On the small-time behavior of subordinators

SHAUL K. BAR-LEV[1], ANDREAS LÖPKER[2] and WOLFGANG STADJE[3]

[1]*Department of Statistics, University of Haifa, Haifa 31905, Israel. E-mail: barlev@stat.haifa.ac.il*
[2]*Department of Economics and Social Sciences, Helmut Schmidt University Hamburg, 22043 Hamburg, Germany. E-mail: lopker@hsu-hh.de*
[3]*Department of Mathematics and Computer Science, University of Osnabrück, 49069 Osnabrück, Germany. E-mail: wolfgang@mathematik.uos.de*

We prove several results on the behavior near $t = 0$ of Y_t^{-t} for certain $(0, \infty)$-valued stochastic processes $(Y_t)_{t>0}$. In particular, we show for Lévy subordinators that the Pareto law on $[1, \infty)$ is the only possible weak limit and provide necessary and sufficient conditions for the convergence. More generally, we also consider the weak convergence of $tL(Y_t)$ as $t \to 0$ for a decreasing function L that is slowly varying at zero. Various examples demonstrating the applicability of the results are presented.

Keywords: Pareto law; regular variation; subordinator; weak limit theorem

1. Introduction

We consider the behavior near $t = 0$ of a stochastic process $(Y_t)_{t>0}$ with values in $(0, \infty)$. Let $F_t(y) = \mathbb{P}(Y_t \le y)$ and $\psi_t(u) = \mathbb{E}(e^{-uY_t})$ be the distribution function and the Laplace–Stieltjes transform (LST) of Y_t and let $\xrightarrow{d}$ denote convergence in distribution.

We start with the following observation from [1], which is not difficult to prove. It states that the convergence of Y_t^{-t} to some nondegenerate random variable (r.v.) with distribution function F^* is equivalent to the weak convergence of the distribution function $u \mapsto 1 - \psi_t(u^{1/t})$ to F^*.

Proposition 1.1 (see [1]). *Assume that $F_t(0) = 0$ for all t and let Y^* be a r.v. with distribution function F^* which is not concentrated at one point. Then $Y_t^{-t} \xrightarrow{d} Y^*$ as $t \to 0$ if and only if $\psi_t(u^{1/t}) \to 1 - F^*(u)$ as $t \to 0$ at all continuity points u of F^*.*

In [1], the applicability of Proposition 1.1 to various examples was demonstrated. In these examples, the limiting distribution F^* turned out to be either a Pareto law with support $[1, \infty)$, or a mixture of such a Pareto law and a point mass at 1, or an exponential law (possibly shifted to the right). In general, any distribution on $(0, \infty)$ can occur as F^* (take $Y_t = (Y^*)^{-1/t}$), but it is a challenging question which F^* appear as limits of 'reasonable' processes Y_t^{-t}.

In this paper, we study the case when $\psi_t(u) = \psi(u)^t$ for some LST ψ; of course this means that F is infinitely divisible, and the $(0, \infty)$-valued process $(Y_t)_{t\ge0}$ (with $Y_0 \equiv 0$) can then be interpreted as an increasing Lévy process (a *subordinator*) with Laplace exponent $\Phi(u)$ defined by $\mathbb{E}(e^{-uY_t}) = e^{-t\Phi(u)}$.

In [2], it is proved that for a subclass of exponential dispersion models (cf. [7]) generated by an infinitely divisible probability measure μ on $[0, \infty)$ and associated with an unbounded Lévy

measure ν satisfying $\nu((x, \infty)) \sim -\gamma \log x$ as $x \to 0$, the limit F^* is a Pareto type law supported on $[1, \infty)$. Our main result below shows that this is indeed the only limit law that can occur for any subordinator. We also give several necessary and sufficient conditions for this convergence to occur. Combining subordinators and fixed r.v.'s one obtains mixtures of a Pareto law and the point mass at 1.

The results presented in this paper enable an approximation of the distribution of Y_t for relatively small values of t. Note that while the distribution of Y_t can be quite complex, the specific limiting Pareto law is rather simple to handle. Such numerical approximation aspects for various distributions F_t are subject of future investigations.

This paper is organized as follows. Some preliminary results are presented in Section 2. Under rather mild conditions on the behavior of ψ_t (which are satisfied for subordinators), it is shown that $Y^* \geq 1$ almost surely. Some other straightforward results concerning the limiting behavior of products and sums of stochastic processes are also presented. In Section 3, we present necessary and sufficient criteria for the convergence $Y_t^{-t} \xrightarrow{d} Y^*$ for subordinators in terms of their characteristics. We also provide an alternative proof of the result of [2]. Section 4 presents several applications. In Section 5, we consider the problem under which conditions $t L(Y_t)$ converges weakly as $t \to 0$, if L is a slowly varying decreasing function with $\lim_{x \to 0} L(x) = \infty$. Clearly, our original question concerns the special case $L(x) = -\log x$.

2. Preliminary results

Our first result deals with the limiting variable Y^*. Under a suitable monotonicity condition on ψ_t, it follows easily that $Y^* \geq 1$.

