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BMO-MARTINGALES AND INEQUALITIES

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1. Introduction and preliminaries. In this paper we shall extend Davis's inequality to some class of semimartingales and characterize BMO-martingales by some inequality, related to the weighted norm inequality.

Let (Ω, F, P) be a complete probability space with an increasing right continuous family $(F_t)_{t\geqq 0}$ of sub-σ-fields of F such that $F = \bigvee_{t\geqq 0} F_t$. We use the same notations $[X, Y]$, X^* and so on as in Meyer [4]. We point out that Fefferman's inequality is valid for semimartingales, that is, we have

$$E\left[\int_{[0,\infty[} |d[X, Y]_s|\right] \leqq \sqrt{2}\, E[[X, X]_\infty^{1/2}]\, ||Y||_{\mathrm{BMO}}$$

for each semimartingale X and BMO-martingale Y.

Let us denote by c a positive constant and by c_x a positive constant depending only on the indicated parameter x. Both letters are not necessarily the same in each occurrence.

2. Generalization of Davis's inequality. We consider a fixed BMO-martingale M such that $1 + \Delta M > \varepsilon$ for some positive constant ε and put

$$M\hat{} = -M + \langle M^c, M^c\rangle + \sum_{0\leqq s\leqq \cdot} (\Delta M_s)^2/(1 + \Delta M_s)\ .$$

For each local martingale X we denote by $\phi(X)$ a semimartingale $X + [X, M\hat{}]$. Now Davis's inequality is extended as in the following.

THEOREM 1. *We have the inequalities*:

$$(1)\qquad E[\phi(X)_\infty^*] \leqq c_M E[[\phi(X), \phi(X)]_\infty^{1/2}]$$

and

$$(2)\qquad E[[\phi(X), \phi(X)]_\infty^{1/2}] \leqq c_M E[\phi(X)_\infty^*]$$

for each local martingale X.

PROOF. By a simple calculation we get

$$(3)\qquad \phi(X) = X - [\phi(X), M]$$

and

$$c_M[X, X] \leqq [\phi(X), \phi(X)] \leqq c_M[X, X] \tag{4}$$

for each local martingale X. The equality (3) implies

$$E[\phi(X)^*_\infty] \leqq E[X^*_\infty] + E\left[\int_{[0,\infty[} |d[\phi(X), M]_s|\right].$$

By Davis's inequality the first term on the right hand side is smaller than $cE[[X, X]^{1/2}_\infty]$. It follows from Fefferman's inequality that the second term is dominated by $cE[[\phi(X), \phi(X)]^{1/2}_\infty]$. Hence, we obtain (1) by (4).

To prove (2), it suffices to show the inequality

$$E[[\phi(X), \phi(X)]_\infty/\phi(X)^*_\infty] \leqq E[\phi(X)^*_\infty] + cE[[\phi(X), \phi(X)]^{1/2}_\infty] \tag{5}$$

when $\phi(X)_0$ is a nonzero constant, as in the proof of Meyer [4, V. T30, p. 350]. Moreover, we may assume $E[[\phi(X), \phi(X)]^{1/2}_\infty] < \infty$ and $E[K^*_\infty] < \infty$, where $K = \phi(X)^2 - [\phi(X), \phi(X)] = 2\phi(X)_- \circ \phi(X) = 2(\phi(X)_- \circ X - \phi(X)_- \circ [\phi(X), M])$. Indeed, a local martingale X belongs locally to H^1 and hence $[\phi(X), \phi(X)]^{1/2}$ does locally to L^1 by (4). This and Fefferman's inequality imply that $(\phi(X)_- \circ [\phi(X), M])^*$ is locally in L^1, because $\phi(X)_-$ is locally bounded. Since $\phi(X)_- \circ X$ is a local martingale, $(\phi(X)_- \circ X)^*$ is locally in L^1. Therefore, K^* is locally in L^1.

