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INNER DERIVATIONS IN THE TENSOR PRODUCTS OF OPERATOR ALGEBRAS

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Let A and B be unital C^*-algebras and $A \otimes B$ be C^*-tensor product with respect to the minimal C^*-crossnorm α. It has been known in the literature ([1], [6] etc) that in general the product structure of the algebra $A \otimes B$ is not compatible with the properties of derivations of the component algebras A and B, and even in the simplest case of separable C^*-algebras in which A is commutative and B is an UHF algebra one still finds badly behaved derivations of $A \otimes B$. However, as far as the von Neumann algebras are concerned there are another positive results in this direction ([4], [8]) and quite recently Akemann and Johnson [2] has shown that every derivation of the C^*-tensor product $A \otimes N$ of a commutative C^*-algebra A and a von Neumann algebra N is inner. It is then easily verified that if M is a finite von Neumann algebra of type 1 with bounded degree and N an arbitrary von Neumann algebra, every derivation of $M \otimes N$ is inner.

The purpose of the present note is to prove a corresponding C^*-version of this result (Theorem 1.1) which contains the above theorem of Akemann and Johnson, with some additional results towards the investigation of the derivations in the algebra $M \otimes N$ for an arbitrary pair of von Neumann algebras.

1. As we shall freely use slice maps in the tensor products of operator algebras we recall first their definitions ([11], [12]). Let $A \otimes B$ be the C^*-tensor product of C^*-algebras A and B. We denote by A^* the dual space of A. Then, to a functional φ of A^* we associate a bounded linear map R_φ, the right slice map, of $A \otimes B$ into B such that $R_\varphi(a \otimes b) = \langle a, \varphi \rangle b$. Similarly, a functional ψ of B^* gives rise to a bounded linear map L_ψ, the left slice map, of $A \otimes B$ into A such that $L_\psi(a \otimes b) = \langle b, \psi \rangle a$. These two kinds of mappings are related in the following way: for an element x in $A \otimes B$ we have

$$\langle x, \varphi \otimes \psi \rangle = \langle R_\varphi(x), \psi \rangle = \langle L_\psi(x), \varphi \rangle .$$

We call this the Fubini principle.

Next, let M and N be von Neumann algebras and denote by $M \bar{\otimes} N$ the von Neumann tensor product of M and N. Let M_* and N_* be preduals of M and N. Then a functional φ of N_* defines a σ-weakly continuous right slice map R_φ of $M \bar{\otimes} N$ into N as an extension of the right slice map R_φ on the C^*-tensor product $M \otimes N$. There are also σ-weakly continuous left slice maps to M with respect to the functionals in N_*. Moreover, for a linear functional φ of M^* we can further define a bounded linear map R_φ of $M \bar{\otimes} N$ into N by the relation,

$$\langle R_\varphi(x), \psi \rangle = \langle L_\psi(x), \varphi \rangle$$

for all $\psi \in N_*$. We call this map R_φ the generalized right slice map for φ. Generalized left slice maps may also be defined in a similar way, but we note that in general we cannot expect the Fubini principle to hold for a pair of generalized slice maps (cf. [12: Theorem 5.1]).

Let x be an element of $M \bar{\otimes} N$. Then it gives rise to the map $r_x: \varphi \in M^* \to R_\varphi(x) \in N$, which is weak $*$-σ-weakly continuous and in particular when x belongs to the C^*-tensor product $M \otimes N$ the map is weak $*$-norm continuous in the bounded sphere of M^*.

Now let Z be the center of N. Then it is known that there is a normal projection ε of norm one of N onto Z. With these notations we have

THEOREM 1.1. *Let A be a unital C^*-algebra whose irreducible representations are finite dimensional with bounded degree. Let δ be a derivation in the algebra $A \otimes N$. Then δ is inner if and only if the restricted derivation $1 \otimes \varepsilon \circ \delta$ in $A \otimes Z$ is inner, where $1 \otimes \varepsilon$ is the product projection of norm one to the subalgebra $A \otimes Z$.*