Proposition 2.1. *Suppose that $Y_t^{-t} \xrightarrow{d} Y^*$ and that there are $s > 0$ and $y > 0$ such that $\psi_t(u)$ is decreasing in t for $t \in [0, s]$ and $u \in [0, y]$. Then $Y^* \geq 1$ almost surely.*

Proof. Let $u < 1$. Then $u^{1/t}$ converges to 0 as $t \to 0$ and if $t > \max\{s, \log u / \log y\}$ then $1 \geq \psi_t(u^{1/t}) \geq \psi_s(u^{1/t})$ and $\psi_s(u^{1/t})$ converges to 1 as $t \to 0$. Since additionally $\psi_t(u^{1/t}) \to 1 - F^*(u)$, it follows that $F^*(u) = 0$ for all $0 \leq u < 1$. $\qquad\square$

Example 1 (Stable densities). Let $\psi_t(u) = \exp(-au^t)$, $a > 0$, be the LST of the positive stable density of type $t \in (0, 1)$. The family $(\psi_t)_{t>0}$ does not satisfy the condition stated in Proposition 2.1. In fact, $\psi_t(u^{1/t}) = 1 - e^{-au}$ for all t so that F^* is the exponential distribution with mean $1/a$, whose support is $[0, \infty)$. Consider however the distributions belonging to the natural exponential families generated by these positive stable densities (with canonical parameter $\theta > 0$). They have LST's $\psi_t(u; \theta) = \exp\{-a[(\theta + u)^t - \theta^t]\}$ and thus satisfy the condition of Proposition 2.1; in this case it is easily checked that F^* has the shifted exponential density $ae^{-a(x-1)}1_{(1,\infty)}(x)$; see also Example 2.5(iii) in [1].

Proposition 2.2. *Let* $(Y_{i,t})_{t>0}$, $i \in \{1, 2\}$, *be two independent families of positive r.v.'s. If* $Y_{i,t}^{-t} \xrightarrow{d} Y_i^*$ *as* $t \to 0$ *for* $i = 1, 2$, *then*

$$(Y_{1,t} Y_{2,t})^{-t} \xrightarrow{d} Y_1^* Y_2^* \quad and \quad (Y_{1,t} + Y_{2,t})^{-t} \xrightarrow{d} \min\{Y_1^*, Y_2^*\}.$$

Proof. The convergence $(Y_{1,t} Y_{2,t})^{-t} \xrightarrow{d} Y_1^* Y_2^*$ is trivial. To prove the second assertion, note that

$$\psi_{1,t}(u^{1/t})\psi_{2,t}(u^{1/t}) \to \mathbb{P}(Y_1^* > u)\mathbb{P}(Y_2^* > u) = \mathbb{P}(\min\{Y_1^*, Y_2^*\} > u)$$

for every u that is a common point of continuity of the functions $u \mapsto \mathbb{P}(Y_i^* > u)$, $i = 1, 2$. But $\psi_{1,t}\psi_{2,t}$ is the LST of the sum $Y_{1,t} + Y_{2,t}$ and the result follows immediately from Proposition 1.1. $\qquad\square$

In particular, suppose that $a_t, b_t > 0$ are positive functions with $a_t \sim a^{-1/t}$ and $b_t \sim b^{-1/t}$ as $t \to 0$, with some constants $a, b > 0$. Then $Y_t^{-t} \xrightarrow{d} Y^*$ implies

$$(a_t Y_t + b_t)^{-t} \xrightarrow{d} \min\{aY^*, b\}.$$

3. Small-time behavior of Lévy subordinators

3.1. The main result

Let $(Y_t)_{t \geq 0}$ be a subordinator, that is, an increasing Lévy process with $Y_0 \equiv 0$ (see Chapter III in [3]). We assume that Y_t has no drift, so that the process has the Lévy–Khintchine-representation $\psi_t(u) \equiv \mathbb{E}(e^{-uY_t}) = e^{-t\Phi(u)}$, where the Laplace exponent Φ is given by

$$\Phi(u) = \int_0^\infty (1 - e^{-ux}) \, d\nu(x).$$

Here ν is the Lévy measure with support $[0, \infty)$, satisfying $\overline{\nu}(x) \equiv \int_x^\infty d\nu(u) < \infty$ and $\int_0^x u \, d\nu(u) < \infty$ for all $x > 0$. In what follows, we write $Y = Y_1$ and $F(x) = \mathbb{P}(Y \leq x)$.

It is known that a driftless subordinator Y_t tends to zero sub-linearly as $t \to 0$, that is, almost surely, Y_t/t tends to zero as $t \to 0$ (Proposition 8 in [3]). Moreover, if $h(t)$ is an increasing function such that $h(t)/t$ is also increasing, then (see [3], Theorem 9)

$$\text{either} \quad \lim_{t \to 0} \frac{Y_t}{h(t)} = 0 \quad \text{a.s.} \quad \text{or} \quad \limsup_{t \to 0} \frac{Y_t}{h(t)} = \infty \quad \text{a.s.}$$

Lévy processes in general possess the small-time ergodic property

$$\lim_{t \to 0} t^{-1}\mathbb{E}(f(Y_t)) = \int f(x) \, d\nu(x)$$

for bounded continuous functions f vanishing in a neighborhood of the origin ([11], Corollary 8.9). Letting $P_t f(x) = \mathbb{E}(f(Y_t)|Y_0 = x)$ and A the infinitesimal generator of the Markov process Y_t, this is nothing else than saying that $P_t f(0) \approx f(0) + t \cdot A f(0)$ as $t \to 0$.

