ON MEROMORPHIC FUNCTIONS WITH REGIONS FREE OF POLES AND ZEROS

BY

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Introduction

In this paper we investigate, from the point of view of Nevanlinna's theory, meromorphic functions with certain restrictions on the location of their poles and zeros. We assume familiarity with Nevanlinna's theory and with its standard notations.

In order to state our results concisely, we introduce two definitions.

DEFINITION 1. *A path L in the complex z-plane is said to be regular if it satisfies the two following conditions:*

(i) *it is possible to represent L by the parametric equation*

$$L: \quad z = z(t) = te^{i\alpha(t)} \quad (t \geqslant t_0 \geqslant 0),$$

where $\alpha(t)$ is a real-valued continuous function;

(ii) *there is a constant $B(\geqslant 1)$ such that, for any pair (t_1, t_2) $(t_0 \leqslant t_1 < t_2)$, the portion of L which lies in $t_1 \leqslant |z| \leqslant t_2$ is rectifiable and of length*

$$s(t_1, t_2) \leqslant B(t_2 - t_1). \tag{1}$$

If it is important to mention the constant B, we shall call a regular curve for which (1) holds *B-regular*.

DEFINITION 2. *Let S be a curvilinear sector, in the z-plane, bounded by an arc of $|z| = t_0$ and two regular paths in $|z| \geqslant t_0$.*

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We say that S has opening $\geqslant c$ if the intersection of every circle $|z| = r(\geqslant t_0)$ and S is an arc of length $\geqslant cr$.

Our simplest result is

THEOREM 1. *Let $L_1, L_2, \ldots, L_s$ be regular curves dividing the plane into s sectors, each of opening $\geqslant c$, for some $c > 0$. Let ξ_1, ξ_2 be two finite distinct complex numbers.*

If $f(z)$ is an entire function of infinite order, then at least one of the equations

$$f(z) = \xi_1, \quad f(z) = \xi_2,$$

has infinitely many roots which do not lie on the paths $L_1, L_2, \ldots, L_s$.

This result will be a corollary of

THEOREM 2. *Let the s B-regular curves*

$$L_j: \quad z = te^{i\alpha_j(t)} \quad (t \geqslant t_0;\ j = 1, 2, \ldots, s;\ \alpha_1(t) < \alpha_2(t) < \ldots < \alpha_s(t) < \alpha_1(t) + 2\pi = \alpha_{s+1}(t)) \qquad (2)$$

divide $|z| \geqslant t_0$ into s sectors, each of which has opening $\geqslant c > 0$.

Suppose that all but a finite number of zeros and poles of the meromorphic function $f(z)$ lie on the curves L_j.

If some $\tau(\tau \neq 0, \tau \neq \infty)$ is a deficient value (in the sense of R. Nevanlinna) of the function $f^{(q)}(z)$, for some non-negative integer $q\,(f^{(0)} \equiv f)$, then the order λ of $f(z)$ is necessarily finite and

$$\lambda \leqslant \lambda_0 = 9\pi B^2/c. \qquad (3)$$

COROLLARY. *Let ξ_1, ξ_2, ξ_3 be three distinct complex values, one of which may be ∞. If all except a finite number of the roots of the equations*

$$f(z) = \xi_1, \quad f(z) = \xi_2, \quad f(z) = \xi_3$$

lie on s regular curves L_j satisfying the same hypothesis as in Theorem 2, then either the order of $f(z)$ does not exceed λ_0, given by (3), or $f(z)$ has no deficient value, finite or infinite.

This corollary follows at once by the application of Theorem 2 to the three functions

$$(f - \xi_1)/(f - \xi_2), \quad (f - \xi_1)/(f - \xi_3), \quad (f - \xi_2)/(f - \xi_3)$$

(easy modification, if one of the ξ's is infinite).

Theorem 1 is a special case of this corollary ($\xi_3 = \infty$, $\delta(\xi_3, f) = 1$).

Theorem 2 generalizes a result obtained by one of us [3; p. 276] in the special case

$$\alpha_j(t) \equiv \text{const.} \quad (j = 1, 2, \ldots, s). \qquad (4)$$

It is then possible to replace (3) by

$$\lambda \leqslant \lambda_0^* = \frac{\pi}{c}. \tag{5}$$

The quotient of Bessel functions

$$f(z) = J_{1/s}(2z^{\frac{1}{2}s}/s)/J_{-1/s}(2z^{\frac{1}{2}s}/s) \quad (2 \leqslant s = \text{integer}) \tag{6}$$

has s finite deficient values (none of which is zero); its zeros and poles are on the lines $\arg z = 2k\pi/s$ $(k = 1, 2, \cdots, s)$ and its order is $s/2$ [5; p. 343]. This shows that the bound λ_0^*, in (5), is "best possible".

The more general bound given in Theorem 2 is not as accurate but still is, in some respects, satisfactory. In the special case of the functions (6), we have $B = 1$, $c = 2\pi/s$, so that (3) yields

$$\lambda \leqslant \lambda_0 = 9\left(\frac{s}{2}\right);$$

this shows that the form of the dependence of λ_0 on c is correct.

The restriction $\tau \neq 0$, $\tau \neq \infty$ in Theorem 2 is essential. This may be seen by considering an entire function $g(z)$ of order λ, $2 < \lambda < +\infty$, all of whose zeros are real. Trivially $\delta(\infty, g) = \delta(\infty, g^{(q)}) = 1$. We have shown elsewhere [4] that $\delta(0, g) > 0$. It is well known [10; p. 22] that for an entire function of finite order $\delta(0, g^{(q)}) \geqslant \delta(0, g)$, so that also $\delta(0, g^{(q)}) > 0$. The function $g(z)$ satisfies the hypotheses of Theorem 2 with $s = 1$, $q \geqslant 0$, $\tau = 0$ or $\tau = \infty$, but the order of g can be arbitrarily large.

It is possible to generalize Theorem 2 by allowing zeros and poles of $f(z)$ to lie off the paths L_j, provided the number of such zeros and poles, in $|z| \leqslant r$, is suitably restricted. In the case (4), of radial lines, such a result was obtained by I. V. Ostrovski [9].

Under the hypotheses of Theorem 2 about the location of zeros and poles, a function of order $\lambda > \lambda_0$ can not have any deficient values other than 0 and ∞. The Theorem gives no information about functions of order $\lambda \leqslant \lambda_0$. In this direction we prove

THEOREM 3. *Let $f(z)$ ($\not\equiv$ const.) be an entire function of finite order λ and let $L_1, L_2, \ldots, L_s$ be the s B-regular paths defined by* (2).

Let $\delta(>0)$ be fixed and let $\bar{n}_\delta(r)$ denote the number of zeros of $f(z)$ which lie in $r_0 \leqslant |z| \leqslant r$ but outside the s sectors $\mathcal{E}_j(\delta)$ $(j = 1, 2, \cdots, s)$ defined by

$$\alpha_j(t) - \delta \leqslant \arg z \leqslant \alpha_j(t) + \delta, \quad r_0 \leqslant |z| = t < +\infty. \tag{7}$$

Assume that for every fixed $\delta(>0)$, *we have*

$$\lim_{r\to\infty} \frac{\bar{n}_\delta(r)}{T(r,f)} = 0, \tag{8}$$

where $T(r,f)$ *denotes Nevanlinna's characteristic function.*

Denote by p *the number of deficient values of* $f(z)$ *other than* 0 *and* ∞. *Then*

$$p \leqslant 2\lambda. \tag{9}$$

Our proof of Theorem 3 also yields

$$p \leqslant s. \tag{10}$$

If the configuration (2) is fixed and if $F(z)$ is an entire function of order $\lambda(\leqslant +\infty)$, with all but a finite number of its zeros on the s paths (2), we may, by combining Theorem 2, Theorem 3 and (10) summarize our results as follows:

If $\lambda = \infty$ or $$\lambda > \frac{9\pi B^2}{c},$$

then $$p = 0.$$

Otherwise $$p = \min\{s, 2\lambda\}.$$

It is not known whether there exist entire functions of finite order with infinitely many deficient values. Assume that such functions exist and that $G(z)$ be one of them. Then, the lemmas and methods of this paper show that the arguments of the zeros of $G(z)$ cannot have a simple behavior. A closer study of the question leads to the following theorem which we state without proof.

THEOREM 4. *Let* $f(z)$ *be an entire function of finite order* λ *and let*

$$a_1, a_2, a_3, \ldots$$

be its zeros of positive modulus.

Put $$a_\nu = |a_\nu| e^{i\omega_\nu} \quad (0 \leqslant \omega_\nu \leqslant 2\pi)$$

and let Ω *be the closure of the set* $\{\omega_\nu\}$.

If Ω *is of measure zero,* $f(z)$ *has at most* 2λ *deficient values other than* 0 *and* ∞.

We conclude this Introduction by an indication of the contents of the following paragraphs.

1. Notation and statement of known lemmas.
2. Statement of principal lemmas.
3. Proof of Theorem 2.
4. Proof of Theorem 3.

The remaining paragraphs 5–9 are devoted to the proofs of the lemmas stated in § 2.

1. Notation, terminology and statement of known results

We use the symbol A to denote a positive absolute constant and the symbol K to denote a positive constant depending on one or more parameters.

Most of our inequalities are only valid for sufficiently large values of certain parameters $m, r, \dots$. We usually indicate this fact by writing, immediately after the relevant inequality, $(m > m_0)$, $(r > r_0)$, ...

The quantities A, K, m_0, r_0, ... are not necessarily the same ones each time they occur. We write $A_1, A_2, \dots, K_1, K_2, \dots$ whenever it seems clearer to preserve the identity of the constants and $K_1(\varkappa, \lambda, \dots)$, $K_2(\varkappa, \lambda, \dots)$, ... if it is useful to list explicitly all the parameters on which the constants depend.

The measurable sets E, which will appear in our proofs are subsets of the positive axis. If E is such a set, we denote by $E(\alpha, \beta)$ its intersection with the interval (α, β) and by $mE(\alpha, \beta)$ the measure of this intersection.

Nevanlinna's notation for the means of $\overset{+}{\log}|f|$ will be extended by the following convention.

If J is a measurable set of values of θ, we write

$$\frac{1}{2\pi}\int_J \overset{+}{\log}|f(re^{i\theta})|\,d\theta = m(r, f; J). \tag{1.1}$$

For the convenience of the reader, we first state as Lemma A some well-known consequences of the fundamental estimates of R. Nevanlinna.

