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A LOCALIZATION THEOREM FOR $\mathscr{D}$-MODULES

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Introduction. Atiyah [1] invented equivariant K-theory, and he proved the so-called *Atiyah-Bott character formula* in collaboration with Bott [2]. Here we try to generalize the Atiyah-Bott character formula to the $\mathscr{D}$-module case, when an algebraic torus T acts on an algebraic variety X.

There are two finiteness conditions for $\mathscr{D}$-modules, namely, coherency and holonomicity. We adopt an approach à la Grothendieck. Therefore we choose holonomicity as a finiteness condition, since holonomicity is preserved under pull-backs and push-forwards.

We should notice that the concept of weakly equivariant $\mathscr{D}$-modules in [5] is better suited for our purposes rather than that of (strongly) equivariant $\mathscr{D}$-modules, because the Grothendieck group of weakly T-equivariant holonomic $\mathscr{D}$-modules on a point is the representation ring $R(T)$ of T, whereas that of (strongly) T-equivariant holonomic $\mathscr{D}$-modules on a point is just $\boldsymbol{Z}$. Therefore we consider the Grothendieck group $K_{T,h}(X)$ of weakly T-equivariant holonomic $\mathscr{D}_X$-modules. However, a difficulty lies in the fact that the nontrivial $R(T)$-module $K_{T,h}(T)$ is free. In order to avoid this difficulty, we put more relations into the Grothendieck group $K_{T,h}(X)$. Namely, we suppose that we have a relation when there exists an exact sequence as T-equivariant $\mathscr{O}_X$-modules. This quotient group of $K_{T,h}(X)$ is denoted by $\tilde{K}_T(X)$.

In Section 1, we develop generalities on weakly equivariant $\mathscr{D}$-modules which are discussed in [10]. In Section 2, we prove the existence of good stratifications, which are appropriate for our purposes. In Section 3, we prove a localization theorem for $\tilde{K}_T(X)$:

THEOREM. *The homomorphism of $R(T)_\Lambda$-modules*

$$(\boldsymbol{R}\Gamma_{X^T})_\Lambda : \tilde{K}_T(X)_\Lambda \longrightarrow \tilde{K}_T(X)_\Lambda$$

is the identity, where Λ is the multiplicatively closed subset of $R(T)$ generated by $1-\chi$ for nontrivial characters χ of T.

In Section 4, we define a formal character morphism

$$\mathrm{ch}: \tilde{K}_T(X) \longrightarrow R(T)_\Lambda ,$$

when a smooth T-variety X satisfies a *positivity condition*. Then we obtain an Atiyah-Bott type character formula for holonomic $\mathscr{D}$-modules as a corollary of the localization thorem.

THEOREM. *Let X be a smooth T-variety with a positivity condition. Then we have a character formula*

$$\mathrm{ch}(\mathcal{M}) = \mathrm{ch}(\boldsymbol{R}\Gamma_{X^T}(\mathcal{M}))$$

for all weakly T-equivariant holonomic $\mathcal{D}_X$-modules $\mathcal{M}$.

Finally, we propose a conjecture.

CONJECTURE. *If X has only finitely many T-fixed points, then we can define a formal character morphism*

$$\mathrm{ch}: \tilde{K}_T(X) \longrightarrow R(T)_\Lambda .$$

As a consequence, we would obtain an Atiyah-Bott type character formula for holonomic $\mathcal{D}$-modules with applications to representations of algebraic groups.

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Joshua [10] independently proved a localization theorem using different techniques for coherent $\mathcal{D}_X$-modules when X is projective.

1. Equivariant $\mathcal{D}$-modules. Let X be a smooth algebraic variety over the complex number field $\boldsymbol{C}$. Throughout this paper, we assume any variety we treat is quasiprojective. We denote by $\mathcal{O}_X$ the structure sheaf of X, by $\mathcal{D}_X$ the sheaf of rings of linear differential operators on X.