We investigate the limiting behavior of Y_t^{-t} as $t \to 0$. Since $\psi_t(u) = \psi(u)^t$ is decreasing in t for fixed $u < 1$, it is an immediate consequence of Proposition 2.1 that the limit in distribution, if it exists, will be concentrated on $[1, \infty)$. The Pareto law $\mathcal{P}_\gamma$ with parameter $\gamma > 0$ has the distribution function

$$\Pi_\gamma(x) = (1 - x^{-\gamma}) 1_{[1,\infty)}(x).$$

Theorem 3.1. *Let Y^* be a positive r.v. which is not concentrated at one point. Let $F^*(x) = \mathbb{P}(Y^* \le x)$ be its distribution function. Then the following statements are equivalent*:

(S1) $Y_t^{-t} \overset{d}{\to} Y^*$ *as $t \to 0$.*
(S2) $t \Phi(u^{1/t}) \to -\log(1 - F^*(u))$ *as $t \to 0$, for all continuity points u of F^*.*
(S3) $Y_t^{-t} \overset{d}{\to} \mathcal{P}_\gamma$ *as $t \to 0$ for some $\gamma > 0$.*

Furthermore, for any $\gamma > 0$ the following statements are equivalent:

(S4) $Y_t^{-t} \overset{d}{\to} \mathcal{P}_\gamma$ *as $t \to 0$.*
(S5) $\Phi(s)/\log s \to \gamma$ *as $s \to \infty$.*
(S6) $\log F(x)/\log x \to \gamma$ *as $x \to 0$.*
(S7) $\overline{v}(x)/\log x \to -\gamma$ *as $x \to 0$.*

If $F(x)$ is absolutely continuous near the origin, that is, if there is a measurable function $f(x)$ such that $F(x) = \int_0^x f(u) \, du$ for all $x \ge 0$ in a neighborhood of the origin, then

(S8) $\log f(x)/\log x \to \gamma - 1$ *as $x \to 0$*

implies (S4)–(S7). *If additionally the density f is monotone near the origin then* (S8) *is equivalent to* (S4)–(S7).

Proof. (S3) $\Rightarrow$ (S1) is obvious.

(S1) $\Leftrightarrow$ (S2). This is clearly equivalent to Proposition 1.1.

(S2) $\Rightarrow$ (S3). Suppose that $t \Phi(e^{z/t}) \to -\log(1 - F^*(e^z))$ for all continuity points e^z of F^*. We know already that $F^*(e^z) = 0$ for $z < 0$. Moreover, if $z > 0$ then

$$\frac{\Phi(e^{z/t})}{z/t} \to -\frac{\log(1 - F^*(e^z))}{z} \tag{3.1}$$

for all continuity points e^z of F^*. Now, the key observation is that the latter limit is necessarily the same for all $z > 0$. Indeed, the left-hand side of (3.1) has the form $h(z/t)$ for some function h so that if (3.1) holds for some $z > 0$ then for arbitrary $z' > 0$ we get, setting $t' = (z/z')t$,

$$\lim_{t \to 0} \frac{\Phi(e^{z'/t})}{z'/t} = \lim_{t' \to 0} \frac{\Phi(e^{z/t'})}{z/t'} = -\frac{\log(1 - F^*(e^z))}{z}.$$

Denote the limit in (3.1) by γ. As F^* attains a value in $(0, 1)$, we have $\gamma \in (0, \infty)$. Then $F^*(e^z) = 1 - e^{-\gamma z}$, that is, $F = \Pi_\gamma$. This completes the proof of the equivalence of (S1)–(S3).

(S4) $\Leftrightarrow$ (S5). This follows by setting $s = e^{z/t}$ in (3.1).

(S6) $\Rightarrow$ (S5). For every $s \geq 0$ and every $z \geq 0$, we have the decomposition

$$\psi(s) = s \int_0^\infty e^{-sx} F(x) \, dx = \int_0^z e^{-x} F(x/s) \, dx + \int_z^\infty e^{-x} F(x/s) \, dx. \tag{3.2}$$

Consequently, $\psi(s) \leq F(z/s) \int_0^z e^{-x} \, dx + \int_z^\infty e^{-z} \, dx = F(z/s)(1 - e^{-z}) + e^{-z}$ and $\psi(s) \geq e^{-z} \int_0^z F(x/s) \, dx + F(z/s)e^{-z}$, yielding the inequalities

$$F(z/s)e^{-z} \leq \psi(s) \leq F(z/s)(1 - e^{-z}) + e^{-z}. \tag{3.3}$$

Since we assume (S6), we have $F(x) = x^{\gamma + o(1)}$ as $x \to 0$. Letting z be a constant on the left-hand side of (3.3), we see that $\psi(s) \geq s^{-\gamma + o(1)}$ as $s \to \infty$. Moreover, by choosing $z = z(s) = (\log s)^2$ on the right-hand side of (3.3) we obtain $z(s) = s^{o(1)}$ and $s^\delta = o(e^{z(s)})$ for any $\delta > 0$, as $s \to \infty$, so that by (3.3)

$$\psi(s) \leq F\left(s^{o(1)-1}\right) + e^{-(\log s)^2} = s^{(o(1)-1)(\gamma + o(1))} + s^{-\log s}.$$

Hence, $\psi(s) = s^{-\gamma + o(1)}$ as $s \to \infty$, which is tantamount to (S5).

(S5) $\Rightarrow$ (S6). By letting $x = z/s$, it follows from (3.3) that

$$\frac{e^z \psi(z/x) - 1}{e^z - 1} \leq F(x) \leq \psi(z/x)e^z. \tag{3.4}$$

Now suppose that $\log \psi(s)/\log s \to -\gamma$ as $s \to \infty$. Letting z be constant on the right-hand side, we obtain that $F(x) \leq x^{\gamma + o(1)}$. Then, by choosing $z(s) = \sqrt{\log s}$, we see that $F(x) \geq x^{\gamma + o(1)}$ and hence $F(x) = x^{\gamma + o(1)}$.

(S5) $\Rightarrow$ (S7). This follows from Lemma 5.17(ii) in [9].