PROOF. If $\delta = \mathrm{ad}(c)$ for some element c in $A \otimes N$, then $(1 \otimes \varepsilon \circ \delta) | A \otimes Z = \mathrm{ad}(1 \otimes \varepsilon(c))$ by [10: Theorem 1], so that the derivation $1 \otimes \varepsilon \circ \delta$ is inner, too. Now suppose the derivation $1 \otimes \varepsilon \circ \delta$ be inner and let c_0 be its generator in $A \otimes Z$. Let $\tilde{A}$ be the enveloping von Neumann algebra of A. We consider the von Neumann algebra $\tilde{A} \bar{\otimes} N$ and extend the derivation δ to the algebra $\tilde{A} \bar{\otimes} N$ (cf. [7: Lemma 3]). Let c_1 be a generator of this extended derivation. Denote by $1 \otimes \varepsilon$ again the normal product projection of norm one of $\tilde{A} \bar{\otimes} N$ to $\tilde{A} \bar{\otimes} Z$. Then obviously $1 \otimes \varepsilon(c_1)$ is a generator of the derivation $1 \otimes \varepsilon \circ \delta$. Hence $c_0 - 1 \otimes \varepsilon(c_1)$ belongs to the center of $\tilde{A} \bar{\otimes} Z$ which coincides with the center of $\tilde{A} \bar{\otimes} N$. Put $\delta_1 = \delta - \mathrm{ad}(c_0)$. Then we have

$$\delta_1 = \mathrm{ad}(c_1) - \mathrm{ad}(c_0) = \mathrm{ad}(c_1) - \mathrm{ad}(1 \otimes \varepsilon(c_1)) = \mathrm{ad}(c_1 - 1 \otimes \varepsilon(c_1)) .$$

Now in order to get the conclusion, it suffices to show that the derivation

δ_1 is inner. Thus we may assume for our argument that δ has the generator c in $\tilde{A} \overline{\otimes} N$ such that $1 \otimes \varepsilon(c) = 0$. Let φ be a bounded linear functional of A and let $\hat{\varphi}$ be its canonical extension to $\tilde{A}$ as a σ-weakly continuous linear functional. We consider the derivation δ_φ in N defined by $\delta_\varphi(x) = R_\varphi(\delta(1 \otimes x))$. Since $\delta(1 \otimes x)$ belongs to $A \otimes N$, the map: $\varphi \in A^* \to \delta_\varphi$ is continuous on the unit sphere S^* of A^* with respect to the weak $*$ topology and point-norm convergence topology in the space of derivations in N. Therefore, the set $\{\delta_\varphi | \varphi \in S^*\}$ is compact for the point-norm convergence topology, hence by [2: Theorem 2.1] it is compact for the norm topology. Thus the point-norm and norm topologies agree on the set $\{\delta_\varphi | \varphi \in S^*\}$. It follows that the map: $\varphi \in S^* \to \delta_\varphi$ is continuous for the weak $*$ and norm topologies. Now the map: $x \in N \to \mathrm{ad}(x)$ induces a homeomorphism between the quotient Banach space N/Z and the space of derivation in N. Hence by Michael's selection theorem [9: Theorem 3.2″] there exists a continuous selection $b(\varphi)$ of S^* to N, that is, $\mathrm{ad}(b(\varphi)) = \delta_\varphi$. On the other hand, $R_{\hat{\varphi}}(c)$ is a generator of δ_φ so that

$$R_{\hat{\varphi}}(c) = b(\varphi) + z(\varphi)$$

for some element $z(\varphi)$ of Z. Consequently we have

$$\varepsilon(b(\varphi)) + z(\varphi) = \varepsilon(R_{\hat{\varphi}}(c)) = R_{\hat{\varphi}}(1 \otimes \varepsilon(c)) = 0 .$$