In this section, we develop generalities on weakly equivariant $\mathcal{D}$-modules which are discussed in [10]. The concept of weakly equivariant $\mathcal{D}$-modules was introduced in [5]. One might refer to [3], [4], [8], [15], or [17] for basics on $\mathcal{D}$-modules. Surveys have been presented in [11] and [14]. We shall follow the notation in [8].

1.1. Notation and definitions. Let X be a smooth algebraic variety over $\boldsymbol{C}$ and G be an algebraic group acting on X throughout this section. Let $q: G\times X \to X$ be the group action and $p: G\times X\to X$ be the projection.

DEFINITION. A weakly G-equivariant $\mathcal{D}_X$-module is a $\mathcal{D}_X$-module $\mathcal{M}$ together with an isomorphism $\alpha: q^*\mathcal{M} \xrightarrow{\sim} p^*\mathcal{M}$ as $\mathcal{O}_G \boxtimes \mathcal{D}_X$-modules with the following cocycle condition: The diagram

$$\begin{tikzcd}
{[q\circ(1_G\times q)]^*\mathscr{M}} \arrow[r, "(1_G\times q)^*\alpha"] \arrow[d, equal] & {[p\circ(1_G\times q)]^*\mathscr{M}} \arrow[r, equal] & {[q\circ p_{23}]^*\mathscr{M}} \arrow[d, "p_{23}^*\alpha"] \\
{[q\circ(\mu\times 1_X)]^*\mathscr{M}} \arrow[r, "(\mu\times 1_X)^*\alpha"'] & {[p\circ(\mu\times 1_X)]^*\mathscr{M}} \arrow[r, equal] & {[p\circ p_{23}]^*\mathscr{M}}
\end{tikzcd}$$

on $G\times G\times X$ is commutative, where μ is the multiplication map of G, p_{23} is the projection $G\times G\times X\to G\times X$ to the second and the third factors.

DEFINITION. Suppose $\mathscr{M}$ and $\mathscr{N}$ are weakly G-equivariant $\mathscr{D}_X$-modules and φ is a $\mathscr{D}_X$-module morphism from $\mathscr{M}$ to $\mathscr{N}$. Then φ is a morphism of weakly G-equivariant $\mathscr{D}_X$-modules when the diagram

$$\begin{tikzcd}
q^*\mathscr{M} \arrow[r, "\sim"] \arrow[d, "q^*\varphi"'] & p^*\mathscr{M} \arrow[d, "p^*\varphi"] \\
q^*\mathscr{N} \arrow[r, "\sim"] & p^*\mathscr{N}
\end{tikzcd}$$

commutes.

LEMMA 1.1. *Suppose $\varphi:\mathscr{M}\to\mathscr{N}$ is a morphism of weakly G-equivariant $\mathscr{D}_X$-modules. Then* $\operatorname{Ker}(\varphi)$ *and* $\operatorname{Coker}(\varphi)$ *are weakly G-equivariant $\mathscr{D}_X$-modules as well.*

The proof is straightforward and left to the reader.

LEMMA 1.2. *The category of $\mathscr{O}_G\boxtimes\mathscr{D}_X$-modules which are quasicoherent as $\mathscr{O}_{G\times X}$-modules is abelian and has enough injectives.*

PROOF. This is a consequence of Lemma 1.1 and general theory by Grothendieck (cf. [7]).

DEFINITION. Let $D^b(\mathscr{D}_X)$ denote the derived category of bounded complexes of $\mathscr{D}_X$-modules on both sides. A weakly G-equivariant complex is a complex $\mathscr{M}\in D^b(\mathscr{D}_X)$ together with an isomorphism $\alpha: q^*\mathscr{M}\xrightarrow{\sim}p^*\mathscr{M}$ in $D^b(\mathscr{O}_G\boxtimes\mathscr{D}_X)$ with the cocycle condition as before.