LEMMA A [7; p. 62 and p. 104]. *Let $f(z)$ be a meromorphic function which does not reduce to a polynomial.*

There is a set E (of values of r) of finite measure, such that $r \notin E$ implies all the following inequalities

$$T(r, f^{(k)}) \leqslant K(T(r, f) + \overset{+}{\log} r) \quad (k = 0, 1, 2, \dots, q+1), \tag{1.2}$$

$$m(r, f^{(k+1)}/f^{(k)}) \leqslant K(\overset{+}{\log} T(r, f) + \overset{+}{\log} r) \quad (k = 0, 1, \dots, q), \tag{1.3}$$

$$m(r, f^{(q+1)}/(f^{(q)} - \tau)) \leqslant K(\overset{+}{\log} T(r, f) + \overset{+}{\log} r). \tag{1.4}$$

We also need the three following lemmas which we have proved elsewhere [5].

LEMMA B. *Let $f(z)$ be a meromorphic function ($f(z) \not\equiv const.$), let $\tau(\neq 0)$ be a complex number and let J be a measurable set of θ, in $0 \leqslant \theta < 2\pi$. Then*

$$m(r, f/f'; J) > m(r, 1/(f-\tau); J) - m(r, f'/f) - m(r, f'/(f-\tau)) - K(\tau). \tag{1.5}$$

LEMMA C [5; p. 322, Lemma III]. *Let $g(z)$ be meromorphic. With each $r(>0)$ we associate a measurable set $I(r)$ (of values of θ) of measure*

$$mI(r) = \mu(r).$$

Then, for

$$1 \leqslant r < R',$$

$$m(r, g; I(r)) \leqslant \frac{11\, R'}{R'-r}\, T(R', g)\, \mu(r) \left[1 + \overset{+}{\log} \frac{1}{\mu(r)}\right]. \tag{1.6}$$

Our next lemma is a special case of Lemma 10.2 [5] ($\alpha = 0$, $\varepsilon = 1$ and $\alpha = 3$, $\varepsilon = 1$, $M = 2$):

LEMMA D. *Let $V(r)$ be a non-negative, non-decreasing, unbounded function defined in $r > r_0$. There is a set E with*

$$mE(\varrho, 2\varrho) \leqslant \frac{8\varrho}{(\log V(\varrho))^{\frac{1}{4}}} \quad (\varrho > \varrho_0)$$

such that outside E simultaneously

$$V\left(r + \frac{r}{\log^2 V(r)}\right) < e V(r),$$

$$V\left(r + \frac{r}{(\log V(r))^{\frac{1}{2}}}\right) < V^2(r).$$

The following Lemma E is obtained from a result of R. Nevanlinna [8; p. 84, formula (14″)] by letting β' (in Nevanlinna's notation) shrink to a point a, putting $\beta'' = \mathcal{B}$.

LEMMA E. *Let G be a domain bounded by a Jordan curve $\mathcal{C}$ consisting of a Jordan arc $\mathcal{B}$ and its complement $\mathcal{A}$ in $\mathcal{C}$. Let $\mathfrak{L}$ be a rectifiable curve in G joining a point $a \in \mathcal{A}$ to a point $b \in \mathcal{B}$. Let z be a point on $\mathfrak{L}$. Let $\varrho(z)$ be the distance of z from $\mathcal{A}$. Then the harmonic measure $\omega(z)$ of $\mathcal{B}$ with respect to G satisfies*

$$\omega(z) \geqslant \frac{1}{2\pi} \exp\left\{-4 \int_z^b \frac{|d\zeta|}{\varrho(\zeta)}\right\},$$

where the integral is taken along $\mathfrak{L}$.

2. Lemmas

Here we state Lemmas needed in the proofs of Theorems 2 and 3. The numbers in brackets refer to the paragraphs in which these Lemmas are proved.

LEMMA 1 [§ 5]. *Let*
$$z = z(u) = u\, e^{i\alpha(u)} \quad (u \geqslant t_0) \tag{2.1}$$
be the parametric equation of a B-regular curve L. Then the point
$$t\, e^{i(\alpha(t)+\gamma)}$$
is at a distance
$$d \geqslant t \left|\sin \tfrac{1}{2}\gamma\right| / B$$
from L.

This Lemma readily yields

LEMMA 2 [§ 5]. *Let* $\mathcal{R}$ *be a denumerable set of circles with centers in* $|z| \geqslant t_1 \geqslant t_0$ *and sum of radii less than* $D(<t_1/B)$.

Let
$$L(\gamma):\quad \zeta(u) = u\, e^{i(\alpha(u)+\gamma)} \quad (-\pi < \gamma \leqslant \pi)$$
be the curve obtained by rotating the B-regular curve (2.1) *through an angle* γ.

Then $L(\gamma)$ *will not meet any circle of* $\mathcal{R}$ *if* γ *lies outside a set of measure* $2\pi BD/t_1$.

LEMMA 3 [§ 6]. *Let* $f(z)$ *be a meromorphic, non-rational function. There is a measurable set E, of values of r, such that*
$$mE(\varrho, 2\varrho) = o(\varrho) \quad (\varrho \to \infty)$$
and such that for $r \notin E$
$$T(r, f) \leqslant A T(r, f^{(q)}) \log^3 T(r, f^{(q)}).$$

LEMMA 4 [§ 7]. *Let* $f(z)$ $(\not\equiv 0)$ *be a meromorphic function and let*
$$d_1, d_2, d_3, \ldots \quad (|d_m| \leqslant |d_{m+1}|)$$
be the sequence of its zeros and poles, each one appearing as often as its multiplicity indicates. Let $H(\geqslant 1)$ *be given and denote by* $\mathcal{R}(H)$ *the union of the discs*
$$\mathcal{R}_m:\quad |z - d_m| \leqslant \frac{1}{Hm^2} \quad (m = 1, 2, 3, \ldots).$$

Then there is an r_0 *such that for*
$$R' > r \geqslant |z| > r_0 \quad z \notin \mathcal{R}(H),$$

we have
$$\left|\frac{f^{(q+1)}(z)}{f(z)}\right| < K_1(q)\left\{\frac{HR'\,T(R',f)}{R'-r}\right\}^{K_2(q)}, \tag{2.2}$$

where $K_1(q)$ and $K_2(q)$ depend only on q.

Let C be a circular arc belonging to the half-plane $x \geqslant 0$ $(z = x + iy)$ and passing through the points $\pm i\alpha$ $(\alpha > 0)$. Let Λ^+ be the closed set (in $x \geqslant 0$) of points bounded by C and by the segment $[-i\alpha, +i\alpha]$ of the imaginary axis.

We define the "lens" Λ to be the smallest set containing Λ^+ and symmetrical with respect to the imaginary axis.

The lens Λ is characterized by $\alpha(>0)$ and by $\beta(0 < \beta < \pi)$, the angle formed by the imaginary axis and the tangent to C at $i\alpha$. Ambiguities concerning the value of β will be removed by the convention that $0 < \beta \leqslant \pi/2$ for convex lenses.

LEMMA 5 [§ 8]. *Let Λ be the lens, in the z-plane, with vertices $\pm i\alpha$ and semi-vertical angle $\beta(0 < \beta < \pi)$.*

Let $H(z)$ be regular in Λ; assume that

$$|H(z)| \leqslant 1 \quad (z \in \Lambda),$$

and
$$\int_{-\alpha+\varepsilon}^{\alpha-\varepsilon} \log|1/H(iy)|\,dy > M^* > 0 \quad (0 < \varepsilon < \alpha). \tag{2.3}$$

Then
$$\log|H(iy)| < -\frac{2M^*}{\pi\alpha\beta}\left\{\frac{\varepsilon}{2\alpha}\right\}^{2\pi/\beta} \quad (|y| \leqslant \alpha - \varepsilon). \tag{2.4}$$

Our last lemma is a straightforward consequence of Ahlfors' distortion theorem.

LEMMA 6 [§ 9]. *Let the domain D in the z-plane be bounded by portions of two regular paths L_1, L_2:*

$$L_j: \quad z = te^{i\alpha_j(t)} \quad (0 \leqslant t < +\infty;\ j = 1, 2), \tag{2.5}$$

and by the two circular arcs

$$z = \varrho_j e^{i\theta} \quad (\alpha_1(\varrho_j) \leqslant \theta \leqslant \alpha_2(\varrho_j), \quad j = 1, 2;\ 0 < \varrho_1 < \varrho_2).$$

Put
$$\Theta(t) = \alpha_2(t) - \alpha_1(t) \tag{2.6}$$

assume that
$$0 < \Theta(t) \leqslant 2\pi \quad (0 < t < +\infty) \tag{2.7}$$

and let t_1, t_2 be such that

$$\varrho_1 \leqslant t_1 < t_2 \leqslant \varrho_2.$$

If $\omega_2(z, t_2)$ denotes the harmonic measure with respect to D of the part of the boundary of D which lies in $|z| \geqslant t_2$ and if

$$re^{i\theta} \in D, \quad t_2/r > e^{9\pi} \tag{2.8}$$

then
$$\omega_2(re^{i\theta}, t_2) < \frac{5e^{4\pi}}{\pi} \exp\left\{-\pi \int_r^{t_2} \frac{dt}{t\Theta(t)}\right\}. \tag{2.9}$$

Similarly, if $\omega_1(z, t_1)$ denotes the harmonic measure with respect to D of the part of the boundary of D which lies in $|z| \leqslant t_1$, then for

$$re^{i\theta} \in D, \quad r/t_1 > e^{9\pi}$$

we have
$$\omega_1(re^{i\theta}, t_1) < \frac{5e^{4\pi}}{\pi} \exp\left(-\pi \int_{t_1}^{r} \frac{dt}{t\Theta(t)}\right). \tag{2.10}$$

3. Proof of Theorem 2

Denoting by λ the order (not necessarily finite) of $f(z)$, we prove Theorem 2 by deducing from the assumption

$$\lambda > 9\pi B^2/c = \lambda_0,$$

the contradiction that $f(z)$ is a polynomial.

