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Area, coarea, and approximation in $W^{1,1}$

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Abstract. Let $\Omega\subset\mathbb{R}^n$ be an arbitrary open set. We characterize the space $W^{1,1}_{\mathrm{loc}}(\Omega)$ using variants of the classical area and coarea formulas. We use these characterizations to obtain a norm approximation and trace theorems for functions in the space $W^{1,1}(\mathbb{R}^n)$.

1. Introduction

Let $\Omega\subset\mathbb{R}^n$ be an open set and let $p\geq 1$. The Sobolev space $W^{1,p}(\Omega)$ consists of all functions $u\in L^p(\Omega)$ whose first order distributional partial derivatives also belong to $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,p;\Omega}=(\|u\|^p_{L^p(\Omega)}+\|Du\|^p_{L^p(\Omega)})^{1/p}, \tag{1}$$

where Du is the distributional gradient of u. When $\Omega=\mathbb{R}^n$ we write $\|\cdot\|_{1,p}$ in place of $\|\cdot\|_{1,p;\mathbb{R}^n}$. The space $W^{1,p}_{\mathrm{loc}}(\Omega)$ consists of all functions u defined on Ω which belong to the space $W^{1,p}(\Omega')$ for every open set Ω' whose closure is a compact subset of Ω. It is not hard to verify that $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ if and only if $u\in W^{1,p}(Q)$ for every open n-cube Q whose closure is contained in Ω. The space $W^{1,p}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ in the norm of $W^{1,p}(\Omega)$. Associated with the space $W^{1,p}(\mathbb{R}^n)$ is the p-capacity γ_p, defined for each set $E\subset\mathbb{R}^n$ as

$$\gamma_p(E)=\inf\{\|u\|^p_{1,p}: u\in W^{1,p}(\mathbb{R}^n) \text{ and } E\subset \operatorname{int}\{u\geq 1\}\}. \tag{2}$$

Here and throughout the paper we abuse notation when we by $\{u\geq 1\}$ mean the set $\{x{:}u(x)\geq 1\}$. It is well known (cf. [6] and [16]) that γ_p is an outer regular outer measure on $\mathbb{R}^n$. Throughout the paper we will write γ in place of γ_1.

In this paper we consider several geometric and analytic properties of functions in the space $W^{1,1}_{\mathrm{loc}}(\Omega)$. The area and coarea formulas for Lipschitz mappings (cf. [7, Theorem 3.2.3] and [7, Theorem 3.2.5]) are fundamental results in geometric measure theory. In Section 3 we consider the area and coarea of functions $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$.

Extensions of the area and coarea formulas to mappings in Sobolev spaces have previously been obtained in [12] and [11]. A basic technical issue in problems of this sort is that such functions u are generally not continuous, and one must use care to formulate the theorem for the so-called precise representative of u. We show that the area and coarea formulas as obtained in [11] may be cast in such a way as to be independent of any particular representative of u, and in fact may be used to characterize the space $W^{1,1}_{\mathrm{loc}}(\Omega)$. Our argument draws ideas from the theory of functions of bounded variation and sets of finite perimeter.

An important property of functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ is that of quasicontinuity: for every $u \in W^{1,p}(\mathbb{R}^n)$ and $\varepsilon > 0$ there exist an open set U and a continuous function v defined on $\mathbb{R}^n$ so that $\gamma_p(U) < \varepsilon$, and v coincides with the precise representative of u off of U. It was proved in [4] and [14] that if $p > 1$, then the approximator v may in fact be selected so that $v \in C(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ and $\|u - v\|_{1,p} < \varepsilon$ in addition to the above stated properties. Thus u may be approximated simultaneously pointwise and in norm by a continuous function v. In Section 4 we give a proof of this result in the case $p = 1$. The argument relies on the results obtained in Section 3, along with a smoothing operator first developed in [5] by Calderón and Zygmund and used in [14].

Finally in Section 5 we characterize the space $W^{1,1}_0(\Omega)$ as a subspace of $W^{1,1}(\Omega)$. Bagby [2] and Havin [10] proved independently that if $u \in W^{1,p}(\mathbb{R}^n)$, $p > 1$, then $u \in W^{1,p}_0(\Omega)$ if and only if u vanishes off Ω in the sense that

$$\lim_{r \to 0^+} \fint_{B(x,r)} |u(y)|\, dy = 0$$

for γ_p-quasievery $x \in \mathbb{R}^n \setminus \Omega$. A variant of this was obtained by the author and Ziemer who proved in [15] that if $u \in W^{1,p}(\Omega)$, then $u \in W^{1,p}_0(\Omega)$ if and only if

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B(x,r) \cap \Omega} |u(y)|\, dy = 0$$

for γ_p-quasievery $x \in \partial\Omega$. This condition may be described by stating that u has *inner trace* 0 at quasievery point $x \in \partial\Omega$. We extend both of these results to the case $p = 1$.

2. Preliminaries

Definition 2.1. Given $E \subset \mathbb{R}^n$ and $s \geq 0$, we denote by $H^s(E)$ the s-dimensional Hausdorff measure of E and by $H^s_\infty(E)$ the *s-dimensional Hausdorff content*, defined as

$$H^s_\infty(E)=\inf\left\{\sum_{k=1}^{\infty}(\operatorname{diam} E_k)^s : E\subset\bigcup_{k=1}^{\infty} E_k\right\}.$$

Observe that $H^s(E)=0$ if and only if $H^s_\infty(E)=0$.

It was proved by Fleming [8] that

$$H^{n-1}(E)=0 \quad \text{if and only if} \quad \gamma(E)=0.$$

In fact, there exist constants C_1 and C_2 depending only on n with the property that

$$C_1 H^{n-1}_\infty(E)\le\gamma(E)\le C_2(H^{n-1}_\infty(E)+H^{n-1}_\infty(E)^{n/(n-1)}) \tag{3}$$

holds for all $E\subset\mathbb{R}^n$. A simple proof of the inequality on the left-hand side of (3) was given in [11]. The inequality on the right-hand side of (3) follows easily from the observation that

$$\gamma(B(x_0,r))\le C_2(r^n+r^{n-1})$$

for all $x_0\in\mathbb{R}^n$ and $r>0$, and a simple covering argument.