Since p and q are flat, we obtain the following:

LEMMA 1.3. *Suppose $\mathscr{M}\in D^b(\mathscr{D}_X)$ is a weakly G-equivariant complex. Then each $H^i(\mathscr{M})$ is a weakly G-equivariant $\mathscr{D}_X$-module.*

DEFINITION. Let $\mathscr{M}$, $\mathscr{M}'\in D^b(\mathscr{D}_X)$ be weakly G-equivariant. A morphism $\varphi:\mathscr{M}\to\mathscr{M}'$ in $D^b(\mathscr{D}_X)$ is a morphism of weakly G-equivariant complexes when the diagram

$$\begin{tikzcd} q^*\mathcal{M} \arrow[r, "\sim"] \arrow[d, "q^*\varphi"'] & p^*\mathcal{M} \arrow[d, "p^*\varphi"] \\ q^*\mathcal{M}' \arrow[r, "\sim"] & p^*\mathcal{M}' \end{tikzcd}$$

commutes in $D^b(\mathcal{O}_G \boxtimes \mathcal{D}_X)$.

NOTATION. The category of weakly G-equivariant complexes of $\mathcal{D}_X$-modules is denoted by $D_G^b(\mathcal{D}_X)$. The category of G-equivariant complexes of $\mathcal{O}_X$-modules is denoted by $D_G^b(\mathcal{O}_X)$.

1.2. Inverse images.

LEMMA 1.4. *Suppose $f\colon X \to Y$ is a morphism of smooth algebraic varieties. Then the diagram*

$$\begin{tikzcd} D^b(\mathcal{D}_{G\times Y}) \arrow[r, "L(1\times f)^*"] \arrow[d, "F"'] & D^b(\mathcal{D}_{G\times X}) \arrow[d, "F"] \\ D^b(\mathcal{O}_G \boxtimes \mathcal{D}_Y) \arrow[r, "L(1\times f)^*"] \arrow[d, "F"'] & D^b(\mathcal{O}_G \boxtimes \mathcal{D}_X) \arrow[d, "F"] \\ D^b(\mathcal{O}_{G\times Y}) \arrow[r, "L(1\times f)^*"] & D^b(\mathcal{O}_{G\times X}) \end{tikzcd}$$

commutes, where F's are forgetful functors.

PROOF. This is clear because we can calculate inverse images of $\mathcal{D}$-modules in the category of $\mathcal{O}$-modules. ∎

PROPOSITION 1.5. *Suppose $f\colon X \to Y$ is a morphism of smooth algebraic G-varieties. Then we can define functors $Lf^*\colon D_G^b(\mathcal{D}_Y) \to D_G^b(\mathcal{D}_X)$ and $Lf^*\colon D_G^b(\mathcal{O}_Y) \to D_G^b(\mathcal{O}_X)$. Moreover, we have the commutative diagram*

$$\begin{tikzcd} D_G^b(\mathcal{D}_Y) \arrow[r, "Lf^*"] \arrow[d, "F"'] & D_G^b(\mathcal{D}_X) \arrow[d, "F"] \\ D_G^b(\mathcal{O}_Y) \arrow[r, "Lf^*"] & D_G^b(\mathcal{O}_X)\,. \end{tikzcd}$$

PROOF. The proof is straightforward and the commutativity of the diagram follows from Lemma 1.4. ■

PROPOSITION 1.6. *Let* $f^!\mathscr{M} := Lf^*\mathscr{M}[\dim X - \dim Y]$. *Then the same statements as in Proposition* 1.5 *hold true for* $f^!$.

1.3. Direct images.

LEMMA 1.7. *Let* $f\colon X \to Y$ *be a morphism of smooth algebraic varieties. Define a functor* $\int_{1\times f}\colon D^b(\mathscr{O}_G \boxtimes \mathscr{D}_X) \to D^b(\mathscr{O}_G \boxtimes \mathscr{D}_Y)$ *by*

$$\int_{1\times f} \mathscr{M} = \boldsymbol{R}(1\times f)_*(\mathscr{O}_G \boxtimes \mathscr{D}_{Y\leftarrow X} \overset{\boldsymbol{L}}{\underset{\mathscr{O}_G \boxtimes \mathscr{D}_X}{\otimes}} \mathscr{M}) .$$

Then the diagram

$$\begin{array}{ccc} D^b(\mathscr{D}_{G\times X}) & \xrightarrow{\int_{1\times f}} & D^b(\mathscr{D}_{G\times Y}) \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ D^b(\mathscr{O}_G \boxtimes \mathscr{D}_X) & \xrightarrow{\int_{1\times f}} & D^b(\mathscr{O}_G \boxtimes \mathscr{D}_Y) \end{array}$$

commutes.