Choose μ so that
$$\lambda_0 < \mu < \lambda. \tag{3.1}$$

Then there exist arbitrarily large ϱ such that

$$T(\varrho, f) > (2\varrho)^{\mu}$$

and consequently
$$T(r, f) > r^{\mu} \tag{3.2}$$

in
$$\varrho < r < 2\varrho.$$

If $\varrho > \varrho_0$, then r can be chosen in such a way that all the following relations hold:

$$T(r, f^{(k)}) < KT(r, f) \quad (k = 0, 1, 2, \ldots, q+1), \tag{3.3}$$

$$m(r, f^{(k+1)}/f^{(k)}) < K \log T(r, f) \quad (k = 0, 1, 2, \ldots, q), \tag{3.4}$$

$$m(r, f^{(q+1)}/(f^{(q)} - \tau)) < K \log T(r, f), \tag{3.5}$$

$$T(r + r\{\log T(r, f^{(k)})\}^{-2}, f^{(k)}) < eT(r, f^{(k)}) \quad (k = 0, 1, 2, \ldots, q+1), \tag{3.6}$$

$$T(r + r(\log T(r, f))^{-\frac{1}{2}}, f) < T^2(r, f), \tag{3.7}$$

$$T(r, f) < AT(r, f^{(q)}) \log^3 T(r, f^{(q)}). \tag{3.8}$$

This assertion is true, because by Lemma A, Lemma D, (3.2) and Lemma 3 the set E of values for which at least one of (3.3)–(3.8) ceases to be true satisfies

$$mE(\varrho, 2\varrho) = o(\varrho) \quad (\varrho \to +\infty).$$

Since τ is a deficient value of $f^{(q)}$, there is a $\varkappa > 0$ such that

$$m\left(R, \frac{1}{f^{(q)} - \tau}\right) > \varkappa T(R, f^{(q)}) \quad (R > r_0;\ q \geqslant 0). \tag{3.9}$$

The curves $L_1, L_2, \ldots, L_s$ divide the region $|z| \geqslant t_0$ into s sectors $S_1, S_2, \ldots, S_s$. Let $J_k = J_k(R)$ be the set of arguments of the arc of $|z| = R$ which lies in S_k. Then (3.9) implies that there is at least one index $k = k(R)$ such that for $J = J_{k(R)}(R)$

$$m\left(R, \frac{1}{f^{(q)} - \tau};\ J\right) > \{\varkappa/s\}\, T(R, f^{(q)}) \quad (R > r_0). \tag{3.10}$$

When $R \to \infty$ through the values of a sequence

$$R_1, R_2, \ldots, R_m, \ldots, \tag{3.11}$$

at least one value of $k(R)$ must be taken infinitely often. Without loss of generality, we may assume it to be $k = 1$, corresponding to the sector $S_1 = S$ given by

$$S: \quad r \geqslant t_0, \quad \alpha_1(r) \leqslant \theta \leqslant \alpha_2(r) \quad (z = re^{i\theta}).$$

In the remainder of the proof, the letter R will always stand for a member of a fixed sequence (3.11), such that for $r = R = R_m$ $(m = 1, 2, \ldots)$, (3.2)–(3.8) hold as well as (3.10) with $J = J_1(R_m)$. It is important to notice that the constants K which will appear in the proof are independent of m.

By Lemma B, (3.10), (3.4), (3.5) and the assumptions $\tau \neq 0, \tau \neq \infty$,

$$m(R, f^{(q)}/f^{(q+1)};\ J) > KT(R, f^{(q)}) - K \log T(R, f). \tag{3.12}$$

The identity
$$\frac{f^{(q)}}{f^{(q+1)}} = \frac{f}{f^{(q+1)}} \cdot \frac{f'}{f} \cdot \frac{f''}{f'} \cdots \frac{f^{(q)}}{f^{(q-1)}}$$

and (3.4) imply

$$m(R, f/f^{(q+1)};\ J) > m(R, f^{(q)}/f^{(q+1)};\ J) - K \log T(R, f). \tag{3.13}$$

Combining (3.12), (3.13), (3.8), (3.3) and using the abbreviation

$$T = T(R, f),$$

we find
$$m(R, f/f^{(q+1)};\ J) > KT(\log T)^{-3}, \tag{3.14}$$

where $R = R_m$ and $m > m_0$.

We now leave m fixed and consider the function

$$h(z)=\frac{f^{(q+1)}(z)}{f(z)} \quad (z=re^{i\theta}),$$

in the curvilinear sector S'

$$S': \quad |z|\geqslant r_0(>t_0), \quad \alpha_1(r)+T^{-\frac{1}{2}}\leqslant\theta\leqslant\alpha_2(r)-T^{-\frac{1}{2}},$$

where r_0 has been chosen so large that S' exists and is free from zeros and poles of $f(z)$. We establish first that if

$$z\in S', \quad r_0\leqslant|z|\leqslant R+\tfrac{3}{4}R\log^{-\frac{1}{2}}T,$$

then
$$|h(z)|\leqslant T^{K_3(q)}. \tag{3.15}$$

This follows from (3.7), Lemma 4 with

$$R'=R+R\log^{-\frac{1}{2}}T, \quad H=T$$

and the remark that, by Lemma 1, the distance between a point of S' and the curves L_1, L_2 is at least

$$r_0\sin(\tfrac{1}{2}T^{-\frac{1}{2}})/B>T^{-1}.$$

Next we show that

$$\int_{\alpha_1(R)+(\log T)^{-6}}^{\alpha_2(R)-(\log T)^{-6}}\log|T^{K_3}/h(Re^{i\theta})|\,d\theta>KT\log^{-3}T \quad (R=R_m,\ m>m_0,\ K_3=K_3(q).) \tag{3.16}$$

If $I=I(R)$ is the union of the two intervals

$$\alpha_1(R)\leqslant\theta\leqslant\alpha_1(R)+(\log T)^{-6},$$
$$\alpha_2(R)-(\log T)^{-6}\leqslant\theta\leqslant\alpha_2(R),$$

then, by Lemma C with $g(z)=1/h(z)$ and

$$R'=\min\{R+R(\log T)^{-2}, \quad R+R(\log T(R,f^{(q+1)}))^{-2}\}$$

combined with (3.6) and (3.3):

$$\begin{aligned} m(R,1/h;I)&\leqslant AT(R',1/h)\,mI(1+\overset{+}{\log}(1/m(I))\log^2 T\\ &\leqslant A\{T(R',f)+T(R',f^{(q+1)})\}\log^{-4}T\log\log T\\ &=o(T\log^{-3}T) \quad (R=R_m\to\infty). \end{aligned} \tag{3.17}$$

By (3.14) and (3.17)

$$m(R,1/h;J-I)>KT\log^{-3}T \quad (R=R_m,\ m>m_0).$$

A fortiori $$m(R, T^{K_3}/h; J - I) > KT \log^{-3} T.$$

This is exactly (3.16) with the log under the integral sign replaced by $\overset{+}{\log}$. But by (3.15),

$$0 \leqslant \log |T^{K_3}/h| = \overset{+}{\log} |T^{K_3}/h|,$$

on $J - I$, and (3.16) is proved.

Let Γ be the arc $z = Re^{i\theta}$ with

$$\alpha_1(R) + \{\log T\}^{-6} \leqslant \theta \leqslant \alpha_2(R) - \{\log T\}^{-6}; \tag{3.18}$$

our next step is to show that

$$\log |h(z)| < -KT \exp(-\{\log T\}^{\frac{2}{3}}) \quad (z \in \Gamma,\ R = R_m;\ m > m_0). \tag{3.19}$$

This is done by an application of Lemma 5. To prepare this application, we first map S' into the ζ-plane by

$$\zeta = \xi + i\eta = \Psi(z) = \log z + \text{const.},$$

in such a way that the insersection of S' with $|z| = R$ is mapped onto the segment

$$\xi = 0, \quad |\eta| \leqslant \alpha' = \tfrac{1}{2}\{\alpha_2(R) - \alpha_1(R)\} - T^{-\frac{1}{2}},$$

of the imaginary axis. Then the arc Γ is mapped on

$$\xi = 0, \quad |\eta| \leqslant \alpha' - (\log T)^{-6} + T^{-\frac{1}{2}}.$$

Let Λ be the lens, in the ζ-plane, bounded by the two circular arcs which pass through the points $\pm i\alpha'$ and make an angle

$$\beta = \frac{1}{4\alpha'}\{\log T\}^{-\frac{1}{2}} \tag{3.20}$$

with the η-axis. If R is large enough, we have $\alpha' > c/3$ and since $T \to \infty$ as $R \to \infty$, it is clear that

$$\lim_{R \to \infty} \beta = 0. \tag{3.21}$$

We prove first that the image $\Psi^{-1}(\Lambda)$ of Λ, in the z-plane, lies in the intersection $\mathcal{D}$ of S' and

$$|z| \leqslant R + \tfrac{3}{4} R(\log T)^{-\frac{1}{2}}.$$

If R is large enough, Λ lies in the parallelogram P defined by

$$|\xi| \leqslant (\alpha' - \eta) \tan \beta \quad (0 \leqslant \eta \leqslant \alpha'),$$

$$|\xi| \leqslant (\alpha' + \eta) \tan \beta \quad (-\alpha' \leqslant \eta \leqslant 0),$$

so that
$$\Psi^{*-1}(P)\subset \mathcal{D} \tag{3.22}$$
implies
$$\Psi^{*-1}(\Lambda)\subset \mathcal{D}. \tag{3.23}$$

Put
$$\zeta_1 = i(\alpha' - \eta_1) \quad (0 \leqslant \eta_1 \leqslant \alpha'),$$
$$\zeta = i(\alpha' - \eta_1) + \xi,$$
then
$$\Psi^{*-1}(\zeta_1) = Re^{i\{\alpha_2(R) - T^{-\frac{1}{2}} - \eta_1\}}$$
$$\Psi^{*-1}(\zeta) = Re^{\xi} e^{i\{\alpha_2(R) - T^{-\frac{1}{2}} - \eta_1\}}.$$

Hence, for $\zeta \in P$,
$$|\Psi^{*-1}(\zeta)| = Re^{\xi} \leqslant Re^{\alpha' \tan\beta} \leqslant (1 + 2\alpha' \tan\beta) R \tag{3.24}$$
provided $\beta(>0)$ is small enough.

