the solutions of particular cases of (1.1) (see, e.g., [11–17]).

A Mönch fixed point theorem [18] and the techniques of the theory of the measure of noncompactness are used to prove the existence of solution of problem (1.1). By using some conditions expressed in terms of the measure of noncompactness which the function f satisfies, we define a completely continuous operator F over the Banach space $C([0,a])$, whose fixed points are solutions of (1.1). The fixed point theorem of Mönch is used to prove the existence of a fixed point of the operator F.

Our fundamental tools are the Kuratowski and Hausdorff measures of noncompactess [19].

For any bounded subset A of E we denote by $\alpha(A)$ *the Kuratowski measure of noncompactness of* A, that is, the infimum of all $\varepsilon > 0$, such that there exists a finite covering of A by sets of diameter smaller than ε.

For any bounded subset A of E we denote by $\alpha_1(A)$ *the Hausdorff measure of noncompactness of* A, that is, the infimum of all $\varepsilon > 0$, such that A can be covered by a finite number of balls of radius smaller than ε.

The properties of measure of noncompactness $\gamma = \alpha$ and $\gamma = \alpha_1$ are as follows:

(i) if $A \subset B$, then $\gamma(A) \le \gamma(B)$;

(ii) $\gamma(A) = \gamma(\overline{A})$, where $\overline{A}$ denotes the closure of A;

(iii) $\gamma(A) = 0$ if and only if A is relatively compact;

(iv) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$;

(v) $\gamma(\lambda A) = |\lambda|\gamma(A)$, $(\lambda \in R)$;

(vi) $\gamma(A + B) \le \gamma(A) + \gamma(B)$;

(vii) $\gamma(\operatorname{conv} A) = \gamma(A)$;

(viii) $\alpha_1(A) \le \alpha(A) \le 2\alpha_1(A)$.

Lemma 1.1 [20]. *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Let, for $t \in I_a$, $H(t) = \{h(t) \in E,\ h \in H\}$. Then $\alpha_C(H) = \sup_{t \in I_a} \alpha(H(t)) = \alpha(H(I_a))$, where $\alpha_C(H)$ denotes the measure of noncompactness in $C(I_a, E)$ and the function $t \mapsto \alpha(H(t))$ is continuous.*

We now gather some well-known definitions and results from the literature, which we will use throughout this paper.

Definition 1.2. A function $f : I_a \times E \times E \to E$ is L^1-*Carathéodory* if the following conditions hold:

(i) the map $t \to f(t,x,y)$ is measurable for all $(x,y) \in E^2$;

(ii) the map $(x,y) \to f(t,x,y)$ is continuous for almost all $t \in I_a$.

Definition 1.3. A function $k : I_a \times I_a \times E \to E$ is L^1-*Carathéodory* if the following conditions hold:

(i) the map $(t,s) \to k(t,s,y)$ is measurable for all $y \in E$;

(ii) the map $y \to k(t,s,y)$ is continuous for almost all $(t,s) \in I_a^2$.

Definition 1.4. A nonnegative real-valued function $(t,s,z) \to h(t,s,z)$ defined on $I_a \times I_a \times E$ is *a Kamke function* if h satisfies the Carathéodory conditions, and for each fixed t, s, the function $z \to h(t,s,z)$ is nondecreasing and for each q, $0 < q \leq a$, the function identically equal to zero is the unique continuous solution of the integral equation $z(t) = \int_0^t h(t,s,z(s))ds$ defined on $[0,q)$.

Definition 1.5 [3, 5]. A family $\mathcal{F}$ of functions F is said to be *uniformly absolutely continuous in the restricted sense on* $A \subseteq [a,b]$ *or in short uniformly* $AC_*(A)$ if for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every F in $\mathcal{F}$ and for every finite or infinite sequence of nonoverlapping intervals $\{[a_i,b_i]\}$ with $a_i, b_i \in A$ and satisfying $\sum_i |b_i - a_i| < \eta$, then $\sum_i \omega(F,[a_i,b_i]) < \varepsilon$, where ω denotes the oscillation of F over $[a_i,b_i]$ (i.e., $\omega(F,[a_i,b_i]) = \sup\{|F(r) - F(s)| : r,s \in [a_i,b_i]\}$).