Now we put $H_t = E[1/\phi(X)^*_\infty | F_t]$. Then H is a bounded martingale and we have $\|\phi(X)_- \circ H\|_{\mathrm{BMO}} \leqq \sqrt{6}$ because of $|\phi(X)_{t-}H_t| \leqq 1$ (see [4, V. T6, p. 335]). By Ito's formula we have $KH = K_- \circ H + H_- \circ K + [K, H] = K_- \circ H + 2(H_-\phi(X)_-) \circ X - 2(H_-\phi(X)_-) \circ [\phi(X), M] + 2[\phi(X), \phi(X)_- \circ H]$. Since the first and second terms on the extreme right hand side are local martingales, we have

$$|E[K_{T_n}H_{T_n}]| \leqq 2E\left[\int_{[0,\infty[} |d[\phi(X), M]_s|\right] + 2E\left[\int_{[0,\infty[} |d[\phi(X), \phi(X)_- \circ H]_s|\right]$$

for some sequence of stopping times T_n with $T_n \uparrow \infty$ as $n \to \infty$. Then the right hand side is smaller than $c_M E[[\phi(X), \phi(X)]^{1/2}_\infty]$ by Fefferman's inequality. Letting $n \to \infty$ we obtain (5). q.e.d.

By applying Garsia's lemma (see [4, V. 24, p. 347]) to the above theorem we have the following:

COROLLARY. *Let f be a continuous increasing convex function on $[0, \infty[$ satisfying $f(0) = 0$ and the growth condition $f(2t) \leqq cf(t)$, $t \geqq 0$. Then we have*

$$c_{M,f}E[f(\phi(X)^*_\infty)] \leqq E[f([\phi(X), \phi(X)]^{1/2}_\infty)] \leqq c_{M,f}E[f(\phi(X)^*_\infty)] \tag{6}$$

for each local martingale X.

3. Characterization of BMO-martingales. Let M be a fixed local martingale such that $1/\varepsilon > 1 + \Delta M > \varepsilon$ for some positive constant ε. In the case $M^{\wedge}$ and $\phi(X)$ in §2 are well-defined. Moreover, the equality (3) and the inequality (4) are still valid. Now we shall characterize BMO-martingales as follows:

THEOREM 2. *In order that M is a* BMO-*martingale, it is necessary and sufficient that the inequality*

$$E[\phi(X)^*_\infty] \leqq cE[X^*_\infty] \tag{7}$$

is valid for all local martingales X.

PROOF. Suppose that M is a BMO-martingale. Then Theorem 1 and (4) imply $E[\phi(X)^*_\infty] \leqq cE[[X, X]^{1/2}_\infty]$. We apply Davis's inequality to the right hand side and obtain (7). We next show the converse. By the equality (3) $[X, M] = [\phi(X), M] + \sum_{0 \leqq s \leqq \cdot} \Delta\phi(X)_s \Delta M_s \Delta M_s$. Since ΔM is bounded, we have

$$\begin{aligned}\int_{[0,\infty[} |d[X, M]_s| &\leqq \int_{[0,\infty[} |d[\phi(X), M]_s| + c \sum_{0 \leqq s < \infty} |\Delta\phi(X)_s \Delta M_s| \\ &\leqq c \int_{[0,\infty[} |d[\phi(X), M]_s| \,.\end{aligned}$$

Hence, for the proof of the converse it suffices to show the following inequality:

$$E\left[\int_{[0,\infty[} |d[\phi(X), M]_s|\right] \leqq cE[X^*_\infty] \tag{8}$$

for each X in H^1. Indeed, we have $E\left[\int_{[0,\infty[} |d[X, M]_s|\right] \leqq cE[X^*_\infty]$ for each X in H^1 and hence M is a BMO-martingale by the duality theorem, i.e., $(H^1)^* = \text{BMO}$. Now we set $D = |d[\phi(X), M]|/d[\phi(X), M]$, which is an optional process with $D^2 = 1$. Let X be in H^1 and consider the stochastic integral $D \circ X$. By the properties of the stochastic integral of optional processes (see [4, V. T19 and 20, pp. 343–345]) and the equality (3) we have