PROOF. For a $\mathscr{D}_{G\times X}$-module $\mathscr{M}$, we have

$$\begin{aligned}(1\times f)_*(\mathscr{D}_{G\times Y\leftarrow G\times X} \underset{\mathscr{D}_{G\times X}}{\otimes} \mathscr{M}) &= (1\times f)_*(\mathscr{D}_G \boxtimes \mathscr{D}_{Y\leftarrow X} \underset{\mathscr{D}_G \boxtimes \mathscr{D}_X}{\otimes} \mathscr{M}) \\ &\simeq (1\times f)_*(\mathscr{O}_G \boxtimes \mathscr{D}_{Y\leftarrow X} \underset{\mathscr{O}_G \boxtimes \mathscr{D}_X}{\otimes} \mathscr{M})\end{aligned}$$

as $\mathscr{O}_G \boxtimes \mathscr{D}_Y$-modules. Hence the lemma follows. ■

PROPOSITION 1.8. *Let* $f\colon X \to Y$ *be a morphism of smooth algebraic G-varieties. If a complex* $\mathscr{M} \in D^b(\mathscr{D}_X)$ *is weakly G-equivariant, then* $\int_f \mathscr{M}$ *is weakly G-equivariant as well.*

PROOF. This follows easily from Lemma 1.7 and the base change theorem (see [8], for instance). ■

Discussion on direct images will be taken up again in Section 3.

1.4. Local cohomologies.

PROPOSITION 1.9. *Suppose* $Z_1 \subset Z_2 \subset X$ *are G-stable closed algebraic subsets. Then we obtain*

$$p^*\boldsymbol{R}\Gamma_{Z_2/Z_1}(\mathscr{M}) = \boldsymbol{R}\Gamma_{(G\times Z_2)/(G\times Z_1)}(p^*\mathscr{M})$$

and

$$q^*\boldsymbol{R}\Gamma_{Z_2/Z_1}(\mathscr{M})=\boldsymbol{R}\Gamma_{(G\times Z_2)/(G\times Z_1)}(q^*\mathscr{M})$$

for all $\mathscr{M}\in D^b(\mathscr{D}_X)$.

PROOF. We can easily check $p^*\Gamma_Z(\mathscr{M})=\Gamma_{G\times Z}(p^*\mathscr{M})$ as $\mathscr{D}_{G\times X}$-modules for any $\mathscr{D}_X$-module $\mathscr{M}$. Hence we obtain $p^*\Gamma_{Z_2/Z_1}(\mathscr{M})=\Gamma_{(G\times Z_2)/(G\times Z_1)}(p^*\mathscr{M})$ as $\mathscr{D}_{G\times X}$-modules.

Let $P:=p^*\circ\Gamma_{Z_2/Z_1}=\Gamma_{(G\times Z_2)/(G\times Z_1)}\circ p^*$. Then we have, in the category $D^b(\mathscr{D}_{G\times X})$, $\boldsymbol{R}P=p^*\circ\boldsymbol{R}\Gamma_{Z_2/Z_1}=\boldsymbol{R}\Gamma_{(G\times Z_2)/(G\times Z_1)}\circ p^*$. Similarly the above formula holds in the categories $D^b(\mathscr{O}_G\boxtimes\mathscr{D}_X)$ and $D^b(\mathscr{O}_{G\times X})$. Moreover, these three $\boldsymbol{R}P$'s are compatible, because $\mathscr{D}_X$ (resp. $\mathscr{D}_G$) is faithfully flat over $\mathscr{O}_X$ (resp. $\mathscr{O}_G$). The same arguments work for "q" as well. ∎