A family $\mathcal{F}$ of functions F is said to *be uniformly generalized absolutely continuous in the restricted sense on* $[a,b]$ *or uniformly* ACG_* if $[a,b]$ is the union of a sequence of closed sets A_i such that on each A_i, the function F is uniformly $AC_*(A_i)$.

In the proof of the main theorem we will apply the following fixed point theorem.

Theorem 1.6 [18]. *Let D be a closed convex subset of E, and let F be a continuous map from D into itself. If for some $x \in D$, the implication*

$$\overline{V} = \operatorname{conv}(\{x\} \cup F(V)) \Longrightarrow V \text{ is relatively compact} \tag{1.2}$$

holds for every countable subset V of D, then F has a fixed point.

2. Henstock-Kurzweil integral in Banach spaces

In this part, we present the Henstock-Kurzweil integral in a Banach space and we give some properties of this integral.

Definition 2.1. Let δ be a positive function defined on the interval $[a,b]$. A tagged interval $(x,[c,d])$ consists of an interval $[c,d] \subset [a,b]$ and a point $x \in [c,d]$. The tagged interval $(x,[c,d])$ is subordinate to δ if $[c,d] \subset [x - \delta(x), x + \delta(x)]$.

The letter P will be used to denote finite collections of nonoverlapping tagged intervals. Let

$$P = \{(s_i, [c_i, d_i]) : 1 \le i \le n\}, \quad n \in N, \tag{2.1}$$

be such a collection in $[a,b]$. Then

(i) the points $\{s_i : 1 \le i \le n\}$ are called the tags of P,

(ii) the intervals $\{[c_i, d_i] : 1 \le i \le n\}$ are called the intervals of P,

(iii) if $(s_i, [c_i, d_i])$ is subordinate to δ for each i, then we write P is sub-δ,

(iv) if $[a,b] = \bigcup_{i=1}^{n} [c_i, d_i]$, then P is called a tagged partition of $[a,b]$,

(v) if P is a tagged partition of $[a,b]$ and if P is sub-δ, then we write P is sub-δ on [a,b],

(vi) if $f : [a,b] \to E$, then $f(P) = \sum_{i=1}^{n} f(s_i)(d_i - c_i)$,

(vii) if F is defined on the subintervals of $[a,b]$, then

$$F(P) = \sum_{i=1}^{n} F([c_i, d_i]) = \sum_{i=1}^{n} (F(d_i) - F(c_i)). \tag{2.2}$$

If $F : [a,b] \to E$, then F can be treated as a function of intervals by defining $F([d, c]) = F(d) - F(c)$. For such a function, $F(P) = F(b) - F(a)$ if P is a tagged partition of $[a,b]$.

Definition 2.2 [1]. A function $f : [a,b] \to E$ is *Henstock-Kurzweil integrable on* $[a,b]$ ($f \in \mathrm{HK}([a,b],E)$) if there exists a vector z in E with the following property: for each $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that $\|f(P) - z\| < \varepsilon$ whenever P is sub-δ on $[a,b]$. The function f is *Henstock-Kurzweil integrable on a measurable set* $A \subset [a,b]$ if $f\chi_A$ is Henstock-Kurzweil integrable on $[a,b]$. The vector z is the Henstock-Kurzweil integral of f.

We note that this definition includes the generalized Riemann integral defined by Gordon in [21].