Definition 2.2. Let $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and let $x\in\mathbb{R}^n$. For $r>0$ we define

$$\bar u_r(x)=\fint_{B(x,r)} u(y)\,dy.$$

We define the precise representative of u by

$$\bar u(x)=\lim_{r\to 0^+}\bar u_r(x)$$

at all points x where the limit exists.

Any point x where $\bar u(x)$ exists is called a Lebesgue point of u. It is well known that almost every point $x\in\mathbb{R}^n$ is a Lebesgue point of a function $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and if $u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, then in fact H^{n-1}-almost every point $x\in\mathbb{R}^n$ is a Lebesgue point of u. We will use the following somewhat stronger fact, see e.g. [6, proof of Theorem 1, pp. 160–162].

Proposition 2.3. *Suppose that $u\in W^{1,1}(\mathbb{R}^n)$. Then $\bar u(x)$ exists for H^{n-1}-almost every $x\in\mathbb{R}^n$, and for every $\varepsilon>0$ there exists an open set U with $H^{n-1}_\infty(U)<\varepsilon$ such that*

$$\lim_{r\to 0^+}\fint_{B(x,r)}|u(y)-\bar u(x)|\,dy=0$$

uniformly for $x\in\mathbb{R}^n\setminus U$.

Note that H^{n-1}_∞ and γ may be used interchangeably in the conclusion of Proposition 2.3.

Definition 2.4. Let $E\subset\mathbb{R}^n$ be measurable. The *density* of E at a point $x\in\mathbb{R}^n$ is the quantity

$$D(E,x)=\lim_{r\to 0^+}\frac{|B(x,r)\cap E|}{|B(x,r)|} \tag{4}$$

provided that the limit exists. The *measure-theoretic interior* of E is the set of all points x where E has density 1, and the *measure-theoretic exterior* of E is the set of all points x where E has density 0. The *measure-theoretic boundary* of E, defined by

$$\partial_M E=\mathbb{R}^n\cap\{x:D(E,x)\neq 0\}\cap\{x:D(E,x)\neq 1\}, \tag{5}$$

consists of all points which are neither measure-theoretic interior nor measure-theoretic exterior points of E.

In our development we will use the space $\mathrm{BV}(\Omega)$ consisting of all functions $u\in L^1(\Omega)$ whose first order distributional partial derivatives are signed Radon measures on Ω with finite total variation. The distributional gradient of a function $u\in\mathrm{BV}(\Omega)$ is the vector-valued measure $Du=(\mu_1,...,\mu_n)$, with total variation measure $\|Du\|$. The total variation $\|Du\|$ is absolutely continuous with respect to Lebesgue measure if and only if each of the measures μ_i is, in which case the partial derivatives may be represented by L^1 functions. This observation implies the following result.

Proposition 2.5. *Let $u\in\mathrm{BV}(\Omega)$. Then $u\in W^{1,1}(\Omega)$ if and only if $\|Du\|$ is absolutely continuous with respect to Lebesgue measure, in which case*

$$\|Du\|(\Omega)=\int_\Omega |Du|\,dx.$$

Definition 2.6. A Lebesgue measurable set $E\subset\mathbb{R}^n$ is said to have *finite perimeter* in Ω if and only if $\chi_E\in\mathrm{BV}(\Omega)$. The perimeter of E in Ω is defined as the quantity

$$P(E,\Omega)=\|D\chi_E\|(\Omega).$$

The following characterization of $\mathrm{BV}(\Omega)$ in terms of the perimeters of level sets was obtained by Fleming and Rishel [9].

Proposition 2.7. *Let $u\in L^1(\Omega)$. Then $u\in\mathrm{BV}(\Omega)$ if and only if*

$$\int_{\mathbb{R}}^{*} P(\{u>t\},\Omega)\,dt<\infty,$$

in which case $t\mapsto P(\{u>t\},\Omega)$ *is measurable and*

$$\|Du\|(\Omega)=\int_{\mathbb{R}} P(\{u>t\},\Omega)\,dt.$$

Here, and throughout the paper, $\int^*$ is used to denote the upper Lebesgue integral. The Hausdorff measure of the measure theoretic boundary of a set E is closely related to its perimeter. We will require the following background results.

Proposition 2.8. ([6, Theorem 1, p. 222]) *Let* $E\subset\mathbb{R}^n$ *be measurable. If* $H^{n-1}(\partial_M E)<\infty$*, then* E *has finite perimeter.*

Proposition 2.9. ([16, Theorem 5.8.1 and Lemma 5.9.5]) *Let* $E\subset\mathbb{R}^n$ *be measurable. If* $P(E,\Omega)<\infty$*, then*

$$P(E,\Omega)=H^{n-1}(\Omega\cap\partial_M E).$$

3. Area and coarea

Throughout this section we assume that $\Omega\subset\mathbb{R}^n$ is an open set, $n\geq 2$. The following extensions of the classical area and coarea formulas to precise representatives of functions in the space $W^{1,1}_{\mathrm{loc}}(\Omega)$ were proved in [11].

Proposition 3.1. (Area formula) *Suppose that* $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$*. Then*

$$H^n(\{(x,y)\in\mathbb{R}^{n+1}: x\in E \text{ and } \bar{u}(x)=y\})=\int_E \sqrt{1+|Du|^2}\,dx$$

for every Lebesgue measurable set $E\subset\Omega$.

Proposition 3.2. (Coarea formula) *Suppose that* $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$*. Then*

$$\int_{\mathbb{R}} H^{n-1}(E\cap\bar{u}^{-1}(t))\,dt=\int_E |Du|\,dx$$

for every measurable set $E\subset\Omega$.