(S7) $\Rightarrow$ (S5). Suppose that $\overline{\nu}(x)/\log x \to -\gamma$ as $x \to 0$. Applying integration by parts, we can write Φ as an ordinary Laplace transform:

$$\Phi(s) = \int_0^\infty e^{-x} \overline{\nu}(x/s) \, dx = \int_0^K e^{-x} \overline{\nu}(x/s) \, dx + \int_K^\infty e^{-x} \overline{\nu}(x/s) \, dx$$
$$= I_K(s) + J_K(s), \qquad \text{say, for every } K > 0. \tag{3.5}$$

Fix an arbitrary $\varepsilon > 0$. By assumption, there is an $s_{K,\varepsilon} > K$ such that $-\gamma - \varepsilon \leq \overline{\nu}(x/s)/\log(x/s) \leq -\gamma + \varepsilon$ for all $x \in (0, K]$ and all $s \geq s_{K,\varepsilon}$. Hence,

$$\int_0^K e^{-x} \overline{\nu}(x/s) \, dx \leq \int_0^K e^{-x} |\log(x/s)|(\gamma + \varepsilon) \, dx$$
$$\leq (\gamma + \varepsilon)\left[\int_0^K e^{-x} \log s \, dx - \int_0^K e^{-x} \log x \, dx\right], \qquad s > s_{K,\varepsilon}.$$

Clearly, $\int_0^K e^{-x}|\log x|\,dx < \infty$. Therefore,

$$\limsup_{s\to\infty} I_K(s)/\log s \le (\gamma + \varepsilon) \int_0^K e^{-x}\,dx \le \gamma + \varepsilon$$

for every $\varepsilon > 0$. Thus,

$$\limsup_{s\to\infty} I_K(s)/\log s \le \gamma. \tag{3.6}$$

For $x > K$ we have $\overline{v}(x/s) \le \overline{v}(K/s)$ so that $J_K(s) \le \overline{v}(K/s)\int_K^\infty e^{-x}\,dx$, yielding

$$\limsup_{s\to\infty} J_K(s)/\log s \le \gamma e^{-K} \qquad \text{for every } K > 0. \tag{3.7}$$

Letting $K \to \infty$ we obtain from (3.5)–(3.7) that

$$\limsup_{s\to\infty} \Phi(s)/\log s \le \gamma.$$

The relation $\liminf_{s\to\infty} \Phi(s)/\log s \ge \gamma$ follows along similar lines. Altogether this proves (S5).

(S8) $\Rightarrow$ (S6). Assume that F is absolutely continuous around the origin with a density f satisfying $\log f(x)/\log x \to \gamma - 1$ as $x \to 0$. Then, given an arbitrary $\varepsilon \in (0, \gamma)$, we have $f(x) \le x^{\gamma-1-\varepsilon}$ for all sufficiently small $x > 0$. Thus,

$$F(x) = \int_0^x f(u)\,du \le \frac{x^{\gamma-\varepsilon}}{\gamma - \varepsilon},$$

which implies $F(x) \le x^{\gamma-(\varepsilon/2)}$ for sufficiently small x. Similarly, it follows that $F(x) \ge x^{\gamma+(\varepsilon/2)}$ for small x, so that indeed $\lim_{x\to0} \log F(x)/\log x = \gamma$.

(S6) $\Rightarrow$ (S8). Finally, suppose that $\lim_{x\to0} \log F(x)/\log x = \gamma$ and F has a monotone density f near 0. First, let f be nondecreasing at 0. Given an arbitrary $\varepsilon > 0$, we obtain

$$f(x) \ge \frac{1}{x}\int_0^x f(u)\,du = \frac{F(x)}{x} \ge x^{\gamma-1+\varepsilon} \qquad \text{for small } x.$$

Similarly,

$$f(x) \le \frac{\int_x^{2x} f(u)\,du}{x} \le \frac{F(2x)}{x} \le 2^{\gamma-(\varepsilon/2)} x^{\gamma-1-(\varepsilon/2)} \qquad \text{for small } x,$$

and the right-hand side is ultimately $\le x^{\gamma-1-\varepsilon}$ as $x \to 0$. If f is nonincreasing near zero we can interchange $\le$ and $\ge$ in the last inequalitites. $\qquad\square$

Remark 1. The implication "(S6) for some $\gamma > 0 \Rightarrow$ (S1)" was already shown in [1].

Remark 2. Some of the above equivalences have counterparts in the theory of regularly varying functions. In particular (S5)$\Leftrightarrow$(S6) has the classical form of a Tauberian theorem of the type of

Theorem 8.1.7 in [4]. However, the functions there are regularly varying while our functions are of type $x^\gamma L(x)$ with some $L(x) = x^{o(1)}$. Note that the class of regularly varying functions is a subclass of the class investigated here. The extra smoothness conditions in the Karamata theory come with the reward of being able to conclude from $f(x) = x^\gamma L(x)$, with L slowly varying, that the Laplace transform of f is of the same form $s^{-\gamma} L'(1/s)$ with a precisely determined function L'. As opposed to this, in our situation the exact form of the $x^{o(1)}$ terms remain unknown, but are not needed anyway. It is also worth mentioning that for regularly varying functions the implication (S8)$\Rightarrow$(S4)–(S7) follows from the monotone density theorem (Theorem 8.1.8 in [4]).

Remark 3. Among the possible limits of Y_t^{-t} as t tends to zero is the somewhat uninteresting limit 1, which is excluded from Theorem 3.1. Loosely speaking, this is the $\gamma = \infty$ case of the theorem. We refrain from stating the corresponding result here.