Definition 2.3 [1]. A function $f : [a,b] \to E$ is *HL integrable* on $[a,b]$ ($f \in \mathrm{HL}([a,b],E)$) if there exists a function $F : [a,b] \to E$, defined on the interval $[a,b]$, satisfying the following property: given $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that if $P = \{(s_i, [c_i, d_i]) : 1 \le i \le n\}$ is a tagged partition of $[a,b]$ sub-δ, then

$$\sum_{i=1}^{n} \|f(s_i)(d_i - c_i) - F([c_i, d_i])\| < \varepsilon. \tag{2.3}$$

We note that by the triangle inequality, $f \in \mathrm{HL}([a,b],E)$ implies $f \in \mathrm{HK}([a,b],E)$. In general, the converse is not rue. For real-valued functions, the two integrals are equivalent.

Definition 2.4. Let $f : I_a \to E$ be Henstock-Kurzweil integrable on $[a,b]$. Then a function $F(t) = \int_a^t f(s)ds$, which is defined on subintervals of $[a,b]$ and the integral is in the sense of Henstock-Kurzweil, is called the *primitive of f.*

For the integral, we have the following theorems.

Theorem 2.5 [1]. *Let $f : [a,b] \to E$. If $f = 0$ almost everywhere on $[a,b]$, then f is HL integrable on $[a,b]$ and $\int_a^b f(t)dt = 0$.*

Theorem 2.6 [1]. *Let $f : [a,b] \to E$ be HL integrable on $[a,b]$ and let $F(x) = \int_a^x f(t)dt$ for each $x \in [a,b]$. Then*

(i) *the function F is continuous on $[a,b]$,*
(ii) *the function F is differentiable almost everywhere on $[a,b]$ and $F' = f$,*
(iii) *f is measurable.*

Theorem 2.7 (see [22, Theorem 5]). *Suppose that $f_n : [a,b] \to E$, $n = 1,2,\ldots$, is a sequence of HL integrable functions satisfying the following conditions:*

(i) *$f_n(x) \to f(x)$ almost everywhere in $[a,b]$, as $n \to \infty$;*
(ii) *the set of primitives of f_n, $\{F_n(x)\}$, where $F_n(x) = \int_a^x f_n(s)ds$, is uniformly ACG_* in n;*
(iii) *the primitives F_n are equicontinuous on $[a,b]$,*

then f is HL integrable on $[a,b]$ and $\int_a^x f_n \to \int_a^x f$ uniformly on $[a,b]$, as $n \to \infty$.

We note that this theorem for Denjoy-Bochner integrals is mentioned in [22] without a proof. It is also true for HL integrals. The proof is similar to that of [5, Theorem 7.6], see also [23, Theorem 4].

Theorem 2.8. *If the function $f : I_a \to E$ is HL integrable, then*

$$\int_I f(t)dt \in |I| \cdot \overline{\operatorname{conv}} f(I), \tag{2.4}$$

where $\overline{\operatorname{conv}} f(I)$ is the closure of the convex of $f(I)$, I is an arbitrary subinterval of I_a, and $|I|$ is the length of I.

The proof is similar to that of [24, Lemma 2.1.3]; see also [25, Theorem 10.4, page 268].

Lemma 2.9 [10]. *Let E_1 be a separable Banach space. Suppose that V is a countable set of HL integrable functions. Let $F = \{\int_0^t x(s)ds,\ x \in V,\ t \in I_a\}$ be an equicontinuous, equibounded, and uniformly ACG_* on I_a. Then $\alpha_1(\int_0^t V(s)ds) \le \int_0^t \alpha_1(V(s))ds$, $t \in I_a$, whenever $\alpha_1(V(s)) \le \varphi(s)$ for $s \in I_a$ a.e. φ is a Lebesgue integrable function and α_1 denotes the Hausdorff measure of noncompactness.*

3. Existence results for integrodifferential equations

We will consider the problem

$$x(t) = x_0 + \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))\,ds\right)dz, \quad t \in I_a,\ x_0 \in E, \tag{3.1}$$

where integrals are taken in the sense of HL.