Hence,

$$z(\varphi) = -\varepsilon(b(\varphi))$$

and $z(\varphi)$ is a Z-valued continuous function on S^*. Thus we know that the map: $\varphi \in A^* \to R_{\hat{\varphi}}(c) \in N$ is a bounded linear map which is continuous for the weak $*$ and norm topologies in the unit sphere S^* of A^*. By the assumption for A we know by [3: Theorem 3.1] that A satisfies the metrical approximation property. Therefore, by [14: Proposition 4] there exists an element a in $A \otimes_\lambda N$ such that $r_\varphi(a) = R_{\hat{\varphi}}(c)$ for every $\varphi \in A^*$, where $A \otimes_\lambda N$ means the tensor product of A and N with respect to the least crossnorm λ and r_φ is the right slice map for φ defined in $A \otimes_\lambda N$. Let τ be the canonical map from $A \otimes N$ to $A \otimes_\lambda N$. Then, as the assumption for A implies that τ is an onto mapping ([15: Proposition 1]), there is an element c_0 in $A \otimes N$ with $\tau(c_0) = a$. We have

$$R_{\hat{\varphi}}(c_0) = R_\varphi(c_0) = r_\varphi(a) = R_{\hat{\varphi}}(c)$$

for every $\varphi \in A^*$ and $c = c_0 \in A \otimes N$. This completes the proof.

A C^*-algebra A is said to be n-homogeneous if every irreducible representation of A is n-dimensional. In this terminology, the commutativity coincides with the 1-homogeneity. The following corollary contains the result of Akemann and Johnson [2: Theorem 2.3] cited in the introduction.

COROLLARY 1.2. *Let A be a unital n-homogeneous C^*-algebra and N an arbitrary von Neumann algebra. Then every derivation of the algebra $A \otimes N$ is inner.*

PROOF. As is easily seen, the algebra $A \otimes Z$ is, in this case, n-homogeneous and every derivation of $A \otimes Z$ is inner (cf. [1: Theorem 3.2]).

COROLLARY 1.3. *Let M be a finite von Neumann algebra of type 1 with bounded degree and N an arbitrary von Neumann algebra. Then every derivation of the algebra $M \otimes N$ is inner.*

PROOF. This is because every derivation of $M \otimes Z$ is inner by the above corollary.

It is to be noticed that with the assumption for A in the theorem we cannot expect in general that every derivation of the algebra $A \otimes Z$ is inner (cf. [1], [5]).

2. Let M and N be von Neumann algebras. In [2] the problem whether or not every derivation of the algebra $M \otimes N$ is inner is considered and it is shown that the algebra $M \otimes N$ shares some properties related to the arguments on derivations for von Neumann algebras ([2: Theorems 4.3 and 4.8]). The difficult part of the problem lies mainly in the point that we do not have any criterion to tell us when an element c of $M \overline{\otimes} N$ belongs to the C^*-tensor product $M \otimes N$. Here we present a candidate for this sort of criteria. Namely, when an element c of $M \overline{\otimes} N$ belongs to $M \otimes N$, the mapping $r_c\colon \varphi \in M^* \to R_\varphi(c) \in N$ is, as we explained before, continuous for the weak $*$ and norm topologies on the unit sphere of M^* and the set $\{R_\varphi(c) \mid \varphi \in M^*, \|\varphi\| \leqq 1\}$ is compact for the norm topology in N. Thus the linear space $\{R_\varphi(c) \mid \varphi \in M^*\}$ is generated by the above compact (convex and circled) set. We are not able to decide whether or not the converse to this assertion is true except in a few cases which cover Corollary 1.3. However, if it would be the case the following proposition would settle the general problem.

PROPOSITION 2.1. *Let δ be a derivation of $M \otimes N$. Then there exists a generator c of δ in $M \overline{\otimes} N$ such that the map r_c is continuous for the weak $*$ and norm topologies on the unit sphere of M^*.*

PROOF. Let Z be the center of N and ε be a normal projection of norm one to Z. Then the derivation $1 \otimes \varepsilon \circ \delta$ in $M \otimes Z$ is inner by

Corollary 1.2. Let c_0 be a generator of $1 \otimes \varepsilon \circ \delta$ in $M \otimes Z$ and put $\delta_1 = \delta - \mathrm{ad}(c_0)$. Then the same arguments as in the proof of Theorem 1.1 show that there exists a generator c_1 of δ_1 with the map r_{c_1} having the required property. Hence, the generator $c = c_0 + c_1$ for δ has the property in the Proposition.