COROLLARY 1.10. *In the same situation as in Proposition* 1.9, *we have the following commutative diagram*:

$$\begin{array}{ccc} D^b_G(\mathscr{D}_X) & \xrightarrow{\boldsymbol{R}\Gamma_{Z_2/Z_1}} & D^b_G(\mathscr{D}_X) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ D^b_G(\mathscr{O}_X) & \xrightarrow{\boldsymbol{R}\Gamma_{Z_2/Z_1}} & D^b_G(\mathscr{O}_X)\,. \end{array}$$

PROOF. This is clear from the proof of the previous proposition. ∎

2. Stratification. In this section, we collect some known facts. From those, we deduce the existence of a good stratification. This stratification enables us to define direct images on the level of a "modified" equivariant Grothendieck group $\tilde{K}_T(\ )$ defined in the next section, and to prove a localization theorem for $\tilde{K}_T(\ )$.

The following lemma is a result due to T. Oda.

LEMMA 2.1 (cf. [18]). *Let T be an algebraic torus and let X be a normal variety on which T acts regularly. Then, for any point x in X, there is a T-stable affine open neighborhood of x.*

DEFINITION. Let G be an algebraic group and X be a G-space. Two points x and y in X are said to be of the same orbit type when the respective isotropy subgroups G_x and G_y are conjugate.

DEFINITION. Let G be an algebraic group and X be a G-space. An open dense subset U of X is called a principal orbit type for (G, X) if all points of U are of the same orbit type.

LEMMA 2.2 (cf. [16, Proposition 5-3]). *Suppose a reductive affine algebraic group G acts regularly on a smooth affine algebraic variety X. Then there exists a principal orbit type for (G, X).*

LEMMA 2.3 (cf. [13, Lemma 0-6]). *Assume that the geometric quotient Y of X by G exists. Then Y is a scheme if and only if the action is separated.*

LEMMA 2.4 (cf. [13, Proposition 1-9]). *Let G be a reductive algebraic group acting on X. Then a universal geometric quotient (Y, ϕ) of $X^s(\mathrm{Pre})$ by G exists, where $X^s(\mathrm{Pre}) = \{x \in X \mid x$ is prestable$\}$. Moreover, ϕ is affine and Y is an algebraic prescheme.*

LEMMA 2.5. *Let T be an algebraic torus. Let X be an affine smooth T-algebraic variety, and assume that the action of T on X is set-theoretically free and the geometric quotient exists and is smooth. Then there exists an affine open T-subvariety Y of X, which is T-isomorphic to $Y/T \times T$, on which T acts by translation on the second factor.*

PROOF. Let $X(R)$ be the set of the weights of $R := \boldsymbol{C}[X]$ with respect to the action of T. We claim that $X(T)$ is generated by $X(R)$ as a $\boldsymbol{Z}$-module. Assume the contrary and put

$$T' = \bigcap_{\lambda \in X(R)} \mathrm{Ker}(\lambda) .$$

Then T' is nontrivial. It is clear that T' acts on R trivially. Hence T' acts on X trivially. This contradicts the assumption.

Now, let $X(T) = \boldsymbol{Z}\lambda_1 + \dots + \boldsymbol{Z}\lambda_m$, and let $f_1, \dots, f_m \in R$ be nonzero weight vectors corresponding to $\lambda_1, \dots, \lambda_m$. Set $Y = (f_1 \neq 0) \cap \dots \cap (f_m \neq 0)$. Then each f_i is invertible on Y. Let $R' = \boldsymbol{C}[Y]$. Fix any basis $\{\mu_1, \dots, \mu_n\}$ of $X(T)$. Then we have invertible weight vectors $g_1, \dots, g_n \in R'$ corresponding to $\mu_1, \dots, \mu_n$. Let $A = \boldsymbol{C}[g_1, g_1^{-1}, \dots, g_n, g_n^{-1}]$. The map

$$R' \ni v \mapsto g_1^{c_1} \dots g_n^{c_n} \otimes g_1^{-c_1} \dots g_n^{-c_n} v \in A \otimes R'^T$$

gives an isomorphism, where v is a weight vector with weight $\mu_1^{c_1} \dots \mu_n^{c_n}$. In fact, we see that A is isomorphic to $\boldsymbol{C}[T]$ as follows: Clearly we have a surjection $\boldsymbol{C}[T] \to A$. If this is not injective, then $g_1^{c_1} \dots g_n^{c_n} = 0$ for some $c_1, \dots, c_n$, because the kernel of the above map is a T-submodule. This contradicts the fact that each g_i is invertible. ∎