Remark 4. The theorem shows that the Pareto distribution is the only possible limit distribution of Y_t^{-t} as $t \to 0$. This can alternatively be deduced as follows. Note that since Y_t is a Lévy process, $Y_t \overset{d}{\sim} \sum_{k=1}^n Y_{k,t/n}$ for any n, where $(Y_{i,.})_{i=1,2,\dots,k}$ are i.i.d. copies of the process $Y_{..}$. It follows from Proposition 2.2 that if $Y_t^{-t} \overset{d}{\to} Y^*$ and $Y_{k,t}^{-t} \overset{d}{\to} Y_k^*$, then (taking the limits Y_k^* to be independent)

$$Y_t^{-t/n} \overset{d}{\sim} \left(\sum_{k=1}^n Y_{k,t/n} \right)^{-t/n} \overset{d}{\to} \min\{Y_k^*, k = 1, 2, \dots, n\}.$$

On the other hand $Y_t^{-t/n} \overset{d}{\to} (Y^*)^{1/n}$, so that $\min\{Y_k^*, k = 1, 2, \dots, n\} \overset{d}{\sim} (Y^*)^{1/n}$. Consequently, letting $F^*(x) = \mathbb{P}(Y^* \leq x)$ and $\overline{F}^*(x) = 1 - F^*(x)$, we have $\overline{F}^*(x^n) = \overline{F}^*(x)^n$ for all $n \in \mathbb{N}$. It follows that for all $q = n/m \in \mathbb{Q}$ with $n, m \in \mathbb{N}$, $\overline{F}^*(x^q) = \overline{F}^*(x^{1/m})^n = (\overline{F}^*(x))^q$. Hence, F^* is a continuous function and $\overline{F}^*(x^r) = \overline{F}^*(x)^r$ for all $r \in [0, \infty)$.

We next show that F^* is strictly monotone (unless $Y^* = 1$ a.s., which is not of interest here). We already know from Proposition 2.1 that F^* is concentrated on $[1, \infty)$. Let $x, y \in [1, \infty)$ with $x \neq y$ and suppose that $F^*(x) = F^*(y)$. It then follows that $F^*(x^r) = F^*(y^r)$ for all $r \in [0, \infty)$, implying $F^*(x) = 1$ constantly for $x \in [1, \infty)$. If this is not the case, the function $g(x) = \log \overline{F}^*(e^x)$ is monotone decreasing on $[1, \infty)$ and satisfies the functional equation $g(ry) = rg(y)$ for $y \in [0, \infty)$, identifying g as $g(x) = -\gamma y$ for some $\gamma > 0$.

Remark 5. Adding a positive drift $ct, c > 0$ to the subordinator Y_t^* changes the limiting behavior dramatically, because $Y_t^{-t} \overset{d}{\to} Y^*$ implies $(ct + Y_t)^{-t} \overset{d}{\to} 1$ by Proposition 2.2.

Remark 6. Suppose that Y_t does not start at zero, but $Y_0 \overset{d}{\sim} B$ instead, where B is a nonnegative r.v. with $q = \mathbb{P}(B = 0) \in (0, 1)$ and Y_t is of the form $Y_t = L_t + B$, where L_t is a subordinator independent of B with $L_t^{-t} \overset{d}{\to} L^*$. If β denotes the LST of B, then $\beta(u^{1/t})$ tends to $\beta(0) = 1$ for $u < 1$ and to q for $u > 1$ as $t \to 0$. Letting φ denote the LST of L_t and F^* the distribution

function of L^*, it follows that

$$\lim_{t \to 0} \psi_t(u^{1/t}) = \lim_{t \to 0} \varphi(u^{1/t})^t \beta(u^{1/t}) = \begin{cases} 1 - F^*(u), & u < 1, \\ q(1 - F^*(u)), & u > 1, \end{cases}$$

so that the limiting distribution of Y_t^{-t} has a atom of mass $1 - q$ at 1 and an atom of mass q at infinity, as expected, since by Proposition 2.1,

$$Y_t^{-t} \xrightarrow{d} \min(L^*, B^*),$$

where L^* and B^* are independent, L^* has a Pareto distribution and B^* attains only the values 1 and infinity. This way we can obtain any mixture of a Pareto distribution $\mathcal{P}_\gamma$ and the point mass at 1 as limiting distribution (with $F^*(x) = q \mathbf{1}_{[0,1)}(x) + (1 - qx^{-1})\mathbf{1}_{[1,\infty)}(x)$).

Remark 7. Example 1 (in Section 2) shows that for parametrized families $(\psi_t)_{t>0}$ not of the infinitely divisible form $\psi(u)^t$ other interesting limit laws can occur (e.g., the shifted exponential distribution). Thus, there may be other limit theorems and characterizations to be explored.

4. Applications

4.1. Explicit examples

Example 2. The following distributions are all infinitely divisible (see [11], Section 2.8). A close look at their distribution functions or densities reveals that either condition (S6) or condition (S8) can be applied so that Y_t^{-t} tends in distribution to $\mathcal{P}_\gamma$ for some $\gamma > 0$. Note that in most cases neither explicit formulas for the convolution powers of F nor simple expressions for $\psi_t(s)$ are known.

- (Gamma process) The Gamma process is a standard example of a subordinator. The density of Y_1 is given by

$$f(x) = x^{\gamma-1} \lambda^\gamma e^{-\lambda x} / \Gamma(\gamma),$$

  where $\lambda > 0$ and $\gamma > 0$. Obviously $f(x) \sim x^{\gamma-1}$ as $x \to 0$, implying that condition (S8) holds.
- (Weibull distribution) If $F(x) = 1 - e^{-x^\gamma}$ then $F(x) \sim x^\gamma$ as $x \to 0$, so that in particular condition (S6) is satisfied.

For the next three distributions, the density $f(x)$ tends to some positive constant as $x \to 0$, so that condition (S8) holds with $\gamma = 1$.