Next we introduce the idea of upper and lower approximate limits. The notation is adapted from [7, Theorem 4.5.9].

Definition 3.3. Let $u\colon\Omega\to\mathbb{R}$ be Lebesgue measurable.
(1) The *upper approximate limit* of u at a point $x\in\Omega$ is

$$\mu_u(x)=\operatorname{ap}\limsup_{y\to x} u(y)=\inf\{s: D(\{u>s\},x)=0\}.$$

(2) The *lower approximate limit* of u at a point $x\in\Omega$ is

$$\lambda_u(x)=\operatorname{ap}\liminf_{y\to x} u(y)=\sup\{s:D(\{u<s\},x)=0\}.$$

(3) The *extended graph* of u over a set $E\subset\Omega$ is

$$\mathcal{G}_u(E)=\{(x,t)\in\mathbb{R}^{n+1}:x\in E \text{ and } \lambda_u(x)\le t\le\mu_u(x)\}.$$

(4) The *extended level set* of u at level t in a set $E\subset\Omega$ is

$$\{x\in E:\lambda_u(x)\le t\le\mu_u(x)\}.$$

Remark 3.4. If $u=v$ almost everywhere in Ω, then by definition $\lambda_u=\lambda_v$ and $\mu_u=\mu_v$ everywhere in Ω. Moreover, if x is a Lebesgue point of u, then $\lambda_u(x)=\bar{u}(x)=\mu_u(x)$. If $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ then H^{n-1}-almost every $x\in\Omega$ is a Lebesgue point of u, which implies that

$$H^n(\mathcal{G}_u(E))=H^n(\{(x,y)\in\mathbb{R}^n:x\in E \text{ and } \bar{u}(x)=y\})$$

and

$$H^{n-1}(E\cap\bar{u}^{-1}(t))=H^{n-1}(\{x\in E:\lambda_u(x)\le t\le\mu_u(x)\})$$

for any set $E\subset\Omega$.

In light of this remark, Propositions 3.1 and 3.2 may be restated as follows.

Proposition 3.5. *Suppose that $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$. Then*

$$H^n(\mathcal{G}_u(E))=\int_E\sqrt{1+|Du|^2}\,dx$$

for every Lebesgue measurable set $E\subset\Omega$.

Proposition 3.6. *Suppose that $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$. Then*

$$\int_{\mathbb{R}}H^{n-1}(\{x\in E:\lambda_u(x)\le t\le\mu_u(x)\})\,dt=\int_E|Du|\,dx$$

for every measurable set $E\subset\Omega$.

The novelty of Propositions 3.5 and 3.6 is that neither formula depends on any particular representative of u. It turns out that both of these formulas have converse statements which may be used to characterize the Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega)$. The following lemma states a general sufficient criterion for membership in $W^{1,1}(\Omega)$.

Lemma 3.7. *Suppose that* $u \in L^1_{\mathrm{loc}}(\Omega)$ *and that there exists* $h \in L^1_{\mathrm{loc}}(\Omega)$ *such that*

$$\int_{\mathbb{R}}^{*} H^{n-1}(\{x \in E : \lambda_u(x) \le t \le \mu_u(x)\})\, dt \le \int_E h\, dx \tag{6}$$

for every measurable set $E \subset \Omega$*. Then* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$*.*

Proof. It suffices to prove that $u \in W^{1,1}(Q)$ for every open n-cube Q compactly contained in Ω. Fix Q, and define $v = u\chi_Q$. Then

$$\|Du\|(Q) = \|Dv\|(Q)$$

and

$$Q \cap \partial_M\{u > t\} = Q \cap \partial_M\{v > t\}.$$

Since v vanishes outside Q, it follows that

$$\partial_M\{v > t\} \subset [Q \cap \partial_M\{u > t\}] \cup \partial Q, \quad t \ne 0. \tag{7}$$

Let $t \in \mathbb{R}$ and let $x \in \partial_M\{u > t\}$. Then $D(\{u > t\}, x) \ne 0$, hence $D(\{u > s\}, x) = 0$ implies $s > t$. Thus $\mu_u(x) \ge t$. Likewise, $D(\{u > t\}, x) \ne 1$ implies that $D(\{u \le t\}, x) \ne 0$, in which case $D(\{u < s\}, x) = 0$ implies $s \le t$. Thus $\lambda_u(x) \le t$. It follows that

$$Q \cap \partial_M\{u > t\} \subset \{x \in Q : \lambda_u(x) \le t \le \mu_u(x)\}. \tag{8}$$

Now, assumption (6) implies that $H^{n-1}(Q \cap \partial_M\{u > t\}) < \infty$ for almost every $t \in \mathbb{R}$, and therefore (7) implies

$$H^{n-1}(\partial_M\{v > t\}) < \infty, \quad \text{a.e. } t \in \mathbb{R}.$$

For all such t, Proposition 2.8 implies that $P(\{v > t\}, Q) < \infty$, and Proposition 2.9 implies in turn that

$$\|D\chi_{\{v>t\}}\|(Q) = H^{n-1}(Q \cap \partial_M\{v > t\}).$$

It follows from (8) and (6) that

$$\begin{aligned}\int_{\mathbb{R}}^{*} \|D\chi_{\{v>t\}}\|(\Omega)\, dt &= \int_{\mathbb{R}}^{*} H^{n-1}(Q \cap \partial_M\{v > t\})\, dt = \int_{\mathbb{R}}^{*} H^{n-1}(Q \cap \partial_M\{u > t\})\, dt \\ &\le \int_{\mathbb{R}}^{*} H^{n-1}(Q \cap \{\lambda_u \le t \le \mu_u\})\, dt \le \int_Q h\, dx < \infty.\end{aligned}$$