To obtain the existence result it is necessary to define a notion of a solution.

Definition 3.1. An ACG_* function $x : I_a \to E$ is said to be *a solution of problem* (1.1) if it satisfies the following conditions:

(i) $x(0) = x_0$;

(ii) $x'(t) = f(t,x(t),\int_0^t k(t,s,x(s))ds)$ for a. e. $t \in I_a$.

Definition 3.2. A continuous function $x : I_a \to E$ is said to be *a solution of the problem* (3.1) if it satisfies $x(t) = x_0 + \int_0^t f(z,x(z),\int_0^z k(z,s,x(s))ds)dz$ for every $t \in I_a$.

Let us observe that each solution x of problem (1.1) is equivalent to the solution of problem (3.1).

Let x be a continuous solution of (1.1). By definition, x is ACG_* function and $x(0) = x_0$. Since, for a.e. $t \in I_a$, $x'(t) = f(t,x(t),\int_0^t k(t,s,x(s))ds)$ and the last is HL integrable, so is differentiable a.e. Moreover, $\int_0^t f(z,x(z),\int_0^z k(z,s,x(s))ds)dz = \int_0^t x'(s)ds = x(t) - x_0$. Thus satisfies (3.1).

Now assume that y is ACG_* function and it is clear that $y(0) = x_0$. By the definition of HL integrals there exists an ACG_* function G such that $G(0) = x_0$ and $G'(t) = f(t,y(t),\int_0^t k(t,s,y(s))ds)$ a.e.

Hence

$$\begin{aligned} y(t) &= x_0 + \int_0^t f\left(z, y(z), \int_0^z k(z,s,y(s))\,ds\right)dz \\ &= x_0 + \int_0^t G'(s)ds = x_0 + G(t) - G(0) = G(t). \end{aligned} \tag{3.2}$$

We obtain $y = G$ and then $y'(t) = f(t,y(t),\int_0^t k(t,s,y(s))ds)$.

For $x \in C(I_a,E)$, we define the norm of x by $\|x\|_C = \sup\{\|x(t)\|, t \in I_a\}$.

Now we present an existence theorem for the problem (1.1) in a separable Banach space E_1.

Let $B = \{x \in C(I_a,E) : \|x\|_C \le \|x_0\|_C + p,\ p > 0\}$. Note that this set is closed and convex

Define the operator $F : C(I_a,E) \to C(I_a,E)$ by

$$F(x)(t) = x_0 + \int_0^t f\left(z,x(z), \int_0^z k(z,s,x(s))\,ds\right)dz, \quad t \in I_a,\ x_0 \in E. \tag{3.3}$$

Let

$$\Gamma = \{F(x) \in C(I_a,E) : x \in B\}. \tag{3.4}$$

Theorem 3.3. *Assume that, for each ACG_* function $x : I_a \to E_1$, functions $k(\cdot,s,x(s))$, $f(\cdot,x(\cdot),\int_0^{(\cdot)} k(\cdot,s,x(s))ds)$ are HL integrable, f and k are L^1-Carathéodory functions. Suppose that there exists a constant d such that*

$$\alpha(f(t,A,C)) \le d \cdot \max\{\alpha(A),\alpha(C)\} \quad \textit{for each bounded subset } A,C \subset E_1,\ t \in I_a, \tag{3.5}$$

where α denotes the Kuratowski measure of noncompactness.

Assume that

$$\alpha(k(t,s,X)) \le h(t,s,\alpha(X)) \quad \textit{for each bounded subset } X \subset E_1,\ 0 \le s \le t \le a, \tag{3.6}$$

where h is a Kamke function.

Moreover, let Γ *be equicontinuous, equibounded, and uniformly* ACG_* *on* I_a. *Then there exists at least one solution of problem (1.1) on* I_c *for some* $0 < c \le a$ *and* $d \cdot c < 1$.