By (3.21), $\beta \to 0$ as $m \to \infty$ and therefore
$$\tan\beta \leqslant \tfrac{3}{2}\beta \quad (m > m_0).$$

Using this inequality and (3.20), in (3.24), we find
$$|\Psi^{*-1}(\zeta)| \leqslant R(1 + \tfrac{3}{4}\{\log T\}^{-\frac{1}{2}}) \quad (m > m_0). \tag{3.25}$$

Also, if ξ is small enough,
$$|\Psi^{*-1}(\zeta) - \Psi^{*-1}(\zeta_1)| = R\,|e^{\xi} - 1| \leqslant 2R\,|\xi| \leqslant 2R[\alpha' - (\alpha' - \eta_1)]\tan\beta = 2R\eta_1 \tan\beta;$$
using again (3.20) and the fact that $T \to \infty$ as $m \to \infty$:
$$|\Psi^{*-1}(\zeta) - \Psi^{*-1}(\zeta_1)| \leqslant \frac{\eta_1}{\alpha'} R(\log T)^{-\frac{1}{2}}.$$

Since
$$\frac{\eta_1}{\alpha'} R(\log T)^{-\frac{1}{2}} \leqslant \tfrac{1}{2}(R/B)\sin(\tfrac{1}{2}\eta_1) \leqslant \tfrac{1}{2}(R/B)\sin(\tfrac{1}{2}\alpha') \quad (m > m_0),$$
it follows by Lemma 1 that $\Psi^{*-1}(\zeta)$ is in a circle with center $\Psi^{*-1}(\zeta_1)$ which does not intersect the boundary curves of S', so that
$$\Psi^{*-1}(\zeta) \in S'.$$

In view of (3.25), this shows that the image of the upper half of P lies[1] in $\mathcal{D}$. The lower half may be treated in a similar way. Hence (3.22) and therefore (3.23) hold for $m > m_0$.

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[1] It is important to observe that (3.25) and the other inequalities for $\Psi^{*-1}(\zeta)$ and $\Psi^{*-1}(\zeta_1)$ hold uniformly for all admissible values of ζ and ζ_1, as soon as $m > m_0$.

If we put
$$H(\zeta) = T^{-K_3} h(z) = T^{-K_3} h(\Psi^{-1}(\zeta)),$$

we have, by (3.15)
$$|H(\zeta)| \leqslant 1, \quad (\zeta \in \Lambda).$$

Rewriting (3.16) as

$$\int_{-\alpha' + (\log T)^{-6} + T^{-\frac{1}{2}}}^{\alpha' - (\log T)^{-6} - T^{-\frac{1}{2}}} \log \left| \frac{1}{H(i\eta)} \right| d\eta > KT(\log T)^{-3} - M^*,$$

defining β by (3.20) and letting

$$\alpha = \alpha', \quad \varepsilon = (\log T)^{-6} - T^{-\frac{1}{2}} > \tfrac{1}{2}(\log T)^{-6},$$

we see that Lemma 5 may be applied to the function $H(\zeta)$ with $\zeta \in \Lambda$.

The assumptions of Theorem 2 imply

$$\frac{c}{3} < \alpha' < \pi \quad (m > m_0),$$

so that (2.4) yields

$$\begin{aligned}
\log |h(Re^{i\theta})| &= \log |H(iy)| + K_3 \log T \\
&< -KT\{\log T\}^{-3} \exp(-A\{\log T\}^{\frac{1}{2}} \log\log T) + K_3 \log T \\
&< -KT \exp(-\{\log T\}^{\frac{2}{3}})
\end{aligned}$$
$$(\alpha_1(R) + (\log T)^{-6} \leqslant \theta \leqslant \alpha_2(R) - (\log T)^{-6};\ m > m_0),$$

which is (3.19).

Next we estimate $\log |h(z)|$ at

$$z = te^{i(\alpha_1(t) - \frac{1}{2}c)} \quad (t \geqslant 2r_0),$$

by applying Lemma E with $G = S''(R)$,

$$S''(R):\ r_0 \leqslant r \leqslant R,\ \alpha_1(r) + (\log T)^{-6} \leqslant \theta \leqslant \alpha_2(r) - (\log T)^{-6}\ (z = re^{i\theta}),$$

and with $\mathcal{B} = \Gamma$ (defined by (3.18)).

For $\mathcal{L}$ we choose the B-regular path

$$s(u) = ue^{i(\alpha_1(u) + \frac{1}{2}c)} \quad (2r_0 \leqslant u \leqslant R). \tag{3.26}$$

Let $\mathcal{C}$ denote the boundary of $S''(R)$, let

$$\mathcal{A} = \mathcal{C} - \mathcal{B},$$

and let $\varrho(s(u))$ denote the shortest distance between $s(u)$ and $\mathcal{A}$.

Considering separately the circular arc and the two B-regular curves which form $\mathcal{A}$, we have, in view of Lemma 1,

$$\varrho(s(u)) \geqslant \min\left\{u\left(1-\frac{r_0}{u}\right),\ \frac{u}{B}\left|\sin\left[\frac{1}{2}\left(\frac{c}{2}-\{\log T\}^{-6}\right)\right]\right|,\right.$$

$$\left.\frac{u}{B}\left|\sin\left[\frac{1}{2}\left\{\alpha_2(u)-\alpha_1(u)-\frac{1}{2}c-(\log T)^{-6}\right\}\right]\right|\right\} \quad (u \geqslant 2r_0). \tag{3.27}$$

In view of the assumptions

$$c \leqslant \alpha_2(u)-\alpha_1(u) \leqslant 2\pi, \quad B \geqslant 1,$$

(3.27) readily yields

$$\varrho(s(u)) \geqslant \frac{u}{B}\min\left\{B\left(1-\frac{r_0}{u}\right),\ \sin\left(\frac{1}{4}c-\frac{1}{2}\{\log T\}^{-6}\right)\right\} > \frac{4uc}{9\pi B} \quad (u \geqslant Kr_0,\ m > m_0).$$

Since the path described by $s(u)$ is B-regular,

$$\int_{s(u)}^{s(R)} \frac{|ds|}{\varrho(s)} \leqslant \int_u^R \frac{B\,dt}{\varrho(s(t))} \leqslant \frac{9\pi B^2}{4c}\int_u^R \frac{dt}{t} = \frac{9\pi B^2}{4c}\log\left(\frac{R}{u}\right) \quad (u \geqslant Kr_0,\ m > m_0). \tag{3.28}$$

By the two-constant theorem [8; p. 42], (3.15) and (3.19),

$$\log|h(s)| < K_3 \log T - \omega KT \exp(-\{\log T\}^{\frac{2}{3}}),$$

where ω is the harmonic measure of $\mathfrak{B}(=\Gamma)$ with respect to $S''(R)$ at the point $s=s(u)$.

By Lemma E and (3.28)

$$\omega \geqslant \frac{1}{2\pi}\exp\left(-4\,\frac{9\pi B^2}{4c}\log\left\{\frac{R}{u}\right\}\right) = \frac{1}{2\pi}\left\{\frac{u}{R}\right\}^{\lambda_0}.$$

But, by (3.2) $$R < \{T\}^{1/\mu},$$

so that $$\omega > \frac{1}{2\pi}u^{\lambda_0}T^{-(\lambda_0/\mu)}$$

$$\log|h(s(u))| < K_3\log T - Ku^{\lambda_0}T^{(1-(\lambda_0/\mu))}\exp(-\{\log T\}^{\frac{2}{3}}). \tag{3.29}$$

As $R=R_m \to \infty$, $T \to \infty$ and the right hand side of (3.29) tends to $-\infty$, by (3.1). Hence

$$f^{(q+1)}(s)/f(s) = 0$$

for every $s = s(u)$ $(u > Kr_0)$. Hence $f^{(q+1)}(z)/f(z)$ vanishes identically, which is only possible if $f(z)$ is a polynomial. This contradicts our assumption that $f(z)$ is of order $\lambda(>\lambda_0)$ and hence completes the proof of Theorem 2.

Proof of Theorem 3

The idea of the proof is as follows. Suppose that the function $f(z)$ satisfies the hypotheses of Theorem 3 and that it has the distinct deficient values

$$\tau_1, \tau_2, \ldots, \tau_p \quad (\tau_j \neq 0, \ \tau_j \neq \infty; \ \ j = 1, 2, \ldots, p).$$

The curves L_j divide the z-plane into sectors S_k. Let $J_k = J_k(r)$ be the set of arguments corresponding to the arc of $|z| = r$ in S_k. Since the τ_j are deficient, there is at least one index $k = k(j, r)$ such that for some fixed $\varkappa > 0$

$$m\left(r, \frac{1}{f - \tau_j}; J_k\right) > \varkappa T(r, f) = \varkappa T(r) \quad (r > r_0, \ k = k(j, r), \ \ j = 1, 2, \ldots, p); \tag{4.1}$$

we may choose

$$\varkappa = \frac{1}{s+1} \min_{1 \leqslant j \leqslant p} \{\delta(\tau_j)\}. \tag{4.2}$$

In (4.1), we have written $T(r)$ instead of $T(r, f)$; from now on this will be done systematically and we shall use the more explicit notation for the characteristics of functions other than f.

From (4.1) we shall deduce that, for some arbitrarily large R, f'/f is small at most points of the intersection D_k of S_k $(k = k(j, R))$ with the annulus

$$e^{-M} R \leqslant |z| \leqslant e^M R \quad (0 < M = \text{const.}). \tag{4.3}$$

Since, by (4.1), $f(z)$ must be close to τ_j for some $z \in S_k$, it will follow, by integration of f'/f, that there is a regular curve C_k in the intersection of the annulus (4.3) with S_k $(k = k(j, R))$ such that

(i) $f(z)$ is close to τ_j on C_k;

(ii) $f'(z)$ is small on C_k.

The curves C_k divide the annulus (4.3) into p sectors. By a method which is closely related to A. J. Macintyre's proof of the Denjoy conjecture [6] we prove that, if

$$p > 2\lambda,$$

$f'(z)$ is so small in one of these new sectors, S', say, that $f(z)$ can not be close to two different τ's at the ends of the arc of $|z| = R$ which lies in S'. This contradicts (i) and shows that the assumption $p > 2\lambda$ is not tenable.

We proceed to the details of the proof.