$$\begin{aligned}E\left[\int_{[0,\infty[} |d[\phi(X), M]_s|\right] &= E\left[\int_{[0,\infty[} D_s d[\phi(X), M]_s\right] \\ &= E\left[\int_{[0,\infty[} D_s d[X, M]_s\right] + E\left[\sum_{0 \leqq s < \infty} D_s \Delta X_s \Delta M^{\wedge}_s \Delta M_s\right] \\ &= E\left[\int_{[0,\infty[} d[D \circ X, M]_s\right] + E\left[\sum_{0 \leqq s < \infty} \Delta(D \circ X)_s \Delta M^{\wedge}_s \Delta M_s\right] \\ &= E[[\phi(D \circ X), M]_\infty] \leqq E[\phi(D \circ X)^*_\infty] + E[(D \circ X)^*_\infty] \,,\end{aligned}$$

which by (7) is not more than $cE[(D\circ X)^*_\infty] \leqq cE[X^*_\infty]$, and obtain (8). q.e.d.

4. **Weighted norm inequality.** Let Z be a P-uniformly integrable martingale with $Z_0 = 1$ and $Z_\infty > 0$ a.s.. We put $Q = Z_\infty \cdot P$ and denote by $E_Q[\cdot]$ the expectation with respect to Q. Moreover, we denote by M the P-local martingale $(1/Z_-)\circ Z$ and use the same notation $M^\wedge$ and $\phi(X)$ as in §2. Then $M^\wedge$ and $\phi(X)$ are Q-local martingales and $M^\wedge = Z_-\circ(1/Z)$. Now we define the conditions (A_∞) and (S) as in Izumisawa, Sekiguchi and Shiota [2]. Combining the results in the above literature with Gehring's lemma in Doléans-Dade and Meyer [3], we see that Z *satisfies the conditions* (A_∞) *and* (S) *if and only if* $M^\wedge$ *is a* BMO-*martingale with respect to* Q *and* $1 + \Delta M^\wedge > \varepsilon$ *for some positive constant* ε. Thus we can rewrite the results in §§2 and 3 as in the following.

THEOREM 1'. *Let f be a continuous increasing convex function on $[0, \infty[$ satisfying $f(0) = 0$ and the growth condition. If Z satisfies the conditions (A_∞) and (S), then we have*

$$c_{Z,f}E_Q[f(X^*_\infty)] \leqq E_Q[f([X, X]^{1/2}_\infty)] \leqq c_{Z,f}E_Q[f(X^*_\infty)]$$

for each P-local martingale X.

The above theorem is an improvement of the result obtained by Izumisawa and the author in [1].

THEOREM 2'. *Suppose that Z is quasi-left-continuous and satisfies the condition (S). Then Z satisfies (A_∞) if and only if the inequality*

$$E_Q[X^*_\infty] \leqq cE_Q[[X, X]^{1/2}_\infty]$$

is valid for all P-local martingales X.

REFERENCES

[1] M. IZUMISAWA AND T. SEKIGUCHI, Weighted norm inequalities for martingales, Séminaire de Probabilités XIII, Université de Strasbourg, Lecture Notes in Math., Springer-Verlag, (to appear).

[2] M. IZUMISAWA, T. SEKIGUCHI AND Y. SHIOTA, Remark on a characterization of BMO-martingales, Tôhoku Math. J., 31 (1979), 281-284.

[3] C. DOLÉANS-DADE AND P. A. MEYER, Inégalités de normes avec poids, Séminaire de Probabilités XIII, Université de Strasbourg, Lecture Notes in Math., Springer-Verlag, (to appear).

[4] P. A. MEYER, Un cours sur les intégrales stochastiques, Séminaire de Probabilités X, Université de Strasbourg, Lecture Notes in Math., 511 (1976), Springer-Verlag, 246-400.

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