Therefore, Proposition 2.7 implies $v \in \mathrm{BV}(Q)$, and

$$\|Dv\|(Q) \leq \int_Q h\,dx.$$

Since u and v coincide on Q it follows that $u \in \mathrm{BV}(Q)$ and

$$\|Du\|(Q) \leq \int_Q h\,dx.$$

This argument may be repeated with any n-cube $Q' \subset Q$, in which case a simple covering argument yields

$$\|Du\|(E) \leq \int_E h\,dx$$

for every Lebesgue measurable set $E \subset Q$. In particular

$$E \longmapsto \|Du\|(E)$$

satisfies Luzin's condition (N). Proposition 2.5 implies that $u \in W^{1,1}(Q)$. $\square$

Theorem 3.8. *Suppose that $u \in L^1_{\mathrm{loc}}(\Omega)$ and that there exists $g \in L^1_{\mathrm{loc}}(\Omega)$ with the property that*

$$\int_{\mathbb{R}} H^{n-1}(\{x \in E : \lambda_u(x) \leq t \leq \mu_u(x)\})\,dt = \int_E g\,dx$$

for every measurable set $E \subset \Omega$. Then $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $|Du| = g$ almost everywhere.

Proof. Appealing to Lemma 3.7 we have $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Proposition 3.6 then implies that

$$\int_E g\,dx = \int_E |Du|\,dx$$

for every Lebesgue measurable set $E \subset \Omega$. Thus $|Du| = g$ almost everywhere. $\square$

Lemma 3.9. *Suppose that $u \in L^1_{\mathrm{loc}}(\Omega)$ and that there exists $h \in L^1_{\mathrm{loc}}(\Omega)$ with the property that*

$$H^n(\mathcal{G}_u(E)) \leq \int_E h\,dx$$

for every measurable set $E \subset \Omega$. Then $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$.

Proof. Let $E\subset\Omega$. Define the projection $p\colon \Omega\times\mathbb{R}\to\mathbb{R}$ by $p(x,t)=t$, so that $\mathrm{Lip}(p)=1$ and

$$\mathcal{G}_u(E)\cap p^{-1}(t)=\{x\in\Omega:\lambda_u(x)\le t\le\mu_u(x)\}\times\{t\}$$

for all $t\in\mathbb{R}$. The Eilenberg inequality (cf. [13, Theorem 7.7]) asserts the existence of a constant C depending only on n with the property that

$$\int_{\mathbb{R}}^{*} H^{n-1}(\mathcal{G}_u(E)\cap p^{-1}(t))\,dt\le CH^n(\mathcal{G}_u(E)).$$

Next let $\pi\colon \mathbb{R}^{n+1}\to\mathbb{R}^n$ denote the projection $\pi(x,t)=x$ so that

$$\pi(\mathcal{G}_u(E)\cap p^{-1}(t))=\{x\in E:\lambda_u(x)\le t\le\mu_u(x)\}.$$

Since Hausdorff measure is non-increasing on projection it follows that

$$H^{n-1}(\{x\in E:\lambda_u(x)\le t\le\mu_u(x)\})\le H^{n-1}(\mathcal{G}_u(E)\cap p^{-1}(t))$$

for all $t\in\mathbb{R}$. Therefore

$$\begin{aligned}\int_{\mathbb{R}}^{*} H^{n-1}(\{x\in E:\lambda_u(x)\le t\le\mu_u(x)\})\,dt&\le\int_{\mathbb{R}}^{*} H^{n-1}(\mathcal{G}_u(E)\cap p^{-1}(t))\,dt\\&\le CH^n(\mathcal{G}_u(E))\le C\int_E h\,dx\end{aligned}$$

for any measurable set $E\subset\Omega$. Finally apply Lemma 3.7 to conclude that $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$. □

Theorem 3.10. *Suppose that $u\in L^1_{\mathrm{loc}}(\Omega)$ and that there exists $g\in L^1_{\mathrm{loc}}(\Omega)$ with the property that*

$$H^n(\mathcal{G}_u(E))=\int_E\sqrt{1+g^2}\,dx$$

for every measurable set $E\subset\Omega$. Then $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $|Du|=|g|$ almost everywhere.

Proof. Lemma 3.9 implies that $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$. It follows from Proposition 3.5 that

$$\int_E\sqrt{1+|Du|^2}\,dx=\int_E\sqrt{1+g^2}\,dx$$

for every measurable set $E\subset\Omega$. Therefore $|Du|=|g|$ almost everywhere. □

Denote the zero extension of a function $u\colon \Omega\to\mathbb{R}$ by

$$u^*(x)=\begin{cases} u(x), & x\in\Omega,\\ 0, & x\notin\Omega.\end{cases} \tag{9}$$

The characterizations obtained above may be used to prove a simple sufficient condition for the zero extension of a function $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ to belong to the space $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 3.11. *Let $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$. If $\overline{u^*}(x)=0$ for H^{n-1}-almost every $x\in\partial\Omega$, then $u^*\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $Du^*=(Du)^*$ almost everywhere.*

Proof. Let $E\subset\mathbb{R}^n$ be a measurable set. In light of Theorem 3.8 it will suffice to show that

$$\int_{\mathbb{R}} H^{n-1}(\{x\in E:\lambda_{u^*}(x)\le t\le\mu_{u^*}(x)\})\,dt=\int_E |(Du)^*|\,dx. \tag{10}$$

By (9) we have $\lambda_{u^*}=\lambda_u$ and $\mu_{u^*}=\mu_u$ in Ω, and by assumption

$$\lambda_{u^*}(x)=\mu_{u^*}(x)=\overline{u^*}(x)=0$$

for H^{n-1}-almost all $x\in\mathbb{R}^n\setminus\Omega$. For any $t\neq0$ it follows that

$$H^{n-1}(E\cap\{\lambda_{u^*}\le t\le\mu_{u^*}\})=H^{n-1}(E\cap\Omega\cap\{\lambda_u\le t\le\mu_u\}),$$

and therefore Proposition 3.6 implies

$$\begin{aligned}\int_{\mathbb{R}} H^{n-1}(E\cap\{\lambda_{u^*}\le t\le\mu_{u^*}\})\,dt&=\int_{\mathbb{R}} H^{n-1}(E\cap\Omega\cap\{\lambda_u\le t\le\mu_u\})\,dt\\ &=\int_{E\cap\Omega}|Du|\,dx=\int_E |Du|^*\,dx.\end{aligned}$$

Since $|Du|^*=|(Du)^*|$ we obtain (10), completing the proof. □

4. An approximation theorem

In this section we will prove the following theorem.