Proof. By equicontinuity and equiboundedness of Γ, there exists a number c, $0 < c \le a$, such that

$$\left\| \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))\,ds\right) dz \right\| \le p \quad \text{for fixed } p > 0,\ t \in I_c,\ x \in B. \tag{3.7}$$

By our assumptions the operator F is well defined and maps B into B,

$$\begin{aligned} \|F(x)(t)\| &= \left\| x_0 + \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))\,ds\right) dz \right\| \\ &\le \|x_0\|_C + \left\| \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))\,ds\right) dz \right\| \le \|x_0\|_C + p. \end{aligned} \tag{3.8}$$

Using Theorem 2.6 we deduce that F is continuous.

Suppose that $V \subset B$ satisfies the condition $\overline{V} = \overline{\operatorname{conv}}(\{x\} \cup F(V))$ for some $x \in B$. We will prove that V is relatively compact, thus (1.2) is satisfied. Theorem 1.6 will ensure that F has a fixed point.

Let, for $t \in I_c$, $V(t) = \{v(t) \in E_1 : v \in V\}$. Since V is equicontinuous, so by Lemma 1.1, $t \mapsto v(t) = \alpha(V(t))$ is continuous on I_c.

For fixed $t \in I_c$, we divide the interval $[0,t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m$, $i = 0,1,\dots,m$. We denote $T_i = [t_i, t_{i+1}]$, $i = 0,1,\dots,m-1$. Let us fix $z \in I_c$. Let $\int_0^z K(s)ds = \{\int_0^z x(s) : x \in K\}$ for any $K \subset C(I_c, E_1)$ and let $\widetilde{k}_z$ denotes the mapping defined by $\widetilde{k}_z(x(s)) = k(z,s,x(s))$ for each $x \in B_1$ and $s \in I_c$. Obviously, $\widetilde{k}_z(V(s)) = k(z,s,V(s))$.

Let

$$\begin{aligned} F(V(t)) &= \{F(x)(t) \in C(I_c, E_1) : x \in V, t \in I_c\} \\ &= \left\{ x_0 + \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))\,ds\right) dz : x \in V, t \in I_c \right\}. \end{aligned} \tag{3.9}$$

By Theorem 2.8 and the properties of the HL integral, we have

$$\begin{aligned} F(x)(t) &= x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f\left(z, x(z), \int_0^z k(z,s,x(s))\,ds\right) dz \\ &\in x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\,\overline{\operatorname{conv}} f\left(z, V(T_i), \int_0^z \widetilde{k}_z(V(s))\,ds\right) \quad \text{for each } x \in V. \end{aligned} \tag{3.10}$$

Therefore, $F(V(t)) \subset x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{\operatorname{conv}} f(z, V(T_i), \int_0^z \widetilde{k}_z(V(s))ds)$.

Using (3.5), (3.6), Lemma 2.9, and the properties of the measure of noncompactness α we have

$$\begin{aligned}\alpha(F(V(t))) &\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\alpha\left(f\left(z, V(T_i), \int_0^z \widetilde{k}_z(V(s))ds\right)\right)\\ &\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d\cdot \max\left\{\alpha(V(T_i)), \alpha\left(\int_0^z \widetilde{k}_z(V(s))ds\right)\right\}.\end{aligned} \tag{3.11}$$

Let us observe that

(i) if $\alpha(V(T_i)) > \alpha(\int_0^z \widetilde{k}_z(V(s))ds)$, then

$$\alpha(V) = \alpha(\overline{\text{conv}}(\{x\}\cup F(V))) \leq \alpha(F(V)) < d\cdot c\cdot \alpha(V), \tag{3.12}$$

because $d\cdot c < 1$, so $\alpha(V) < \alpha(V)$, a contradiction;