From now on we assume that T is an algebraic torus and X is a smooth algebraic variety on which T acts regularly.

REMARK. Since T is separated, the action of T is separated.

PROPOSITION 2.6. *The space X has a good stratification*

$$X = \coprod_{i=0}^{l} X_i$$

such that $X_i = V_i \times T/T_i$, where each V_i is an affine smooth variety and T acts on V_i trivially. Moreover, each X_i has an affine T-stable open neighborhood U_i such that X_i is closed in U_i, and the normal bundle sheaf $\mathscr{N}_{X_i|U_i}$ is a free $\mathscr{O}_{X_i}$-module.

PROOF. From Lemmas 2.1 and 2.2, there exists an open dense subset W_0 in X such that $T_x = T_y$ for all $x, y \in W_0$. Denote this common stabilizer by T_0. Since $W_0^s(\mathrm{Pre}) = W_0$, W_0 has the geometric quotient by Lemma 2.4. Thanks to Lemma 2.5, we can take V_0 in such a way that V_0 is an affine open smooth subscheme of $W_0/(T/T_0)$ and $W_0 \supset V_0 \times T/T_0 =: X_0$.

Set $Y_1 := X - X_0$. Then T acts on $(Y_1)_{\mathrm{reg}}$. Let $(Y_1)_{\mathrm{reg}} = Z_1 \amalg \cdots \amalg Z_m$ be the decomposition of $(Y_1)_{\mathrm{reg}}$ into its connected components. We can find an open smooth T-stable affine subvariety $V_1' \times T/T_1$ of Z_1 such that T acts on V_1' trivially. Put $X_0' := X_0 \amalg (V_1' \times T/T_1)$. Then X_0' is an open T-stable subvariety of X. Since we have a *slice* in $V_1' \times T/T_1$, we can assume $\mathscr{N}_{(V_1' \times T/T_1)|X}$ to be a free $\mathscr{O}_{V_1' \times T/T_1}$-module. Take any point $x_1 \in V_1' \times T/T_1$. From Lemma 2.1, there exists an affine open T-stable neighborhood U_{x_1} of x_1 in X_0'. Since $V_1' \times T/T_1$ is closed in X_0', $(X_1 := (V_1' \times T/T_1) \cap U_{x_1}, U_1 := U_{x_1})$ satisfies the condition. Repeat this process to obtain the desired stratification. ■

3. Localization theorem. In this section, we define a modified equivariant Grothendieck group $\tilde{K}_T(\ )$. After we look at some examples of $\tilde{K}_T(X)$, we show that we can define direct images for good closed immersions on the level of $\tilde{K}_T(\ )$. Since we can define inverse images on the level of $\tilde{K}_T(\ )$ as in Section 1, we can deduce a localization theorem from the existence of a good stratification in Section 2 and the computation of $\tilde{K}_T(T/T' \times V)$.

3.1. Definition of $\tilde{K}_T(X)$. Let G be an algebraic group which acts on a smooth algebraic variety X. Then $D^b_{G,h}(\mathscr{D}_X)$ denotes the subcategory of $D^b_G(\mathscr{D}_X)$ consisting of all complexes whose cohomology groups are all holonomic.