Theorem 4.1. *Let $u\in W^{1,1}(\mathbb{R}^n)$ and let $\varepsilon>0$. Then there exists an open set $U\subset\mathbb{R}^n$ with $\gamma(U)<\varepsilon$ and a function $v\in W^{1,1}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ with the property that $\|u-v\|_{1,1}<\varepsilon$ and $\bar u(x)=v(x)$ for all $x\in\mathbb{R}^n\setminus U$.*

This theorem extends the classical notion of quasicontinuity in the space $W^{1,1}(\mathbb{R}^n)$. The approximator v is constructed using a smoothing procedure developed by Calderón and Zygmund. The following was proved in [5].

Proposition 4.2. *Let $U \subset \mathbb{R}^n$ be an open set with $|U|<1$. Then there exist a function $\delta \in C^\infty(U)$ and positive constants C_1 and C_2 depending only on n (and in particular independent of U) with the property that*

$$C_1 \operatorname{dist}(x, \partial U) \le \delta(x) \le \operatorname{dist}(x, \partial U)$$

and

$$\sup_{x \in U} |D\delta(x)| \le C_2.$$

For the remainder of this section we will denote by C_n a generic constant whose value may change from line to line, but whose value in any specific instance depends only on n.

Proposition 4.3. *Suppose that $u \in L^1(U)$, $w: U \to \mathbb{R}$ is measurable, and that*

$$|w(x)| \le \fint_{B(x,\delta(x)/2)} |u(z)|\, dz.$$

Then $w \in L^1(U)$ and $\|w\|_1 \le C_n \|u\|_1$.

Proof. Integrate the stated inequality over U and apply Fubini's theorem to obtain

$$\begin{aligned} \int_U |w(x)|\, dx &\le C_n \int_U \int_U \delta(x)^{-n} \chi_{B(x,\delta(x)/2)}(z) |u(z)|\, dz\, dx \\ &= C_n \int_U |u(z)| \int_U \delta(x)^{-n} \chi_{B(x,\delta(x)/2)}(z)\, dx\, dz. \end{aligned} \tag{11}$$

Given $x, z \in U$, we have $z \in B\big(x, \frac{1}{2}\delta(x)\big)$ if and only if $x \in B\big(z, \frac{1}{2}\delta(x)\big)$, in which case $\operatorname{dist}(z, \partial U) \ge \frac{1}{2}\delta(x)$ and

$$\operatorname{dist}(z, \partial U) \le |z-x| + \operatorname{dist}(x, \partial U) \le C_n \delta(x).$$

It follows that

$$\delta(x)^{-n} \chi_{B(x,\delta(x)/2)}(z) \le C_n \delta(z)^{-n} \chi_{B(z,C_n\delta(z))}(x),$$

and therefore

$$\int_U \delta(x)^{-n} \chi_{B(x,\delta(x)/2)}(z)\, dx \le C_n$$

for every $z \in U$. With reference to (11) we conclude that

$$\int_U |w(x)|\, dx \le C_n \int_U |u(x)|\, dx. \quad \square$$

Next we define a smoothing operator on $L^1_{\mathrm{loc}}(U)$ which is bounded in the Sobolev norm. The argument presented here is adapted from that given in [14]. Let $\varphi\in C_0^\infty(B(0,1))$ have the property that $P=P*\varphi_\varepsilon$ for every $\varepsilon>0$ and every degree one polynomial P, where $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$. For $x\in U$ and $z\in\mathbb{R}^n$ define

$$\psi_z(x)=\varphi_{\delta(x)/2}(x-z). \tag{12}$$

Since δ and φ are smooth it is clear that $\psi_z\in C^\infty(U)$ for all $z\in\mathbb{R}^n$. Moreover it can be shown that $|D\psi_z(x)|\le C_n\delta(x)^{-n-1}$ for all $x\in U$. Given $u\in L^1_{\mathrm{loc}}(U)$ we define the smoothing Su of u by

$$Su(x)=\int_{\mathbb{R}^n}\psi_z(x)u(z)\,dz. \tag{13}$$

It follows from the construction that $Su\in C^\infty(U)$. We will show that S is bounded on $W^{1,1}(U)$. The proof will use the following result of Bojarski and Hajłasz [3].

Proposition 4.4. *Let $B\subset\mathbb{R}^n$ be an open ball and let $u\in W^{1,1}(B)$. Let*

$$T_Bu(y)=⨍_B u(z)+Du(z)\cdot(y-z)\,dz. \tag{14}$$

Then

$$|u(y)-T_Bu(y)|\le C_n\int_B\frac{|a-Du(z)|}{|y-z|^{n-1}}\,dz \tag{15}$$

for almost all $y\in B$, and for any vector $a\in\mathbb{R}^n$.