(ii) if $\alpha(V(T_i)) < \alpha(\int_0^z \widetilde{k}_z(V(s))ds)$, then

$$\begin{aligned}\alpha(V) &< \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d\cdot \alpha\left(\int_0^z \widetilde{k}_z(V(s))ds\right) \leq 2dc\alpha_1\left(\int_0^z \widetilde{k}_z(V(s))ds\right)\\ &\leq 2dc\int_0^z \alpha_1(k(z,s,V(s))ds) \leq 2dc\int_0^z \alpha(k(z,s,V(s))ds) \leq 2dc\int_0^z h(z,s,V(s))ds,\end{aligned} \tag{3.13}$$

since $V = \overline{\text{conv}}(\{x\}\cup F(V))$ so

$$v(t) \leq 2dc\int_0^z h(z,s,v(s))ds. \tag{3.14}$$

Hence applying now a theorem on differential inequalities we get $v(t) = \alpha(V(t)) = 0$. By Arzelà-Ascoli theorem, V is relatively compact. So, by Theorem 1.6, F has a fixed point which is a solution of the problem (1.1). □

For real-valued Banach space E, we have the following theorem.

Theorem 3.4. *Assume that for each ACG_* function $x : I_a \to E$, functions $k(\cdot,s,x(s)), f(\cdot, x(\cdot), \int_0^{(\cdot)} k(\cdot,s,x(s))ds)$ are HL integrable, f and k are L^1-Carathéodory functions. Suppose that there exists a constant d_1 such that*

$$\alpha(f(t,A,C)) \leq d_1\cdot \max\{\alpha(A),\alpha(C)\} \quad \textit{for each bounded subset } A, C\subset E,\ t\in I_a. \tag{3.15}$$

Assume that there exists a continuous function $d_2 : I_a\times I_a \to R_+$ such that

$$\alpha(k(I,I,X)) \leq \sup_{s\in I} d_2(t,s)\alpha(X) \quad \textit{for each bounded subset } X\subset E,\ t,s\in I,\ I\subset I_a, \tag{3.16}$$

and the zero function is the unique continuous solution of the inequality

$$p(t) \le d_1 \cdot c \cdot \sup_{z\in I_c} \int_0^c d_2(z,s)p(s)ds \quad \text{on } I_c. \tag{3.17}$$

Moreover, let Γ *be equicontinuous, equibounded, and uniformly* ACG_* *on* I_a*. Then there exists a solution of problem (1.1) on* I_c *for some* $0 < c \le a$ *and* $d_1 \cdot c < 1$.

Proof. By equicontinuity and equiboundedness of Γ, there exists a number c, $0 < c \le a$, such that

$$\left\| \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))ds\right)dz \right\| \le p \quad \text{for fixed } p > 0,\ x \in B,\ t \in I_c. \tag{3.18}$$

By our assumptions the operator F is well defined and maps B into B,

$$\begin{aligned} \|F(x)(t)\| &= \left\| x_0 + \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))ds\right)dz \right\| \\ &\le \|x_0\|_C + \left\| \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))ds\right)dz \right\| \le \|x_0\|_C + l. \end{aligned} \tag{3.19}$$

Using Theorem 2.6 we deduce that F is continuous.

Suppose that $V \subset B$ satisfies the condition $\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$ for some $x \in B$. We will prove that V is relatively compact, thus (1.2) is satisfied. Theorem 1.6 will ensure that F has a fixed point.

Let, for $t \in I_c$, $V(t) = \{v(t) \in E : v \in V\}$. Since V is equicontinuous, so by Lemma 1.1, $t \mapsto \nu(t) = \alpha(V(t))$ is continuous on I_c.

For fixed $t \in I_c$ we divide the interval $[0,t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$ where $t_i = it/m$, $i = 0,1,\ldots,m$, and for fixed $z \in [0,t]$ we divide the interval $[0,z]$ into m parts: $0 = z_0 < z_1 < \cdots < z_m = z$, where $z_j = jz/m$, $j = 0,1,\ldots,m$.