DEFINITION. We define $L_G(X)$ to be the quotient of the free abelian group generated by all objects $\mathscr{M} \in D^b_{G,h}(\mathscr{D}_X)$ divided by the relation $[\mathscr{M}] = [\mathscr{L}] + [\mathscr{N}]$, if there are morphisms $\mathscr{L} \to \mathscr{M}$, $\mathscr{M} \to \mathscr{N}$ and $\mathscr{N} \to \mathscr{L}[+1]$ in $D^b_G(\mathscr{O}_X)$, which induce the triangle

$$\begin{tikzcd} & \mathscr{M} \arrow[dr] & \\ \mathscr{L} \arrow[ur] & & \mathscr{N} \arrow[ll, "+1"] \end{tikzcd}$$

in $D^b(\mathscr{O}_X)$.

DEFINITION. We define $\tilde{K}_G(X)$ to be the submodule of $L_G(X)$ generated by objects concentrated in degree 0.

REMARK. We have a natural isomorphism $\tilde{K}_G(X) \xrightarrow{\sim} L_G(X)$ since, for $\mathscr{M} \in D^b_{G,h}(\mathscr{D}_X)$, we have $[\mathscr{M}] = \sum_i (-1)^i [H^i(\mathscr{M})]$ in $L_G(X)$.

LEMMA 3.1. *Suppose $f: X \to Y$ is a morphism of smooth algebraic G-varieties. Then we can define maps from $\tilde{K}_G(Y)$ to $\tilde{K}_G(X)$ by sending $[\mathscr{M}]$ to $[Lf^*(\mathscr{M})]$ and $[f^!(\mathscr{M})]$,* *respectively. They will be denoted by* $\boldsymbol{L}f^*$ *and* $f^!$, *respectively.*

PROOF. This follows easily from Proposition 1.5. ■

LEMMA 3.2. *Suppose* $Z_1 \subset Z_2$ *are* G*-stable closed subsets of* X. *Then we can define an endomorphism of* $\tilde{K}_G(X)$ *by sending* $[\mathscr{M}]$ *to* $[\boldsymbol{R}\Gamma_{Z_2/Z_1}(\mathscr{M})]$. *This endomorphism will be denoted by* $\boldsymbol{R}\Gamma_{Z_2/Z_1}$ *again.*

PROOF. This follows easily from Corollary 1.10. ■

Denote by $R(G)$ the representation ring of G. Clearly we have $\tilde{K}_G(\{\mathrm{pt}\}) = R(G)$. Therefore $\tilde{K}_G(X)$ becomes an $R(G)$-module by tensoring with $f^*(\lambda)$, where $f\colon X \to \{\mathrm{pt}\}$ and λ is a finite dimensional G-module.

PROPOSITION 3.3. (1) *Let* $g\colon X \to Y$ *be a morphism of smooth* G*-varieties. Then* $\boldsymbol{L}g^*\colon \tilde{K}_G(Y) \to \tilde{K}_G(X)$ *is an* $R(G)$*-morphism.*

(2) *Let* $Z_1 \subset Z_2$ *be two closed subsets of* X. *Then* $\boldsymbol{R}\Gamma_{Z_2/Z_1}\colon \tilde{K}_G(X) \to \tilde{K}_G(X)$ *is an* $R(G)$*-morphism.*

PROOF. (1) It is not hard to see $\alpha_{\boldsymbol{L}g^*\mathscr{N}} = \boldsymbol{L}(1 \times g)^*\alpha_{\mathscr{N}}$. Let k be the map $Y \to \{\mathrm{pt}\}$. Then we can check by a simple computation that $\alpha_{\boldsymbol{L}g^*(k^*(\lambda) \otimes \mathscr{N})} = \alpha_{f^*(\lambda)} \otimes \alpha_{\boldsymbol{L}g^*\mathscr{N}}$. This means that $\boldsymbol{L}g^*$ is an $R(G)$-morphism.