Lemma 4.5. *Let $u\in W^{1,1}(U)$. Then $Su\in W^{1,1}(U)$ and $\|Su\|_{1,1;U}\le C_n\|u\|_{1,1;U}$.*

Proof. Let $x\in U$. By (13) we have

$$|Su(x)|\le ⨍_{B(x,\delta(x)/2)}|u(z)|\,dz,$$

so Proposition 4.3 implies that $Su\in L^1(U)$ and $\|Su\|_1\le C_n\|u\|_1$. On the other hand, if P is a polynomial with degree one then

$$Su(y)=P(y)+\int_{\mathbb{R}^n}\psi_z(y)(u(z)-P(z))\,dz$$

for all $y\in U$ because φ_ε commutes with P. This implies

$$DSu(x)=DP(x)+\int_{\mathbb{R}^n}D\psi_z(x)(u(z)-P(z))\,dz, \tag{16}$$

and therefore

$$|DSu(x)| \le |DP(x)| + \frac{C_n}{\delta(x)} \fint_{B(x,\delta(x)/2)} |u(z)-P(z)|\,dz. \tag{17}$$

Let $B=B\big(x,\frac{1}{2}\delta(x)\big)$ and define $P(y)=T_B u(y)$, so that

$$|DP(x)| \le \fint_B |Du(z)|\,dz. \tag{18}$$

On the other hand, Proposition 4.4 with $a=0$ implies

$$|u(z)-P(z)| \le C_n \int_B \frac{|Du(w)|}{|w-z|^{n-1}}\,dw$$

for almost every $z\in B$, and Fubini's theorem implies in turn that

$$\begin{aligned}\int_B |u(z)-P(z)|\,dz &\le C_n \int_B \int_B \frac{|Du(w)|}{|w-z|^{n-1}}\,dw\,dz \\ &= C_n \int_B |Du(w)| \int_B \frac{1}{|w-z|^{n-1}}\,dz\,dw.\end{aligned}$$

Now, if $w,z\in B\big(x,\frac{1}{2}\delta(x)\big)$ then $z\in B(w,\delta(x))$, and thus

$$\int_B \frac{1}{|w-z|^{n-1}}\,dz \le \int_{B(w,\delta(x))} \frac{1}{|w-z|^{n-1}}\,dz = C_n\delta(x).$$

It follows that

$$\int_B |u(z)-P(z)|\,dz \le C_n\delta(x) \int_B |Du(w)|\,dw. \tag{19}$$

Finally, we combine (17), (18), and (19) to conclude

$$|DSu(x)| \le C_n \fint_{B(x,\delta(x)/2)} |Du(w)|\,dw.$$

As above, Proposition 4.3 implies that $DSu\in L^1(U)$ and that $\|DSu\|_1 \le C_n\|Du\|_1$. Thus $Su\in W^{1,1}(U)$, and

$$\|Su\|_{1,1;U} \le C_n\|u\|_{1,1;U}. \quad \square$$

Proof of Theorem 4.1. After these preliminaries we are prepared to prove Theorem 4.1. We divide the proof into several steps. Let $u\in W^{1,1}(\mathbb{R}^n)$ and let $\varepsilon>0$ be given. Fix $\delta>0$.

Step 1. Definition of U and v. Let $K\subset\mathbb{R}^n$ be a closed set with $\gamma(\mathbb{R}^n\setminus K)<\delta$ such that $\bar{u}(x)$ exists for all $x\in K$ and

$$\fint_{B(x,r)} |u(y)-\bar{u}(x)|\,dy\to 0,\quad \text{as } r\to 0^+, \tag{20}$$

uniformly for $x\in K$, cf. Proposition 2.3 above. Define $U=\mathbb{R}^n\setminus K$. We may assume with no loss of generality that $|U|<1$. Let Su denote the smoothing of u in U, and define v by

$$v(x)=\begin{cases} Su(x), & x\in U,\\ \bar{u}(x), & x\in K.\end{cases}$$

Clearly $v=\bar{u}$ is continuous on K, $v\in W^{1,1}(U)$, and by Lemma 4.5,

$$\|v\|_{1,1;U}\le C_n\|u\|_{1,1;U}.$$

Step 2. The function v is continuous. Since $v|_U$ and $v|_K$ are continuous and U is open, it suffices to show that

$$\lim_{\substack{x\to y\\ x\in U}} v(x)=v(y) \tag{21}$$

at each point $y\in K$. Let $y\in K$ and let $x\in U$. Let $x'\in K$ satisfy $|x-x'|=\operatorname{dist}(x,\partial U)$. Then $|x-x'|\le|x-y|$, hence

$$|y-x'|\le|x-x'|+|x-y|\le 2|x-y|.$$

Since $\delta(x)\le|x-x'|$ and

$$v(x)-v(x')=\int_{\mathbb{R}^n}\psi_z(x)(u(z)-\bar{u}(x'))\,dz,$$

we have

$$|v(x)-v(x')|\le\fint_{B(x,\delta(x)/2)}|u(z)-\bar{u}(x')|\,dz\le\fint_{B(x',3|x-x'|/2)}|u(z)-\bar{u}(x')|\,dz.$$

It follows that

$$\begin{aligned}|v(x)-v(y)|&\le|v(x')-v(y)|+|v(x)-v(x')|\\&\le|v(x')-v(y)|+\fint_{B(x',3|x-x'|/2)}|u(z)-\bar{u}(x')|\,dz.\end{aligned}$$

Since $|x'-y|\le 2|x-y|$ and $|x-x'|\le|x-y|$, the continuity of $v|_K$ and the uniformity of the limit (20) imply that

$$|v(x')-v(y)|+\fint_{B(x',|x-x'|)}|u(z)-\bar{u}(x')|\,dz\to 0,$$

as $|x-y|\to 0^+$. This establishes (21), and proves the continuity of v at y.

Step 3. We must show that the piecewise definition of v implies $v\in W^{1,1}(\mathbb{R}^n)$. By construction $v-u\in W^{1,1}(U)$. Let $x\in K$. Then

$$\fint_{B(x,r)} |v(y)-u(y)|\,dy \le \fint_{B(x,r)} |v(y)-v(x)|\,dy + \fint_{B(x,r)} |u(y)-\bar{u}(x)|\,dy,$$

hence

$$\lim_{r\to 0^+} \fint_{B(x,r)} |v(y)-u(y)|\,dy = 0$$

by (20) and the continuity of v. Since $v-u=0$ a.e. on K, we have $(v-u)^*=(v-u)$, where * denotes the zero extension off U as in (9) above. It follows that

$$\overline{(v-u)^*}(x) = \overline{(v-u)}(x) = 0$$

for all $x\in K$. Theorem 3.11 implies $(v-u)^*\in W^{1,1}(\mathbb{R}^n)$, hence

$$v = (v-u)^* + u \in W^{1,1}(\mathbb{R}^n).$$

Step 4. Norm approximation. Observe that

$$\|u-v\|_{1,1} = \|u-v\|_{1,1;U} \le \|u\|_{1,1;U} + \|v\|_{1,1;U} \le C\|u\|_{1,1;U}.$$

Finally $\delta>0$ must be specified. Simply select δ so that $\delta<\varepsilon$ and $\gamma(U)<\delta$ implies $C\|u\|_{1,1;U}<\varepsilon$. This concludes the proof of the theorem. $\square$

A consequence of Theorem 4.1 is a fairly straightforward proof of the following result.