Let ν be any finite fixed number such that

$$\lambda < \nu < \lambda + 1; \tag{4.4}$$

then,
$$T(r)\, r^{-\nu} \to 0,$$

as $r \to \infty$. Hence we can find an increasing, unbounded sequence $r_1, r_2, \ldots, r_m, \ldots$, such that

$$T(r)\, r^{-\nu} \leqslant T(r_m)\, r_m^{-\nu} \quad (r \geqslant r_m;\ m = 1, 2, \ldots). \tag{4.5}$$

With the equations (2) for the L_j, we shall denote by S_k the sector

$$r > t_0, \quad \alpha_k(r) < \theta < \alpha_{k+1}(r) \quad (z = re^{i\theta});$$

by $S_k(\delta)$ the sector

$$r > t_0, \quad \alpha_k(r) + \delta < \theta < \alpha_{k+1}(r) - \delta \quad (0 \leqslant \delta < \tfrac{1}{16} c);$$

by $J_k(\delta)$ the set of arguments of the arc of $|z| = r$ in $S_k(\delta)$ and by $I_k(\delta)$ the complement of $J_k(\delta)$ in $J_k = J_k(0)$. We apply Lemma C to the function $1/(f - \tau_j)$, with $R' = 2r$, $I(r) = I_k(2\delta)$ and

$$r_m \leqslant r \leqslant 2 r_m.$$

This yields

$$m(r, 1/(f - \tau_j);\ I_k(2\delta)) \leqslant 22\, T(2r, 1/(f - \tau_j))\, 4\delta \left(1 + \overset{+}{\log}\left(\frac{1}{4\delta}\right)\right).$$

Using the first fundamental theorem and (4.5), we obtain

$$m(r, 1/(f - \tau_j);\ I_k(2\delta)) \leqslant 90(4)^{\nu}\, T(r)\, \delta \left(1 + \overset{+}{\log}\left(\frac{1}{4\delta}\right)\right) < \frac{\varkappa}{2}\, T(r)$$

provided
$$\delta < \delta_1 = \delta_1(\varkappa, \lambda), \quad m > m_0.$$

Hence, by (4.1),
$$m(r, 1/(f - \tau_j);\ J_k(2\delta)) > \tfrac{1}{2}\, \varkappa T(r). \tag{4.6}$$

We may assume that $f(z)$ is not a polynomial (since non-constant polynomials have no finite deficient values) and hence

$$\log r = o(T(r)) \quad (r \to \infty). \tag{4.7}$$

Combining (4.6), Lemma B, the estimate

$$m(r, f'/f) + m(r, f'/(f - \tau)) = O(\log r)$$

and (4.7), we obtain
$$m(r, f(z)/f'(z);\ J_k(2\delta)) > \frac{\varkappa}{3} T(r), \tag{4.8}$$
for
$$r_m \leqslant r \leqslant 2r_m,\ m > m_0;\ \ k = k(j, r);\ \ j = 1, 2, \ldots, p;\ \ 0 \leqslant \delta < \delta_1.$$

We choose now a constant $M(\geqslant 2)$. For the proof of (10) we take $M = 2$. For the proof of (9) we shall obtain a contradiction if we assume
$$\lambda < \nu < \frac{p}{2}, \tag{4.9}$$
and if we choose M so large that
$$e^M > 16 A_2^2 = U, \quad A_2 = 5e^{4\pi}/\pi, \tag{4.10}$$
and
$$-\frac{K_4}{4} + 4^{\lambda+2} A_2 e^{-M(\frac{1}{2}p-\nu)} + 4A_2 e^{-\frac{1}{2}M} < 0, \tag{4.11}$$

where the constant K_4 (defined in (4.38)) depends only on the function $f(z)$, on the configuration of the paths L_k and on $\varkappa$ (defined by (4.2)). We shall see, in fact, that K_4 (as well as two auxiliary constants K_5 and K_6 wich appear in (4.21) and (4.24), respectively) may be characterized completely in terms of λ, c, B, $\varkappa$. It is essential to observe that these constants depend neither explicitly nor implicitly on the parameters m and M.

Our next task is the investigation of the function $f'(z)/f(z)$ in the annulus
$$\tfrac{1}{3} e^{-M} r_m \leqslant |z| \leqslant 3e^M r_m.$$

By Lemma 4 (with $H = 1$, $q = 0$, $R' = 2r$)
$$|f'(z)/f(z)| < A\{T(2r)\}^A \quad (r > r_0), \tag{4.12}$$
outside a set $\mathfrak{R}$ of discs with sum of radii less than 1. Therefore, since $f(z)$ is of finite order, we can find an integer $h = h(\lambda)$ (depending only on the order λ of $f(z)$) such that
$$|z^{-h} f'(z)/f(z)| < 1 \quad (|z| > r_0,\ z \notin \mathfrak{R}). \tag{4.13}$$

It follows now from Lemma 2, that there exist some δ ($\frac{1}{2}\delta_1 < \delta < \delta_1$) and some r_0 such that (4.13) holds on the boundaries of the $S_k(\delta)$ ($k = 1, 2, \ldots, s$), for $|z| > r_0$. From now on we assume that δ has been chosen in this way and we shall make no further changes in the choice of δ. It is also easily seen that there are two circles
$$|z| = R' = R'_m;\quad 2e^M r_m < R' < 3e^M r_m, \tag{4.14}$$
and
$$|z| = r' = r'_m;\quad e^{-M} r_m/3 < r' < \tfrac{1}{2} e^{-M} r_m,$$
on which (4.13) holds.

Consider now
$$g(z) = z^{-h} f'(z) P(z)/f(z), \tag{4.15}$$
where
$$P(z) = \prod_{\mu=1}^{n} \frac{(z-a_\mu)}{2R'},$$
is the product, taken over all the poles of f'/f which lie in $|z| \leqslant R'$ but outside the sectors $\mathcal{E}_j(\delta)$ $(j = 1, 2, \ldots, s)$ defined by (7).

The function $g(z)$ is regular in the intersection of $r_0 \leqslant |z| \leqslant R'$ with every $S_k(\delta)$ $(k = 1, 2, \ldots, s)$. In $|z| \leqslant R'$
$$|P(z)| \leqslant 1. \tag{4.16}$$
By (4.13), (4.16) and the maximum modulus principle
$$|g(z)| < 1 \quad (z \in D_k), \tag{4.17}$$
where D_k is defined by the inequalities
$$D_k: \quad r' \leqslant r \leqslant R'; \quad \alpha_k(r) + \delta \leqslant \theta \leqslant \alpha_{k+1}(r) - \delta.$$

By a well-known lemma of H. Cartan
$$\prod_{\mu=1}^{n} |z - a_\mu| > (bR')^n$$
outside circles the sum of whose diameters is less than $4ebR'$. In $|z| \leqslant R'$ and outside the circles
$$|P(z)| = \prod_{\mu=1}^{n} \frac{|z-a_\mu|}{2R'} \geqslant (\tfrac{1}{2}b)^n. \tag{4.18}$$

If b is chosen less than some $b_0(c, B, M)$ (c and B as in the statement of Theorem 2), then it is possible to choose

(i) curves C_k $(k = 1, 2, \ldots, s)$ given by
$$C_k: \quad z = z(t) = te^{i(\alpha_k(t) + \gamma_k)} \quad (r' < t < R')$$
with
$$\tfrac{1}{4}c < \gamma_k < \tfrac{1}{2}c$$
on which (4.18) holds; this follows from Lemma 2;

(ii) a circle $|z| = R_m$ with
$$r_m \leqslant R_m \leqslant \tfrac{3}{2} r_m \tag{4.19}$$
on which (4.18) is satisfied.

By (4.15), (4.16) and (4.8)
$$m(R_m, 1/g;\ J_k(2\delta)) > m(R_m, f(z)/f'(z);\ J_k(2\delta)) > \frac{\varkappa}{3} T(R_m) \tag{4.20}$$
for
$$m > m_0, \ k = k(j, R_m), \ j = 1, 2, \ldots, p.$$

Next we use Lemma 5 and Lemma E to show that $g(z)$ is small on C_k $(k=k(j, R_m))$ and on the arcs $J_k(2\delta)$ of $|z|=R_m$ $(k=k(j, R_m))$.

We note first, by repeating the arguments following (3.19), that the image of D_k by

$$\zeta = \log z + \text{const.}$$

contains a lens Λ whose center line is formed by the vertical segment which is the image of $R_m e^{i\theta}$ $(\theta \in J_k(\delta))$ and whose boundary is formed by the two circular arcs through the endpoints of this segment making a sufficiently small constant angle β with it. We choose $\beta = 1/40B$ and apply Lemma 5 with this value of β and

$$H(\zeta)=g(z), \quad \varepsilon=\delta \quad (\tfrac{1}{2}\delta_1<\delta<\delta_1), \quad M^*=\frac{\varkappa}{3}T(R_m),$$

$$\alpha=\tfrac{1}{2}\{\alpha_{k+1}(R_m)-\alpha_k(R_m)\}-\delta>\tfrac{3}{8}c.$$

This yields
$$\log|g(z)|<-K_5 T(R_m) \quad (z\in \mathcal{B}_m(k)), \tag{4.21}$$

where $\mathcal{B}_m(k)$ is given by

$$z=R_m e^{i\theta}, \quad \theta\in J_k(2\delta), \quad k=k(j,R_m), \tag{4.22}$$

and where the constant K_5 may be chosen as

$$K_5=\frac{2\varkappa}{3\pi^3}\exp\left(-\frac{2\pi}{\beta}\log\left\{\frac{4\pi}{\delta_1}\right\}\right)$$

$(\beta=\beta(B)$ and $\delta_1=\delta_1(\varkappa,\lambda))$.

Next we apply Lemma E, first to the part of D_k in $|z|\geqslant R_m$ then to the part of D_k in $|z|\leqslant R_m$. In both cases $\mathcal{B}_m(k)$ is the arc (4.22) and $\mathcal{C}$ is a portion of the curve C_k. It is easily verified, with the aid of Lemma 1, that for any point ζ on C_k, with

$$e^{-M}R_m\leqslant|\zeta|\leqslant e^{M}R_m, \tag{4.23}$$

we have
$$\varrho(\zeta)>|\zeta|/K_6(c,B). \tag{4.24}$$

From now on, we denote by C_k' the portion of C_k which satisfies the condition (4.23).