- (Pareto-type distribution) $f(x) = \frac{a}{(1+x)^{a+1}}$, with $a > 0$.
- (F-distribution) $f(x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} x^{b-1}(1 + x)^{-a-b}$ with $a, b > 0$.
- (Cauchy distribution on $(0, \infty)$) $f(x) = \frac{2}{\pi} \frac{1}{1 + x^2}$.

Example 3 (Generalized gamma process). Let μ be a σ-finite measure on $[0, \infty)$ and suppose that the Lévy measure is given by

$$\nu(\mathrm{d}x) = \frac{1}{x} \int_0^\infty \mathrm{e}^{-xy} \, \mathrm{d}\mu(y) \, \mathrm{d}x.$$

The associated Lévy process is called a generalized gamma process (see [6]) and μ is the so-called *Thorin measure*. If μ is a finite measure and $\gamma = \mu([0, \infty))$, then $\overline{\nu}(x)/\log x \to -\gamma$ as $x \to 0$, by dominated convergence. It then follows from criterion (S7) of the theorem that $Y_t^{-t} \overset{d}{\to} \mathcal{P}_\gamma$. Note that the Gamma process corresponds to the case where μ is the Dirac measure with mass γ at $y = \lambda$.

More examples of generalized gamma processes can be found in [5] (the complete Bernstein functions f correspond to our function Φ, $\tau(\mathrm{d}s)$ corresponds to $\nu(\mathrm{d}s)/s$ and $\rho(\mathrm{d}t)$ to $\mu(t)\,\mathrm{d}t/t$ in our paper). For instance, if the Thorin measure is given by $\mu(\mathrm{d}t) = 1_{(0,\gamma)}(t)\,\mathrm{d}t$ then $\Phi(s) = (x + \gamma)\log(x + \gamma) - x \log x - \gamma \log \gamma$ and hence $\Phi(s)/\log x \to \gamma$. Note that indeed $\gamma = \mu([0, \infty))$. The corresponding Lévy measure is given by $\nu(\mathrm{d}x) = \frac{1 - \mathrm{e}^{-\gamma x}}{x}\,\mathrm{d}x$.

Example 4 (cf. [1,2]). Let the density of Y_t be given by $f_t(x) = \mathrm{e}^{-x} x^{-1} t I_t(x)$, where I_t is the modified Bessel function of order one. Then the Laplace exponent is given by

$$\Phi(s) = \log\left(1 + s - \sqrt{s^2 + 2s}\right)$$

and since $2s(1 + s - \sqrt{s^2 + 2s}) \to 1$ as $s \to \infty$ it follows that $\Phi(s)/\log s \to 1$ and hence $Y_t^{-t} \overset{d}{\to} \mathcal{P}_1$ by criterion (S5).

Example 5. We coin the name *Dickman process* for a subordinator with Lévy measure $\mathrm{d}\nu(x) = \gamma x^{-1} 1_{(0,1]}(x)\,\mathrm{d}x$, where $\gamma > 0$ is some parameter. The infinitely divisible distribution function F associated with ν is the generalized Dickman distribution as defined in [10]. This F appears for example, as

- the distribution of a random variable X satisfying $X \overset{d}{\sim} U^{1/\gamma}(X + 1)$, where U is a uniform random variable on $[0, 1]$ independent of X,
- the limiting distribution of $\sum_{i=1}^n (U_1 U_2 \cdots U_i)^{1/\gamma}$, where $U_1, U_2, \ldots$ are independent uniform random variables on $[0, 1]$.

The name 'Dickman distribution' is due to the fact that for $\gamma = 1$ the density of F is given by $f(x) = \mathrm{e}^{-C}\rho(x)$, where C is Euler's constant and ρ is the generalized Dickman function. This function is implicitly defined by $\rho(z) = 1$ for $z \in [0, 1]$ and $z\rho'(z) = \rho(z - 1)$ for $z > 1$. Since $\overline{\nu}(x) = -\gamma \log x$ for x small enough, criterion (S7) of Theorem 3.1 can be applied; we have $Y_t^{-t} \overset{d}{\to} \mathcal{P}_\gamma$.

4.2. Subordination

If X_t is another subordinator with Laplace exponent $\varphi(s)$ and X_t and Y_t are independent, then both subordinate processes $A_t = X_{Y_t}$ and $B_t = Y_{X_t}$ are again subordinators. Their Laplace expo-

nents are

$$\phi_A(s) = \Phi(\varphi(s)) \quad \text{and} \quad \phi_B(s) = \varphi(\Phi(s)),$$

respectively. Suppose that $Y_t^{-t} \xrightarrow{d} \mathcal{P}_\gamma$ as $t \to 0$ and that $\delta > 0$. It follows immediately from the representations

$$\frac{\phi_A(s)}{\log s} = \frac{\Phi(\varphi(s))}{\log \varphi(s)} \frac{\log \varphi(s)}{\log s} \quad \text{and} \quad \frac{\phi_B(s)}{\log s} = \frac{\varphi(\Phi(s))}{\Phi(s)} \frac{\Phi(s)}{\log s}$$

and criterion (S5) that

$$A_t^{-t} \xrightarrow{d} \mathcal{P}_{\gamma\delta} \qquad \text{as } t \to 0 \quad \Longleftrightarrow \quad \lim_{s\to\infty} \frac{\log \varphi(s)}{\log s} = \delta$$

and

$$B_t^{-t} \xrightarrow{d} \mathcal{P}_{\gamma\delta} \qquad \text{as } t \to 0 \quad \Longleftrightarrow \quad \lim_{s\to\infty} \frac{\varphi(s)}{s} = \delta.$$

Example 6 (Subordination with α-stable processes). Suppose that $\varphi(s) = s^\alpha$ is the Laplace exponent of an α-stable subordinator X_t. Then it follows that

$$A_t^{-t} \xrightarrow{d} \mathcal{P}_{\gamma\alpha} \quad \Longleftrightarrow \quad Y_t^{-t} \xrightarrow{d} \mathcal{P}_\gamma.$$

For $\alpha = 1$, we deduce that $B_t^{-t} \xrightarrow{d} \mathcal{P}_\gamma$ if and only if $Y_t^{-t} \xrightarrow{d} \mathcal{P}_\gamma$, but in this case we just deal with the trivial case of deterministic drift $X_t = t$ and $B_t = Y_t$.