Theorem 4.6. *Let $u\in W^{1,1}(\mathbb{R}^n)$ and suppose that $\{\varphi_j\}_{j=1}^\infty$ is a sequence of continuous functions in $W^{1,1}(\mathbb{R}^n)$ which converges to u in the $W^{1,1}$ norm. Then there exists a subsequence φ_{j_k} with the property that*

$$H^{n-1}(\{x : \varphi_{j_k}(x) \not\to \bar{u}(x)\}) = 0.$$

Proof. Let $j,k\ge 1$ and define

$$E_{j,k} = \left\{ x : |\varphi_j(x) - \bar{u}(x)| \ge \frac{1}{k} \right\}.$$

Let $\delta>0$. Choose an open set U and a function $v\in W^{1,1}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ with the property that $v(x)=\bar{u}(x)$ for all $x\in\mathbb{R}^n\setminus U$, $\gamma(U)<\delta$, and $\|\bar{u}-v\|_{1,1}<\delta$. If $x\in E_{j,k}\setminus U$, then $|\varphi_j(x)-v(x)|=|\varphi_j(x)-\bar{u}(x)|\ge 1/k$, so that

$$(k+\delta)|\varphi_j(x) - v(x)| > 1.$$

Since $|\varphi_j - v|$ is continuous, this implies that $|\varphi_j - v| \geq 1$ on a neighborhood of $E_{j,k} \setminus U$. Thus

$$\gamma(E_{j,k} \setminus U) \leq (k+\delta) \|\varphi_j - v\|_{1,1}.$$

It follows that

$$\begin{aligned}\gamma(E_{j,k}) &\leq \gamma(E_{j,k} \setminus U) + \gamma(U) \leq (k+\delta)\|\varphi_j - v\|_{1,1} + \delta \\ &\leq (k+\delta)\|\varphi_j - \bar{u}\|_{1,1} + (k+\delta)\|\bar{u} - v\|_{1,1} + \delta \leq (k+\delta)\|\varphi_j - \bar{u}\|_{1,1} + (k+\delta)\delta + \delta.\end{aligned}$$

Now pass to the limit as $\delta \to 0$ to conclude that $\gamma(E_{j,k}) \leq k\|\varphi_j - \bar{u}\|_{1,1}$. Choose j_k so that

$$\gamma(E_{j_k,k}) \leq \frac{1}{2^k}.$$

Let $F^1 = \bigcup_{k=1}^{\infty} E_{j_k,k}$. Then $\gamma(F^1) \leq 1$ and $x \notin F^1$ implies that $\varphi_{j_k,k}(x) \to \bar{u}(x)$. Label this subsequence by $\{\varphi_j^1\}_{j=1}^{\infty}$. Now apply a diagonalization procedure. Inductively, having obtained a set F^m with $\gamma(F^m) < 1/m$ and a sequence φ_j^m with the property that $\varphi_j^m \to \bar{u}$ off F^m, repeat the argument above to find a set F^{m+1} with $\gamma(F^{m+1}) < 1/(m+1)$ and a subsequence $\{\varphi_j^{m+1}\}_{j=1}^{\infty}$ of $\{\varphi_j^m\}_{j=1}^{\infty}$ with the property that $\varphi_j^{m+1} \to \bar{u}$ off F^{m+1}. The sequence $\{\varphi_j^j\}_{j=1}^{\infty}$ is the desired subsequence, converging to $\bar{u}$ off a set F with $\gamma(F) = 0$. $\square$

Corollary 4.7. *Suppose that $u \in W_0^{1,1}(\Omega)$. Then $\bar{u}(x) = 0$ for H^{n-1}-almost every $x \in \mathbb{R}^n \setminus \Omega$.*

Proof. By definition there exists a sequence $\{\varphi_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\Omega)$ with the property that $\varphi_j \to u$ in $W^{1,1}(\mathbb{R}^n)$. If $x \in \mathbb{R}^n \setminus \Omega$, then $\varphi_j(x) = 0$ for all x. By the preceding theorem, this implies that $\bar{u}(x) = 0$ for H^{n-1}-almost all $x \in \mathbb{R}^n \setminus \Omega$. $\square$

5. Trace theorems

The proof of the following theorem closely follows the argument given in Section 9.2 of [1].

Theorem 5.1. *Let $u \in W^{1,1}(\mathbb{R}^n)$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Then $u \in W_0^{1,1}(\Omega)$ if and only if $\bar{u}(x) = 0$ for H^{n-1}-almost all $x \in \mathbb{R}^n \setminus \Omega$.*

Proof. Suppose first that $u \in W^{1,1}(\mathbb{R}^n) \cap W_0^{1,1}(\Omega)$. Corollary 4.7 implies that $\bar{u}(x) = 0$ for H^{n-1}-almost all $x \in \mathbb{R}^n \setminus \Omega$.