In the context of the tensor product $M \otimes_\lambda N$ for λ-norm it is almost evident that the property cited in the above proposition is symmetric. In our case, however, the element c is in $M \bar{\otimes} N$ and there is no canonical map from $M \bar{\otimes} N$ to $M \otimes_\lambda N$. Therefore the following remark should be considered.

PROPOSITION 2.2. *Take an element c in $M \bar{\otimes} N$. Then the map r_c is continuous for the weak $*$ and norm topologies if and only if the map 1_c satisfies the same continuity on the unit sphere of N^*.*

PROOF. It suffices to show that the property of c for generalized right slice maps implies that for left slice maps. Thus suppose the conclusion does not hold and let S^* be the unit sphere of N^*. Then there is a positive $\varepsilon > 0$ and a net $\{\psi_\alpha\}$ in S^* such that $\psi_\alpha \to \psi$ in the weak $*$ topology and $\|L_\psi(c) - L_{\psi_\alpha}(c)\| \geqq \varepsilon$. For each index α choose a functional φ_α in the unit sphere of M_* such that

$$|\langle L_\psi(c) - L_{\psi_\alpha}(c), \varphi_\alpha\rangle| > \varepsilon/2 .$$

Passing to the subnet, we may assume that the net $\{\varphi_\alpha\}$ converges to a functional φ in the weak $*$ topology. Now by the assumption for c and $\{\psi_\alpha\}$, we can find an index α_0 such that

$$\|R_\varphi(c) - R_{\varphi_{\alpha_0}}(c)\| < \varepsilon/8 \quad \text{and} \quad |\langle R_\varphi(c), \psi - \psi_{\alpha_0}\rangle| < \varepsilon/4 .$$

We have the following contradiction:

$$\begin{aligned}\varepsilon/2 \leqq |\langle R_{\varphi_{\alpha_0}}(c), \psi - \psi_{\alpha_0}\rangle| &\leqq |\langle R_{\varphi_{\alpha_0}}(c) - R_\varphi(c), \psi - \psi_{\alpha_0}\rangle| \\ &+ |\langle R_\varphi(c), \psi - \psi_{\alpha_0}\rangle| \leqq \|R_{\varphi_{\alpha_0}}(c) - R_\varphi(c)\| \; \|\psi - \psi_{\alpha_0}\| + \varepsilon/4 < \varepsilon/2 ,\end{aligned}$$

where the first inequality holds because the pairs of functionals $(\varphi_{\alpha_0}, \psi)$ and $(\varphi_{\alpha_0}, \psi_{\alpha_0})$ satisfy the Fubini principle by the definition of generalized slice maps.

For a derivation δ in the algebra $M \otimes N$, one may settle the problem if we impose a condition on δ. For instance, we have

PROPOSITION 2.3. *Keep the same notations as above. If there exists a maximal abelian subalgebra A of M for which $A \otimes N$ is invariant for δ, then δ is inner.*

PROOF. Let c be a generator of δ in $M \bar{\otimes} N$ and b be a generator of $\delta | A \otimes N$ in $A \otimes N$ by Corollary 1.2. Then, by the assumption the element $c - b$ belongs to the relative commutant of $A \otimes N$ in $M \bar{\otimes} N$ which coincides with the algebra $A \bar{\otimes} Z$ by the commutation theorem for von Neumann tensor products. Hence, any generator of δ belongs to $A \bar{\otimes} N$. Take a generator c of δ which satisfies the property in Proposition 2.1. Then regarding c as an element of $A \bar{\otimes} N$ one easily verifies that the map: $\varphi \in A^* \to R_\varphi(c) \in N$ is continuous for the weak $*$ and norm topologies. It follows that c belongs to the algebra $A \otimes N$ which is the space of all N-valued continuous functions on the spectrum of A. This completes the proof.

[Added in proof. August 19, 1979] The author is kindly informed by C. J. K. Batty on his paper "Derivations of tensor products of C^*-algebras, J. London Math. Soc. 17 (1978), 129–140", in which he proved a theorem (Theorem 3.5) closely related to our Theorem 1.1. In fact, although his result does not imply Theorem 1.1, with aids of the results in [2] one can modify his proof so that the arguments imply our theorem.

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