4.3. Exponential dispersion models

For each $\theta \geq 0$, we define a new Lévy measure $\nu^{(\theta)}$ by exponentially tilting ν, that is, we let

$$d\nu^{(\theta)}(x) = e^{-\theta x} \, d\nu(x).$$

The Laplace exponent of the associated Lévy process $Y_t^{(\theta)}$ is given by the difference

$$\Phi^{(\theta)}(s) = \Phi(\theta + s) - \Phi(\theta).$$

The new LST $\psi_t^{(\theta)}(s) = \mathbb{E}(e^{-sY_t})$ is related to $\psi(s)$ via

$$\psi_t^{(\theta)}(s) = \left(\frac{\psi(\theta + s)}{\psi(\theta)} \right)^t.$$

Accordingly, the distribution of $Y_t^{(\theta)}$ is given by

$$F_t^{(\theta)}(dx) = \frac{e^{-sx} F^{t*}(dx)}{(\int_0^\infty e^{-su} \, dF(u))^t},$$

where F^{t*} denotes the distribution with LST $(\int_0^\infty e^{-su} \, dF(u))^t$. The class $\{F_t^{(\theta)}, t \geq 0, \theta \geq 0\}$ is called an exponential dispersion model (see [2]).

By writing

$$\frac{\Phi^{(\theta)}(s)}{\log s} = \frac{\Phi(\theta + s)}{\log s} - \frac{\Phi(\theta)}{\log s} = \frac{\Phi(\theta + s)}{\log s} + o(1), \qquad \text{as } s \to \infty$$

we see that $(Y_t^{(\theta)})^{-t} \to \mathcal{P}_\gamma$ if and only if $Y_t^{-t} \to \mathcal{P}_\gamma$.

5. A generalization

In the preceding sections, we have studied the convergence of $-t \log Y_t$ to $X^* = \log Y^*$ as $t \to 0$. In this section, we consider the more general case

$$t L(Y_t) \xrightarrow{d} X^*, \qquad t \to 0,$$

where $L : (0, \infty) \to (-\infty, \infty)$ is some decreasing function satisfying $\lim_{y \to 0} L(y) = \infty$. Let

$$L(\infty) \equiv \lim_{y \to \infty} L(y) \in [-\infty, \infty)$$

and denote by $L^{-1} : (-\infty, \infty) \to (0, \infty)$ the (decreasing) inverse function of L, with the convention that $L^{-1}(x) = \infty$ for $x \leq L(\infty)$, in which case $1/L^{-1}(x) = 0$.

The next result is the counterpart of Proposition 1.1, now for the general case where L is not the negative logarithm. To impose suitable conditions on L, we need the definition of slow variation. The function L is called slowly varying at zero if $\lim_{x \to 0} L(\lambda x)/L(x) = 1$ for all $\lambda > 0$. If this holds one can show that the inverse function is rapidly varying at infinity, that is,

$$\frac{L^{-1}(w)}{L^{-1}(y)} \to \left(\frac{w}{y}\right)^\infty \equiv \begin{cases} 0, & w > y, \\ 1, & w = y, \\ \infty, & w < y, \end{cases}$$

and the convergence is necessarily uniform for w outside of intervals $(y - \delta, y + \delta)$, $\delta > 0$ (for both concepts see [4]). With these prerequisites, we can show the following proposition.

Proposition 5.1. *Suppose that $L : (0, \infty) \to (-\infty, \infty)$ is decreasing with $\lim_{y \to 0} L(y) = \infty$ and that L is slowly varying at zero. Let X^* be a random variable which is not concentrated at one point. Then*

$$t L(Y_t) \xrightarrow{d} X^* \qquad \text{as } t \to 0$$

if and only if

$$\psi_t\left(\frac{1}{L^{-1}(u/t)}\right) \to 1 - H^*(u) \qquad \text{as } t \to 0 \tag{5.1}$$

for all continuity points u of the distribution function $H^*(u) = \mathbb{P}(X^* \leq u)$.

Proof. For $u < 0$, we always have $\psi_t(\frac{1}{L^{-1}(u/t)}) \to 1$ as $t \to 0$, so we restrict ourselves to $u > 0$. Let $F_t(x) = \mathbb{P}(Y_t \leq x)$ and let H_t denote the distribution function of $X_t \equiv tL(Y_t)$. Since $H_t(x) = 1 - F_t(L^{-1}(x/t))$ for $t > 0$, it follows that

$$\psi_t\left(\frac{1}{L^{-1}(u/t)}\right) = \int_0^\infty \exp\left(-\frac{y}{L^{-1}(u/t)}\right) dF_t(y)$$

$$= \int_{L(\infty)}^\infty \exp\left(-\frac{L^{-1}(x/t)}{L^{-1}(u/t)}\right) dH_t(x).$$

Hence, $\psi_t(\frac{1}{L^{-1}(u/t)}) = \mathbb{E}(\zeta_t(X_t, u))$, where $\zeta_t(x, u) = \exp(-L^{-1}(x/t)/L^{-1}(u/t))$. Since L is slowly varying at zero, it follows that L^{-1} is rapidly varying at ∞, implying that $\lim_{t\to 0}\zeta_t(x, u) = 1_{\{x>u\}}$ for $x \neq u$. Furthermore, we obtain that $\lim_{t\to 0} \zeta_t(c_t(x), u) = 1_{\{x>u\}}$ for any function $c_t(x)$ with $c_t(x) \to x$ as $t \to 0$, since $x > u$ implies that $c_t(x) > u$ eventually as $t \to 0$ (and $x < u$ implies that $c_t(x) < u$ eventually).