Conversely, assume that $\bar{u}(x)=0$ for H^{n-1}-almost all $x\in\mathbb{R}^n\setminus\Omega$. Fix $\varepsilon>0$. It will suffice to prove that there exists $w\in W_0^{1,1}(\Omega)$ with $\|u-w\|_{1,1}<\varepsilon$. Define

$$K=\mathbb{R}^n\setminus\Omega \quad \text{and} \quad B=K\setminus\{x:\bar{u}(x)=0\}.$$

For every positive integer j, appeal to Theorem 4.1 to select $v_j\in W^{1,1}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ so that $\gamma(\{v_j\neq u\})<1/j$ and $\|u-v_j\|_{1,1}<1/j$. Define

$$E_j=\{v_j\neq\bar{u}\}\cup B,$$

let $V_j\supset E_j$ be an open set with $\gamma(V_j)<1/j$, and let $\varphi_j\in W^{1,1}(\mathbb{R}^n)$ have the property that $0\le\varphi\le1$, $\varphi_j=1$ on V_j, and

$$\int_{\mathbb{R}^n}(|\varphi_j|+|D\varphi_j|)\,dx<\frac{1}{j}. \tag{22}$$

Let $0<\delta<1$ and define the truncation

$$T_\delta(x)=\begin{cases}\delta^{-1}-\delta, & \text{if } x>\delta^{-1},\\ x-\delta, & \text{if } \delta\le x\le\delta^{-1},\\ 0, & \text{if } |x|<\delta,\\ x+\delta, & \text{if } -\delta^{-1}<x<-\delta,\\ -\delta^{-1}+\delta, & \text{if } x<-\delta^{-1}\end{cases}$$

so that T_δ is Lipschitz, $|DT_\delta|\le1$, and $\|T_\delta v-v\|_{1,1}\to0$ as $\delta\to0^+$ for any $v\in W^{1,1}(\mathbb{R}^n)$. Since v_j is continuous and vanishes on $K\setminus V_j$, it follows that $T_\delta v_j$ vanishes on a neighborhood of $K\setminus V_j$. As $\varphi_j=1$ on V_j and V_j is open we conclude that

$$w_{\delta,j}=T_\delta v_j(1-\varphi_j)$$

vanishes on a neighborhood of K. Moreover, since $\bar{u}=v_j$ off V_j, we may write $w_{\delta,j}=T_\delta u(1-\varphi_j)$ for all δ and j. This implies that

$$\|u-w_{\delta,j}\|_{1,1}=\|u-T_\delta u+(T_\delta u)\varphi_j\|_{1,1}\le\|u-T_\delta u\|_{1,1}+\|(T_\delta u)\varphi_j\|_{1,1}. \tag{23}$$

Choose δ sufficiently close to 0 so that

$$\|u-T_\delta u\|_{1,1}<\varepsilon/2. \tag{24}$$

To estimate $\|(T_\delta u)\varphi_j\|_{1,1}$ we note that $|(T_\delta u)\varphi_j|\le\delta^{-1}|\varphi_j|$ and

$$|D((T_\delta u)\varphi_j)|\le|D(T_\delta u)|\,|\varphi_j|+|T_\delta u|\,|D\varphi_j|\le|Du|\,|\varphi_j|+\delta^{-1}|D\varphi_j|$$

because $|DT_\delta|\le 1$. This implies that

$$\begin{aligned}\|(T_\delta u)\varphi_j\|_{1,1} &= \int_{\mathbb{R}^n} (|(T_\delta u)\varphi_j|+|D((T_\delta u)\varphi_j)|)\,dx\\ &\le \frac{1}{\delta}\int_{\mathbb{R}^n} (|\varphi_j|+|D\varphi_j|)\,dx+\int_{\mathbb{R}^n} |Du|\,|\varphi_j|\,dx,\end{aligned}$$

and by (22) we may choose j sufficiently large so that

$$\|(T_\delta u)\varphi_j\|_{1,1}<\varepsilon/2. \tag{25}$$

Finally, we may combine (23), (24), and (25) to conclude that

$$\|u-w_{\delta,j}\|_{1,1}<\varepsilon.$$

This implies that $w_{\delta,j}\in W^{1,1}(\mathbb{R}^n)$. Since $w_{\delta,j}$ vanishes on a neighborhood of K it follows that $w_{\delta,j}\in W_0^{1,1}(\Omega)$. This completes the proof. □

Finally we present a variant of Theorem 5.1 which extends the main result of [15] to $p=1$.

Theorem 5.2. *Let $u\in W^{1,1}(\Omega)$. Then $u\in W_0^{1,1}(\Omega)$ if and only if*

$$\lim_{r\to 0^+} \frac{1}{r^n}\int_{B(x,r)\cap\Omega} |u(y)|\,dy=0 \tag{26}$$

for H^{n-1}-almost all $x\in\partial\Omega$.

Proof. If $u\in W_0^{1,1}(\Omega)$, then $u^*\in W^{1,1}(\mathbb{R}^n)$. Theorem 5.1 implies that

$$\lim_{r\to 0^+} \frac{1}{r^n}\int_{B(x,r)\cap\Omega} |u(y)|\,dy=\lim_{r\to 0^+} \frac{1}{r^n}\int_{B(x,r)} |u^*(y)|\,dy=0$$

for H^{n-1}-almost all $x\in\mathbb{R}^n\setminus\Omega$, and in particular for H^{n-1}-almost all $x\in\partial\Omega$. Conversely, if (26) holds, then

$$\lim_{r\to 0^+} \frac{1}{r^n}\int_{B(x,r)} |u^*(y)|\,dy=\lim_{r\to 0^+} \frac{1}{r^n}\int_{B(x,r)\cap\Omega} |u(y)|\,dy=0$$

for H^{n-1}-almost all $x\in\partial\Omega$, and thus

$$\lim_{r\to 0^+} \fint_{B(x,r)} |u^*(y)|\,dy=0$$

for H^{n-1}-almost all $x\in\mathbb{R}^n\setminus\Omega$ since u^* vanishes outside Ω. Theorem 3.11 implies that $u^*\in W^{1,1}(\mathbb{R}^n)$, and Theorem 5.2 implies in turn that $u^*\in W_0^{1,1}(\Omega)$. Since u and u^* coincide on Ω we conclude that $u\in W_0^{1,1}(\Omega)$, as desired. □

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