Let $V([z_j,z_{j+1}]) = \{u(s) : u \in V,\ z_j \le s \le z_{j+1}\}$, $j = 0,1,\ldots,m-1$. By Lemma 1.1 and the continuity of ν there exists $s_j \in I_j = [z_j,z_{j+1}]$ such that

$$\alpha(V([z_j,z_{j+1}])) = \sup\{\alpha(V(s)) : z_j \le s \le z_{j+1}\} := \nu(s_j). \tag{3.20}$$

By Theorem 2.8 and the properties of the HL integral we have

$$\begin{aligned} F(u)(t) &= x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f\left(z, u(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} k(z,s,u(s))ds\right)dz \\ &\in x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{\text{conv}} f\left(z, V(I_i), \sum_{j=0}^{m-1} (z_{j+1} - z_j)\overline{\text{conv}} k(z, I_j, V([z_j,z_{j+1}]))\right). \end{aligned} \tag{3.21}$$

Using (3.15), (3.16), and the properties of measure of noncompactness α, we have

$$\begin{aligned}\alpha(F(V(t))) &\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\alpha\left(f\left(z,V(I_i),\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}k(z,I_j,V([z_j,z_{j+1}]))\right)\right)\\ &\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot\max\left\{\alpha(V(I_i)),\alpha\left(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}k(z,I_j,V([z_j,z_{j+1}]))\right)\right\}.\end{aligned} \tag{3.22}$$

Let us observe that

(iii) if $\alpha(V(I_i)) > \alpha(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}k(z,I_j,V([z_j,z_{j+1}])))$, then

$$\alpha(V)=\alpha(\overline{\text{conv}}(\{x\}\cup G(V)))\leq\alpha(F(V))<d_1\cdot c\cdot\alpha(V), \tag{3.23}$$

because $d_1\cdot c<1$ so $\alpha(V)<\alpha(V)$, a contradiction;

(iv) if $\alpha(V(I_i)) < \alpha(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}k(z,I_j,V([z_j,z_{j+1}])))$, then

$$\begin{aligned}\alpha(V) &< \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot\alpha\left(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}k(z,I_j,V([z_j,z_{j+1}]))\right)\\ &\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\alpha(k(z,I_j,V([z_j,z_{j+1}])))\\ &\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\sup_{s\in I_j}d_2(z,s)\alpha(V([z_j,z_{j+1}]))\\ &= d_1\cdot c\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\cdot d_2(z,p_j)v(s_j)\\ &= d_1\cdot c\cdot\left[\sum_{j=0}^{m-1}(z_{j+1}-z_j)\cdot d_2(z,p_j)v(p_j)\right.\\ &\qquad\left.+\sum_{j=0}^{m-1}(z_{j+1}-z_j)(d_2(z,p_j)(v(s_j)-v(p_j)))\right].\end{aligned} \tag{3.24}$$

By continuity of v we have $v(s_j)-v(p_j)<\varepsilon$ and $\varepsilon\to 0$ if $m\to\infty$, so

$$v(t)=\alpha(V(t))\leq d_1\cdot c\cdot\sup_{z\in I_c}\int_0^c d_2(z,s)v(s)ds. \tag{3.25}$$

By (3.17) we have $v(t)=\alpha(V(t))=0$ for $t\in I_c$.

Using Arzelá-Ascoli theorem, we obtain that V is relatively compact. By Theorem 1.6 the operator F has a fixed point. This means that there exists a solution of the problem (1.1). □

References

[1] S. S. Cao, "The Henstock integral for Banach-valued functions," *Southeast Asian Bulletin of Mathematics*, vol. 16, no. 1, pp. 35–40, 1992.

[2] V. G. Čelidze and A. G. Džvaršeĭšvili, *Theory of the Denjoy Integral and Some of Its Applications*, Tbilis. Gos. Univ., Tbilisi, Russia, 1978.