($\Rightarrow$) Suppose first that $X_t \overset{d}{\to} X^*$. We can apply the continuous mapping theorem in the form of Theorem 4.27 in [8]. It follows that $\zeta_t(X_t, u) \overset{d}{\to} 1_{\{x>u\}}$ for any continuity point u of H^*. Since $\zeta_t(x, u) \in [0, 1]$, we have $\mathbb{E}(\zeta_t(X_t, u)) \to \mathbb{E}(1_{\{x>u\}}) = 1 - H^*(u)$ by dominated convergence.

($\Leftarrow$) If on the other hand (5.1) holds, then $\mathbb{E}(\zeta_t(X_t, u)) \to 1 - H^*(u)$ and

$$\left|\mathbb{P}(X_t \leq u) - \left(1 - H^*(u)\right)\right|$$

$$\leq |\mathbb{P}(X_t \leq u) - \mathbb{E}(\zeta_t(X_t, u))| + |1 - H^*(u) - \mathbb{E}(\zeta_t(X_t, u))|$$

$$= \left|\mathbb{E}\left(1_{\{X_t \leq u\}} - \zeta_t(X_t, u)\right)\right| + |1 - H^*(u) - \mathbb{E}(\zeta_t(X_t, u))|.$$

The second term on the right-hand side tends to zero as $t \to 0$. Regarding the first term, for every $\varepsilon > 0$ and $\delta \in (0, u)$, we have for all t large enough (by uniform convergence for rapidly varying functions) that

$$\left|\mathbb{E}\left(1_{\{X_t \leq u\}} - \zeta_t(X_t, u)\right)\right|$$

$$= \int_0^{u-\delta} \left|\mathbb{E}\left(1 - \zeta_t(x, u)\right)\right| dH_t(x) + \int_{u-\delta}^{u+\delta} \left|1_{\{x \leq u\}} - \zeta_t(x, u)\right| dH_t(x)$$

$$+ \int_{u+\delta}^\infty |\mathbb{E}(\zeta_t(x, u))| dH_t(x)$$

$$\leq \varepsilon\left(H_t(u - \delta) + 1 - H_t(u + \delta)\right) + H_t(u + \delta) - H_t(u - \delta).$$

Thus, if u is a continuity point of H^* it follows that $|\mathbb{E}(1_{\{X_t \leq u\}} - \zeta_t(X_t, u))|$ tends to zero too, yielding $|\mathbb{P}(X_t \leq u) - (1 - H^*(u))| \to 0$. $\square$

We can now state the main result of this section, again for an arbitrary subordinator Y_t. The proof follows along the lines of that of Theorem 3.1.

Theorem 5.1. *Let L, H^* and X^* be as in Proposition 5.1 and that $\gamma > 0$. Let $\mathcal{E}_\gamma$ be a r.v. with exponential distribution function $E_\gamma(x) = 1 - e^{-\gamma x}$, $x \geq 0$. Then the following statements are equivalent:*

1. $t L(Y_t) \xrightarrow{d} X^*$ *as $t \to 0$.*
2. $t L(Y_t) \xrightarrow{d} \mathcal{E}_\gamma$ *as $t \to 0$.*
3. $\Phi(1/s)/L(s) \to \gamma$ *as $s \to 0$.*
4. $t \Phi(1/L^{-1}(u/t)) \to \log(1 - H^*(u))$ *as $t \to 0$, for all continuity points u of H^*.*

Candidates other than $-\log x$ that satisfy the conditions of the theorem are for example the functions $-(\log x)^{2k+1}$, $k = 1, 2, \ldots$.

Acknowledgement

References

[1] Bar-Lev, S.K. and Enis, P. (1987). Existence of moments and an asymptotic result based on a mixture of exponential distributions. *Statist. Probab. Lett.* **5** 273–277. MR0896458

[2] Bar-Lev, S.K. and Letac, G. (2010). The limiting behavior of some infinitely divisible exponential dispersion models. *Statist. Probab. Lett.* **80** 1870–1874. MR2734253

[3] Bertoin, J. (1996). *Lévy Processes. Cambridge Tracts in Mathematics* **121**. Cambridge: Cambridge Univ. Press. MR1406564

[4] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge: Cambridge Univ. Press. MR0898871

[5] Jacob, N. and Schilling, R.L. (2005). Function spaces as Dirichlet spaces (about a paper by W. Maz'ya and J. Nagel). *Z. Anal. Anwendungen* **24** 3–28. MR2146549

[6] James, L.F., Roynette, B. and Yor, M. (2008). Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surv.* **5** 346–415. MR2476736

[7] Jørgensen, B. (2006). Dispersion models. In *Encyclopedia of Statistical Sciences*. New York: Wiley.

[8] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). New York: Springer. MR1876169

[9] Kyprianou, A.E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications. Universitext*. Berlin: Springer. MR2250061

[10] Penrose, M.D. and Wade, A.R. (2004). Random minimal directed spanning trees and Dickman-type distributions. *Adv. in Appl. Probab.* **36** 691–714. MR2079909

[11] Sato, K.i. (1999). *Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics* **68**. Cambridge: Cambridge Univ. Press. MR1739520