By (4.24) and by the B-regularity of C_k

$$\int_{R_m e^{i\theta_m}}^{z}\frac{|d\zeta|}{\varrho(\zeta)}\leqslant BK_6\left|\int_{R_m}^{|z|}\frac{dt}{t}\right|=BK_6\left|\log\frac{|z|}{R_m}\right| \quad (z\in C_k';\ R_m e^{i\theta_m}\in C_k').$$

Therefore, by (4.17), (4.21), Lemma E and the two-constant theorem

$$\log|g(z)| < -\frac{K_5}{2\pi}\exp\left(-4BK_6\left|\log\frac{|z|}{R_m}\right|\right)T(R_m) \quad (z\in C_k',\ k=k(j,r_m)). \tag{4.25}$$

We now deduce from (4.21) and (4.25) similar inequalities with g replaced by f'/f. For the degree n of $P(z)$, we have, by (8)

$$n \leqslant \bar{n}_\delta(R') = o(T(R')).$$

By (4.14) and (4.19)
$$\frac{R'}{R_m} \leqslant \frac{R'}{r_m} \leqslant 3e^M, \tag{4.26}$$

and in view of (4.5)

$$n < o(T(3e^M r_m)) = o(\{3e^M\}^\nu T(r_m)) = o(T(R_m)). \tag{4.27}$$

Combining (4.15), (4.18), (4.26) and (4.27), we obtain

$$\log\left|\frac{f'}{f}\right| \leqslant \log|g(z)| + h\log\{3e^M\} + h\log R_m + o(T(R_m)) \quad (|z| \leqslant R_m' = R'). \tag{4.28}$$

Now (4.21), (4.28) and (4.7) yield

$$\left|\frac{f'(z)}{f(z)}\right| < \exp(-\tfrac{1}{2}K_5 T(R_m)) \quad (m > m_0,\ z\in \mathcal{B}_m(k)). \tag{4.29}$$

Similarly, using (4.25) instead of (4.21), we have

$$\log\left|\frac{f'(z)}{f(z)}\right| < -\frac{K_5}{7}\exp\left(-4BK_6\left|\log\frac{|z|}{R_m}\right|\right)T(R_m) \quad (m>m_0,\ z\in C_k',\ k=k(j,R_m)). \tag{4.30}$$

By (4.6), with $r = R_m$, there must be a point z_1 on $\mathcal{B}_m(k(j,R_m))$ such that

$$|f(z_1) - \tau_j| < \varepsilon$$

for any assigned $\varepsilon(>0)$, provided $m > m_0$.

If z is any other point of $\mathcal{B}_m(k)$, then by integration of (4.29) along $\mathcal{B}_m(k)$, keeping (4.7) in mind,

$$|\log f(z) - \log f(z_1)| < 2\pi R_m \exp\left(-\frac{K_5}{2}T(R_m)\right) = o(1) \quad (m \to +\infty),$$

and so for any assigned $\varepsilon(0<\varepsilon<\tfrac{1}{2})$

$$|f(z) - \tau_j| < 2\varepsilon < 1 \quad (z\in\mathcal{B}_m(k),\ k=k(j,R_m),\ m>m_0). \tag{4.31}$$

By choosing $\varepsilon(>0)$ small enough, we see that the index $k(j, R_m)$ cannot have the same value for different values of j. This proves (10) and also shows that all the p curves C_k lie in distinct sectors S_k. The proof is valid with $M=2$ and hence does not depend on the assumption (4.9).

By integrating $f'(z)/f(z)$ along C'_k $(k=k(j,R_m))$ from the point of intersection z_2 of C'_k with $\mathcal{B}_m(k)$ to the point z and remembering that C'_k is a B-regular curve, we obtain, in view of (4.30),

$$|\log f(z) - \log f(z_2)| < B(e^M R_m - R_m) \exp\left(-\frac{K_5}{7} e^{-4BK_6M} T(R_m)\right) \quad (m > m_0,\ z \in C'_k).$$

Hence, by (4.31) and (4.7) we have, for any assigned $\varepsilon(>0)$,

$$|f(z) - \tau_j| < \varepsilon \quad (m > m_0,\ z \in C'_k,\ k = k(j, R_m)).$$

These inequalities and (4.30) imply

$$\log|f'(z)| < -\frac{K_5}{8} \exp\left(-4BK_6 \left|\log\frac{|z|}{R_m}\right|\right) T(R_m)$$
$$(m > m_0,\ z \in C'_k,\ k = k(j, R_m),\ j = 1, 2, \dots, p). \tag{4.32}$$

We have already seen that the curves C_k do not intersect, since they lie in different sectors S_k. Therefore they divide the annulus (4.3) (with $R=R_m$) into p different domains. Let S^* be a typical one of these domains and let $t\Theta(t)$ be the length of the arc of $|z|=t$ which lies in S^*.

Our aim is to estimate $f'(z)$ in S^* by means of Lemma 6. Let $A_1 = e^{9\pi}$ and let A_2 and $U(>A_1)$ be the quantities which appear in (4.10).

Denote by Γ_1 the part of the boundary of S^* in

$$R_m/U < |z| < UR_m,$$

by Γ_2 the boundary arc of S^* on $|z| = e^M R_m$, by Γ_3 the boundary arc of S^* on $|z| = e^{-M} R_m$ and by Γ_4 the part of the boundary of S^* which does not belong to $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

We denote by $\omega_j(z)$ the harmonic measure of Γ_j with respect to S^* $(j = 1, 2, 3, 4)$.

Then, by Lemma 6,

$$\omega_2(R_m e^{i\theta}) + \omega_3(R_m e^{i\theta}) + \omega_4(R_m e^{i\theta}) < A_2 \exp\left\{-\pi \int_{R_m/U}^{R_m} \frac{dt}{t\Theta(t)}\right\} + A_2 \exp\left\{-\pi \int_{R_m}^{UR_m} \frac{dt}{t\Theta(t)}\right\}$$
$$< 2A_2 U^{-\frac{1}{2}} \quad \left(A_2 = \frac{5e^{4\pi}}{\pi},\ U = 16A_2^2\right)$$

since

$$\Theta(t) \leqslant 2\pi.$$

Hence, in view of (4.10),

$$\omega_1(R_m e^{i\theta}) = 1 - \omega_2 - \omega_3 - \omega_4 > \tfrac{1}{2}. \tag{4.33}$$

Similarly,

$$\omega_3(R_m e^{i\theta}) < A_2 e^{-\frac{1}{2}M} \tag{4.34}$$

and

$$\omega_2(R_m e^{i\theta}) < A_2 \exp\left\{-\pi\int_{R_m}^{e^M R_m} \frac{dt}{t\Theta(t)}\right\}. \tag{4.35}$$

We now show that for at least one of the sectors S^*

$$\pi\int_{R_m}^{e^M R_m} \frac{dt}{t\Theta(t)} \geqslant \tfrac{1}{2} pM. \tag{4.36}$$

For (the index j refers to the p different sectors S^*)

$$p^2 = \left\{\sum_{j=1}^{p} (\Theta_j(t))^{\frac{1}{2}} (\Theta_j(t))^{-\frac{1}{2}}\right\}^2 \leqslant 2\pi \sum_{j=1}^{p} (\Theta_j(t))^{-1},$$

by Schwarz's inequality and the obvious fact that $\sum \Theta_j = 2\pi$. Hence

$$\tfrac{1}{2} p^2 M = \tfrac{1}{2} p^2 \int_{R_m}^{e^M R_m} \frac{dt}{t} \leqslant \sum_{j=1}^{p} \pi \int_{R_m}^{e^M R_m} \frac{dt}{t\Theta_j(t)},$$

which is impossible, unless (4.36) holds for at least one S^*. For such an S^*

$$\omega_2(R_m e^{i\theta}) \leqslant A_2 e^{-\frac{1}{2}pM}. \tag{4.37}$$

On Γ_1 and Γ_4 (4.32) holds, so that

$$\log|f'(z)| < -K_4 T(R_m, f) \quad \left(z \in \Gamma_1,\ K_4 = \frac{K_5}{8} e^{-4BK_6 \log(16A_2^2)}\right), \tag{4.38}$$

$$\log|f'(z)| < 0 \quad (z \in \Gamma_4). \tag{4.39}$$

By Nevanlinna's inequality

$$\sup_{|z|=r} \log|f'(z)| \leqslant \frac{2r+r}{2r-r} m(2r, f') = 3m(2r, f') \quad (r > r_0),$$

and, for non-rational functions of finite order,

$$m(t, f') \leqslant m(t, f) + m(t, f'/f) < \tfrac{4}{3} T(t) \quad (t > r_0).$$

Therefore (in view of $2e^{-M} \leqslant 2e^{-2} < 1$)

$$\log|f'(z)| < 4T(2e^{-M} R_m) < 4T(R_m) \quad (z \in \Gamma_3), \tag{4.40}$$

and $\log|f'(z)| < 4T(2e^M R_m) \leqslant 4T(3e^M r_m) \leqslant (4)(3^\nu)e^{\nu M}T(r_m) < 4^{\lambda+2}e^{\nu M}T(R_m) \quad (z \in \Gamma_2)$, (4.41)

by (4.4), (4.5), (4.19) and the fact that $T(r)$ is an increasing function.

Now a bounded function, harmonic in S^*, with the following boundary values:

$$-K_4 T(R_m) \text{ on } \Gamma_1, \qquad 4^{\lambda+2}e^{\nu M}T(R_m) \text{ on } \Gamma_2,$$
$$4T(R_m) \text{ on } \Gamma_3, \qquad 0 \text{ on } \Gamma_4,$$

dominates the subharmonic function $\log|f'(z)|$ at each point of S^*.

Hence

$$\log|f'(R_m e^{i\theta})| < -\omega_1 K_4 T(R_m) + \omega_2 4^{\lambda+2}e^{\nu M}T(R_m) + 4\omega_3 T(R_m) \quad (R_m e^{i\theta} \in S^*, \ m > m_0).$$

The estimates (4.33), (4.34) and (4.37) now give

$$\log|f'(R_m e^{i\theta})| < \left\{-\frac{K_4}{2} + 4^{\lambda+2}A_2 e^{-M(\frac{1}{2}p-\nu)} + 4A_2 e^{-\frac{1}{2}M}\right\} T(R_m),$$

and hence, in view of (4.11),

$$|f'(R_m e^{i\theta})| < \exp\left\{-\frac{K_4}{4}T(R_m)\right\} \quad (R_m e^{i\theta} \in S^*, \ m > m_0). \tag{4.42}$$

Let ζ_1 and ζ_2 be the endpoints of the arc of $|z| = R_m$ in S^*; then, by choosing adequately $\varepsilon(>0)$, in (4.31), it is obvious that $|f(\zeta_1) - f(\zeta_2)|$ stays above a fixed positive bound (as $m \to \infty$).

On the other hand, by integrating (4.42),

$$|f(\zeta_1) - f(\zeta_2)| \leqslant 2\pi R_m \exp\left(-\frac{K_4}{4}T(R_m)\right)$$

and, in view of (4.7), the right-hand side of this inequality tends to 0 as $m \to +\infty$.

This contradiction shows that $p \leqslant 2\lambda$, since otherwise we could always select a ν satisfying (4.4) and (4.9) and an M satisfying (4.10) and (4.11). We have thus proved (9).