[3] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, vol. 4 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 1994.

[4] R. Henstock, *The General Theory of Integration*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, NY, USA, 1991.

[5] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, vol. 2 of *Real Analysis*, World Scientific, Teaneck, NJ, USA, 1989.

[6] T. S. Chew, "On Kurzweil generalized ordinary differential equations," *Journal of Differential Equations*, vol. 76, no. 2, pp. 286–293, 1988.

[7] T. S. Chew and F. Flordeliza, "On $x' = f(t,x)$ and Henstock-Kurzweil integrals," *Differential Integral Equations*, vol. 4, no. 4, pp. 861–868, 1991.

[8] I. Kubiaczyk and A. Sikorska-Nowak, "Differential equations in Banach space and Henstock-Kurzweil integrals," *Discussiones Mathematicae. Differential Inclusions*, vol. 19, no. 1-2, pp. 35–43, 1999.

[9] J. Kurzweil, "Generalized ordinary differential equations and continuous dependence on a parameter," *Czechoslovak Mathematical Journal*, vol. 7, no. 82, pp. 418–449, 1957.

[10] A. Sikorska-Nowak, "Retarded functional differential equations in Banach spaces and Henstock-Kurzweil integrals," *Demonstratio Mathematica*, vol. 35, no. 1, pp. 49–60, 2002.

[11] R. P. Agarwal, M. Meehan, and D. O'Regan, "Positive solutions of singular integral equations—a survey," *Dynamic Systems and Applications*, vol. 14, no. 1, pp. 1–37, 2005.

[12] R. P. Agarwal, M. Meehan, and D. O'Regan, *Nonlinear Integral Equations and Inclusions*, Nova Science Publishers, New York, NY, USA, 2001.

[13] R. P. Agarwal and D. O'Regan, "Existence results for singular integral equations of fredholm type," *Applied Mathematics Letters*, vol. 13, no. 2, pp. 27–34, 2000.

[14] S. Krzyśka, "On the existence of continuous solutions of Urysohn and Volterra integral equations in Banach spaces," *Demonstratio Mathematica*, vol. 28, no. 2, pp. 353–360, 1995.

[15] M. Meehan and D. O'Regan, "Existence theory for nonlinear Volterra integrodifferential and integral equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 31, no. 3-4, pp. 317–341, 1998.

[16] D. O'Regan, "Existence results for nonlinear integral equations," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 3, pp. 705–726, 1995.

[17] D. O'Regan and M. Meehan, *Existence Theory for Nonlinear Integral and Integrodifferential Equations*, vol. 445 of *Mathematics and Its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 1998.

[18] H. Mönch, "Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 4, no. 5, pp. 985–999, 1980.

[19] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, vol. 60 of *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1980.

[20] A. Ambrosetti, "Un teorema di esistenza per le equazioni differenziali negli spazi di Banach," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 39, pp. 349–361, 1967.

[21] R. A. Gordon, "Riemann integration in Banach spaces," *The Rocky Mountain Journal of Mathematics*, vol. 21, no. 3, pp. 923–949, 1991.

[22] G. Ye, P. Y. Lee, and C. Wu, "Convergence theorems of the Denjoy-Bochner, Denjoy-Pettis and Denjoy-Dunford integrals," *Southeast Asian Bulletin of Mathematics*, vol. 23, no. 1, pp. 135–143, 1999.

[23] A. P. Solodov, "On conditions for the differentiability almost everywhere of absolutely continuous Banach-valued functions," *Moscow University Mathematics Bulletin*, vol. 54, no. 4, pp. 29–32, 1999.

[24] R. H. Martin Jr., *Nonlinear Operators and Differential Equations in Banach Spaces*, Robert E. Krieger, Melbourne, Fla, USA, 1987.

[25] A. Alexiewicz, *Functional Analysis*, PWN, Warsaw, Poland, 1969.

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