5. Proof of Lemmas 1 and 2

We choose the determination of γ so that $|\gamma/2| \leqslant \frac{1}{2}\pi$ and notice that if $\gamma = 0$ there is nothing to prove. We may therefore assume

$$\varrho = \frac{t|\sin(\gamma/2)|}{B} > 0. \tag{5.1}$$

If the lemma were not true, it would be possible to find $u(\geqslant t_0)$ and $t(\geqslant t_0)$ such that

$$|te^{i(\alpha(t)+\gamma)} - ue^{i\alpha(u)})| < \varrho. \tag{5.2}$$

This implies $$|t-u| < \varrho,$$

and, by the definition of regular curve

$$\Delta = |te^{i\alpha(t)} - ue^{i\alpha(u)}| \leqslant B|t-u| < B\varrho. \tag{5.3}$$

By the triangle inequality, (5.1) and (5.2),

$$\Delta \geqslant |te^{i[\alpha(t)+\gamma]} - te^{i\alpha(t)}| - |te^{i[\alpha(t)+\gamma]} - ue^{i\alpha(u)}| > 2B\varrho - \varrho \quad (\varrho > 0).$$

Since $B \geqslant 1$, this contradicts (5.3) and proves Lemma 1.

To prove Lemma 2, we consider a disc

$$|z - te^{i[\alpha(t)+\Psi]}| \leqslant \eta \quad (t \geqslant t_1 \geqslant t_0) \tag{5.4}$$

and notice that it will not intersect the curve

$$L(\gamma): \quad \zeta(u) = ue^{i[\alpha(u)+\gamma]} \quad (u \geqslant t_0)$$

if the distance d between the center $te^{i[\alpha(t)+\Psi]}$ of (5.4) and $L(\gamma)$ exceeds η,

Hence, in wiew of Lemma 1, there is no intersection unless

$$\eta \geqslant d \geqslant \frac{t\,|\sin \frac{1}{2}(\gamma - \Psi)|}{B} \geqslant \frac{t_1\,|\sin \frac{1}{2}(\gamma - \Psi)|}{B}.$$

Choosing adequately the determination of Ψ this implies

$$\eta \geqslant \frac{t_1|\gamma - \Psi|}{\pi B}. \tag{5.5}$$

The lemma is now obvious since (5.5) restricts the values of γ to an interval of length $2\pi B\eta/t_1$.

6. Proof of Lemma 3

Let $b_1, b_2, \ldots, b_k$ denote the poles of modulus less than one and

$$b_{k+1}, b_{k+2}, \ldots \quad (1 \leqslant |b_{k+1}| \leqslant |b_{k+2}| \leqslant \ldots),$$

the remaining poles of $f^{(q)}(z)$ (each pole being repeated as often as indicated by its multiplicity).

By the Poisson-Jensen formula,

$$2 < |z| < R$$

implies

$$\log|f^{(q)}(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f^{(q)}(Re^{i\varphi})| \frac{R^2-|z|^2}{R^2+|z|^2-2R|z|\cos(\theta-\varphi)}\,d\varphi$$
$$+ k\log\left(\frac{4R}{|z|}\right) + \sum_{1\leq|b_m|\leq R} \log\left|\frac{R^2-z\bar{b}_m}{R(z-b_m)}\right| \quad (\theta=\arg z). \tag{6.1}$$

Let $\mathcal{R}$ denote the union of the discs

$$\mathcal{R}_m:\quad |z-b_m| \leq \frac{|b_m|}{m^2} \quad (m\geq k+1). \tag{6.2}$$

Then, if $\qquad 2<|z|\leq r<R, \quad z\notin\mathcal{R},$

$$\left|\frac{R^2-z\bar{b}_m}{R(z-b_m)}\right| \leq \frac{2R^2m^2}{R|b_m|} \leq \frac{2R[n(R,f^{(q)})]^2}{|b_m|}, \tag{6.3}$$

so that (6.3) and (6.1) imply

$$\log|f^{(q)}(z)| \leq \frac{R+r}{R-r}m(R,f^{(q)}) + k\log(2R) + n(R,f^{(q)})\log 2$$
$$+ 2n(R,f^{(q)})\log n(R,f^{(q)}) + N(R,f^{(q)}). \tag{6.4}$$

Now for $R'>R\geq 1$ and any meromorphic function $g(z)$

$$n(R,g) \leq \frac{R'}{R'-R}\int_R^{R'} \frac{n(u,g)}{u}\,du \leq \frac{R'}{R'-R}N(R',g). \tag{6.5}$$

By Lemma D, with $V(r)=T(r,f^{(q)})$,

$$T(r+r\{\log T(r,f^{(q)})\}^{-2}, f^{(q)}) < eT(r,f^{(q)}) \tag{6.6}$$

provided r lies outside an exceptional set E_1 with

$$mE_1(\varrho, 2\varrho) = o(\varrho) \quad (\varrho\to\infty).$$

Let $\qquad R' = r + r\{\log T(r,f^{(q)})\}^{-2}, \quad R=\tfrac{1}{2}(R'+r).$

Then we obtain from (6.4), (6.5) (with $g=f^{(q)}$) and (6.6)

$$\log|f^{(q)}(z)| \leq AT(r,f^{(q)})\{\log T(r,f^{(q)})\}^3 \quad ((z\notin\mathcal{R},\ r_0\leq|z|\leq r,\ r\notin E_1). \tag{6.7}$$

Seen from the origin, the discs $\mathcal{R}_m$ subtend angles of sum not greater than

$$2\sum_{m=1}^{\infty}\arcsin\left(\frac{1}{m^2}\right) < 2\frac{\pi}{2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^3}{6} < 2\pi.$$

Therefore we can find a ray

$$\arg z = \Psi, \quad r \geqslant r_0 \tag{6.8}$$

which does not intersect $\mathcal{R}$. It is also easily verified that the set E_2 of values of r such that $|z| = r$ intersects $\mathcal{R}$ satisfies

$$mE_2(\varrho, 2\varrho) = o(\varrho) \quad (\varrho \to \infty).$$

If $r \notin E_2$, then we can join $z_1 = re^{i\theta}$ to $z_0 = r_0 e^{i\Psi}$ (same r_0 as in (6.8)) by a path Γ consisting of an arc of the circle $|z| = r$ and part of the ray (6.8).

Now
$$f(z_1) = \frac{1}{(q-1)!}\int_{z_0}^{z_1} (z_1 - \zeta)^{q-1} f^{(q)}(\zeta)\, d\zeta + O(|z_1|^{q-1}),$$

where the integral is taken along Γ. The length of Γ is, at most, equal to $(\pi + 1)r$, and hence (6.7) yields

$$|f(re^{i\theta})| < Ar^q \exp\{AT(r, f^{(q)})(\log T(r, f^{(q)}))^3\} + O(r^{q-1}) \quad (r \notin \{E_1 \cup E_2\};\ r > r_0). \tag{6.9}$$

Since $\log r = o(T(r, f^{(q)}))$, we find, by taking logarithms in (6.9),

$$m(r, f) < AT(r, f^{(q)})(\log T(r, f^{(q)}))^3 \quad (r > r_0,\ r \notin \{E_1 \cup E_2\}). \tag{6.10}$$

Since at every point where $f(z)$ has a pole $f^{(q)}$ has a pole of at least the same order,

$$N(r, f) \leqslant N(r, f^{(q)}) \leqslant T(r, f^{(q)}) \quad (r \geqslant 1). \tag{6.11}$$

The Lemma now follows from (6.10) and (6.11).

7. Proof of Lemma 4

An easy induction on q starting from

$$\frac{f''}{f} = D\frac{f'}{f} + \left(\frac{f'}{f}\right)^2 \quad \left(D = \frac{d}{dz}\right)$$

shows that $f^{(q+1)}/f$ is expressible as a polynomial in f'/f and its successive derivatives $D^k(f'/f)$ $(k = 1, 2, \dots, q)$. The coefficients of the polynomial are integers depending on q only. It is therefore enough to prove

$$|D^k(f'/f)| < K_7(q)\left(\frac{HR'\,T(R', f)}{R' - r}\right)^{K_8(q)} \quad (k = 0, 1, 2, \dots, q;\ z \notin \mathcal{R}(H),\ r_0 < |z| \leqslant r < R'). \tag{7.1}$$

There is nothing to prove, if $f(z)$ is a constant. We may therefore suppose $T(r, f)$ unbounded.

By $(k+1)$ differentiations of the Poisson-Jensen formula for $\log f(z)$ we find [8; p. 222], for $|z| \leqslant r < R$,

$$\left| D^k \frac{f'}{f}(z) \right| \leqslant k! \sum_{|d_m| \leqslant R} \left\{ \frac{1}{|z-d_m|^{k+1}} + \frac{1}{(R-r)^{k+1}} \right\} + \frac{(k+1)!\, 2R}{(R-r)^{k+2}} \{m(R,f) + m(R,1/f)\}.$$

Now, if $z \notin \mathcal{R}(H)$, the typical term in the sum on the right hand side is less than

$$m^{2k+2} H^{k+1} + \frac{1}{(R-r)^{k+1}} < 2 \frac{H^{k+1}(n(R))^{2k+2} R^{k+1}}{(R-r)^{k+1}} \qquad (R \geqslant 1),$$

where $n(R) = n(R,f) + n(R,1/f)$. The number of terms in the sum is $n(R)$. Therefore,

$$\left| D^k \frac{f'}{f}(z) \right| < \frac{2(k!)\, H^{k+1} R^{k+1} (n(R))^{2k+3}}{(R-r)^{k+1}} + \frac{(k+1)!\, 2R}{(R-r)^{k+2}} \{m(R,f) + m(R,1/f)\}. \tag{7.2}$$

Since $R \geqslant 1$, $H \geqslant 1$ and $\{R/(R-r)\} > 1$, (7.2) implies

$$\left| D^k \frac{f'(z)}{f(z)} \right| < \frac{R^{q+2}}{(R-r)^{q+2}} \{2(q!)\, H^{q+1} \{n(R)\}^{2q+3} + (q+1)!\, 2[2T(R,f) + O(1)]\}$$

$$(k = 0, 1, 2, \ldots, q). \tag{7.3}$$

We choose now $$R = \tfrac{1}{2}(r + R')$$

and estimate $n(R)$ by $N(R',f) + N(R',1/f)$, using (6.5). This yields

$$n(R) < \frac{A R' \, T(R',f)}{R' - r} \qquad (r > r_0). \tag{7.4}$$

Using (7.4) in (7.3), we obtain (7.1).

8. Proof of Lemma 5

The function $$u + iv = w = \Omega(z) = \left\{ \frac{i\alpha + z}{i\alpha - z} \right\}^{\frac{\pi}{\beta}} \tag{8.1}$$

maps the interior of Λ on $\quad |\arg w| < \pi, \quad |w| > 0.$

The interval $$-\alpha + \varepsilon < y < \alpha - \varepsilon$$

of the y-axis is mapped on the interval

$$u_1 < u < 1/u_1$$

of the u-axis, where $$u_1 = \left\{ \frac{\varepsilon}{2\alpha - \varepsilon} \right\}^{\frac{\pi}{\beta}} < 1.$$

Let
$$\Psi(w) = H(\Omega^{-1}(w)),$$
$$\log|H(z)| = \log|\Psi(w)| = \Phi(w),$$

where $\Psi(w)$ is regular in each of the half-planes $v > 0$ and $v < 0$. Moreover, $\Psi(w)$ is continuous and bounded in $v \geqslant 0$ as well as in $v \leqslant 0$. Under our assumptions we also have
$$\Phi(w) \leqslant 0 \quad (-\pi \leqslant \arg w \leqslant +\pi). \tag{8.2}$$

As an immediate consequence of the Poisson-Jensen formula for a half-plane [2; p. 93], we have, for $v > 0$, or $v < 0$
$$\Phi(iv) \leqslant \frac{|v|}{\pi}\int_0^\infty \Phi(u)\frac{du}{u^2+v^2}. \tag{8.3}$$

In the right-hand side of (8.3) we have omitted an integral involving $\Phi(ue^{i\pi})$ or $\Phi(ue^{-i\pi})$; this is possible in view of (8.2).

By (8.2) and (8.3)
$$\Phi(iv) \leqslant \frac{|v|}{\pi}\int_{u_1}^{1/u_1} \Phi(u)\frac{du}{u^2+v^2}.$$

Expressing $\Phi(u)$ and du in terms of y, by means of (8.1),
$$\Phi(iv) \leqslant \frac{2\alpha|v|}{\beta}\int_{-\alpha+\varepsilon}^{\alpha-\varepsilon} \log|H(iy)|\frac{u}{u^2+v^2}\cdot\frac{dy}{(\alpha^2-y^2)}. \tag{8.4}$$

In $u_1 < u \leqslant 1$
$$\frac{u}{u^2+v^2} > \frac{u_1}{1+v^2},$$

and in $1 \leqslant u < 1/u_1$
$$\frac{u}{u^2+v^2} = \frac{1/u}{1+(v/u)^2} > \frac{u_1}{1+v^2}.$$

Hence (8.4) implies
$$\Phi(iv) \leqslant \frac{2\alpha u_1|v|}{\beta(1+v^2)}\int_{-\alpha+\varepsilon}^{\alpha-\varepsilon} \log|H(iy)|\frac{dy}{\alpha^2},$$

(because $\log|H(iy)| \leqslant 0$) and in view of (2.3),
$$\Phi(iv) \leqslant -\frac{2u_1 M^*}{\alpha\beta}\frac{|v|}{1+v^2} \quad (-\infty < v < +\infty). \tag{8.5}$$

The Poisson-Jensen formula for the half-plane $u > 0$ now yields
$$\Phi(t) \leqslant \frac{t}{\pi}\int_{-\infty}^{+\infty} \Phi(iv)\frac{dv}{v^2+t^2} \leqslant -\frac{4u_1 M^*}{\pi\alpha\beta}\int_0^\infty \frac{tv\,dv}{(1+v^2)(t^2+v^2)} \quad (t > 0). \tag{8.6}$$

Observing that, for $t \neq 1$

$$I = \int_0^\infty \frac{v\,dv}{(1+v^2)(t^2+v^2)} = \frac{1}{2(t^2-1)} \int_0^\infty \left\{ \frac{1}{1+v^2} - \frac{1}{t^2+v^2} \right\} d(v^2)$$

we obtain
$$I = \frac{\log t^2}{2(t^2-1)},$$

which, properly interpreted, is also valid for $t=1$.

Using this result in (8.6), we find

$$\Phi(t) \leqslant -\frac{2u_1 M^*}{\pi\alpha\beta} \left[\frac{t \log t^2}{t^2-1} \right] = -\frac{2u_1 M^*}{\pi\alpha\beta} \left[\frac{\log t - \log(1/t)}{t-(1/t)} \right] \quad (t>0).$$

For
$$u_1 < t < 1/u_1,$$

an application of the mean value theorem of the differential calculus now gives

$$\Phi(t) \leqslant -\frac{2u_1^2 M^*}{\pi\alpha\beta} = -\frac{2M^*}{\pi\alpha\beta} \left(\frac{\varepsilon}{2\alpha-\varepsilon} \right)^{2\pi/\beta} \leqslant -\frac{2M^*}{\pi\alpha\beta} \left(\frac{\varepsilon}{2\alpha} \right)^{2\pi/\beta},$$

which is the assertion of Lemma 5.

9. Proof of Lemma 6

Let S be the (open) curvilinear sector (extending from 0 to ∞) which contains D and is bounded by the curves (2.5).

Let Σ be the part of S in $|z| < t_2$ and let C be an arc of its boundary defined by

$$C: \quad |z| = t_2, \quad \alpha_1(t_2) < \arg z < \alpha_2(t_2).$$

We map S, by
$$s = \log z = \log t + i\theta,$$

onto a region Ω to which we shall apply Ahlfors' distortion theorem.

Let
$$w = u + iv = \varphi(s) = \varphi(\log z) = \Phi(z) = U(z) + iV(z) \tag{9.1}$$

map Ω conformally on the strip

$$-\infty < u < +\infty, \quad -\frac{\pi}{2} < v < \frac{\pi}{2},$$

in such a way that $U(z) \to -\infty$ as $|z| \to 0$ and $U(z) \to +\infty$ as $|z| \to +\infty$.

Put
$$U_2 = \inf_{z \in C} U(z).$$

By Ahlfors' theorem [1; p. 10] and the definition (2.6), we see that if

$$0 < t < t_2 < +\infty,$$

and if
$$\int_{\log t}^{\log t_2} \frac{d\sigma}{\Theta(e^\sigma)} = \int_t^{t_2} \frac{d\tau}{\tau\Theta(\tau)} > 2, \tag{9.2}$$

then
$$U_2 - U(te^{i\psi}) \geqslant \pi \int_t^{t_2} \frac{d\tau}{\tau\Theta(\tau)} - 4\pi \tag{9.3}$$

for
$$te^{i\psi} \in \Sigma.$$

By (2.7) and (2.8)

$$\int_r^{t_2} \frac{d\tau}{\tau\Theta(\tau)} \geqslant \frac{1}{2\pi} \int_r^{t_2} \frac{d\tau}{\tau} = \frac{1}{2\pi} \log(t_2/r) \geqslant \frac{9}{2}. \tag{9.4}$$

This shows that (9.2) is satisfied with $t = r$ and hence (9.3) is valid with $te^{i\psi} = re^{i\theta}$. We thus have

$$U_2 - U(re^{i\theta}) \geqslant \tfrac{1}{2}\pi, \tag{9.5}$$

and
$$U(re^{i\theta}) - U_2 \leqslant 4\pi - \pi \int_r^{t_2} \frac{d\tau}{\tau\Theta(\tau)}. \tag{9.6}$$

Two applications of Carleman's principle [8; p. 69] show that

$$\omega_2(z, t_2) < \omega(z, C; \Sigma),$$

where $\omega(z, C; \Sigma)$ is the harmonic measure of C with respect to Σ, at the point $z = re^{i\theta}$. By the invariance of harmonic measure under conformal mapping

$$\omega(z, C; \Sigma) = \omega(U(z) + iV(z),\ \Phi(C);\ \Phi(\Sigma)),$$

where $\Phi(C)$ and $\Phi(\Sigma)$ denote the images of C and Σ under the mapping $w = \Phi(z)$ given by (9.1). A further application of Carleman's principle shows, in view of (9.5), that

$$\omega(U(z) + iV(z),\ \Phi(C);\ \Phi(\Sigma)) < \tilde{\omega}(U(z) + iV(z)),$$

where $\tilde{\omega}(w)$ is the harmonic measure of the boundary segment

$$u = U_2, \quad -\tfrac{1}{2}\pi < v < \tfrac{1}{2}\pi, \tag{9.7}$$

with respect to the semi-infinite strip Z

$$Z: \quad u < U_2, \quad -\tfrac{1}{2}\pi < v < \tfrac{1}{2}\pi.$$

The function
$$\zeta = \xi + i\eta = e^{w - U_2}$$

maps the closure of Z on the closure of the semi-disc Z',

$$Z': \quad |\zeta|<1, \quad \xi>0,$$

in such a way that the circular boundary

$$|\zeta|=1, \quad \xi>0 \tag{9.8}$$

corresponds to (9.7). It is easily verified that, at $\zeta(\in Z')$, the harmonic measure of the arc (9.8), with respect to Z', is given by

$$\operatorname{Re}\left\{2-\frac{2}{\pi i}\log\frac{\zeta+i}{\zeta-i}\right\}=2\left(1-\frac{1}{\pi}\arg\frac{\zeta+i}{\zeta-i}\right)=2\left(1-\frac{\chi}{\pi}\right),$$

where χ is the angle subtended at ζ by the line-segment

$$\xi=0, \quad -i\leqslant\eta\leqslant i.$$

Hence using again the invariance of harmonic measure under conformal transformation,

$$\begin{aligned}\tilde{\omega}(U+iV)&=2-\frac{2}{\pi}\arctan\left\{\frac{1+\eta}{\xi}\right\}-\frac{2}{\pi}\arctan\left\{\frac{1-\eta}{\xi}\right\}\\&=\frac{2}{\pi}\left[\arctan\left\{\frac{\xi}{1+\eta}\right\}+\arctan\left\{\frac{\xi}{1-\eta}\right\}\right]\leqslant\frac{2}{\pi}\left(\frac{\xi}{1+\eta}+\frac{\xi}{1-\eta}\right)\\&=\frac{4\xi}{\pi(1-\eta^2)}=\frac{4e^{U-U_2}\cos V}{\pi(1-e^{2(U-U_2)}\sin^2 V)}\leqslant\frac{4e^{U-U_2}}{\pi[1-e^{2(U-U_2)}]}\qquad(U+iV=U(z)+iV(z)).\end{aligned}$$

Using (9.5) in the denominator and (9.6) in the numerator, we obtain

$$\omega_2(z,t_2)<\tilde{\omega}(U(z)+iV(z))\leqslant\frac{4e^{4\pi}}{\pi(1-e^{-\pi})}\exp\left\{-\pi\int_r^{t_2}\frac{d\tau}{\tau\Theta(\tau)}\right\},$$

which implies (2.9). The proof of (2.10) is similar.

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