(2) We also see $\alpha_{\boldsymbol{R}\Gamma_{Z_2/Z_1}(\mathscr{M})} = \boldsymbol{R}\Gamma_{G \times Z_2/G \times Z_1} \circ (\alpha_{\mathscr{M}})$. Again by a simple computation we obtain $\alpha_{\boldsymbol{R}\Gamma_{Z_2/Z_1}(\mathscr{M} \otimes f^*(\lambda))} = \alpha_{\boldsymbol{R}G_{Z_2/Z_1}(\mathscr{M})} \otimes \alpha_{f^*(\lambda)}$. Therefore $\boldsymbol{R}\Gamma_{Z_2/Z_1}\colon \tilde{K}_G(X) \to \tilde{K}_G(X)$ is an $R(G)$-morphism. ■

From now on, we assume $G = T$, an algebraic torus. Denote by $K_T(X)$ the Grothendieck group of all T-equivariant locally free $\mathscr{O}_X$-modules of finite rank.

PROPOSITION 3.4. *Assume* X *to be a homogeneous* T*-variety with* $X = T/T'$ *for a closed subgroup* $T' \subset T$. *Then* $\tilde{K}_T(X) = K_T(X) = R(T')$.

PROOF. Let $\mathscr{M}$ be a weakly T-equivariant nonzero holonomic $\mathscr{D}_X$-module. By looking at the support of $\mathscr{M}$, it is clear that $\mathscr{M}$ is an $\mathscr{O}_X$-coherent module. Hence we have a natural map $\tilde{K}_T(X) \to K_T(X) = R(T')$. Let $\lambda \in X(T)$ and let f be the map from X to $\{\mathrm{pt}\}$. Then a weakly T-equivariant $\mathscr{D}_X$-module $f^*(\lambda)$ is isomorphic to $\mathscr{O}(\bar{\lambda})$ as T-equivariant $\mathscr{O}_X$-modules by multiplying the image $\bar{\lambda}$ of λ in $X(T')$, where $\mathscr{O}(\bar{\lambda})$ is the sheaf of sections of the vector bundle $T \times^{T'} \boldsymbol{C}_{-\bar{\lambda}} \to X$. Thus the natural map $\tilde{K}_T(X) \to K_T(X)$ is surjective.

Suppose a weakly T-equivariant $\mathscr{D}_X$-module $\mathscr{M}$ is isomorphic to $\bigoplus_{i=1}^m \mathscr{O}(\bar{\lambda}_i)$ as T-equivariant $\mathscr{O}_X$-modules. Then clearly $[\mathscr{M}] = \sum_{i=1}^m [f^*(\lambda_i)]$ in $\tilde{K}_T(X)$. Hence the maps $\tilde{K}_T(X) \ni [f^*(\lambda)] \mapsto [\mathscr{O}(\bar{\lambda})] \in K_T(X)$ and $K_T(X) \ni [\mathscr{O}(\bar{\lambda})] \mapsto [f^*(\lambda)] \in \tilde{K}_T(X)$ (this is well defined by Lemma 3.5) are inverse to each other. Therefore we obtain the proposition. ■

LEMMA 3.5. *Let* $X = T/T' \times V$, *where* T *acts on* V *trivially. Let* λ, $\lambda' \in X(T)$ *and* *assume that* $\lambda_{|T'}=\lambda'_{|T'}$. *Then we have* $f^*(\lambda)\xrightarrow{\sim}f^*(\lambda')$ *as* T*-equivariant* $\mathcal{O}_X$*-modules, where* $f\colon X\to\{\mathrm{pt}\}$.

PROOF. The function λ/λ' on T/T' can be extended trivially to a function on X. Define an automorphism φ of the $\mathcal{O}_X$-module $\mathcal{O}_X$ by the formula $\varphi\colon \mathcal{O}_X\ni h\mapsto (\lambda/\lambda')h\in\mathcal{O}_X$. Then φ gives an isomorphism between $f^*(\lambda')$ and $f^*(\lambda)$ as T-equivariant $\mathcal{O}_X$-modules. ∎

COROLLARY 3.6. *Let* Λ *be the multiplicatively closed subset of* $R(T)$ *generated by* $1-\lambda$ *for nontrivial characters* λ *of* T. *Then we obtain* $\tilde{K}_T(T/T'\times V)_\Lambda=0$, *unless* $T'=T$.

PROOF. Unless $T'=T$, there exists a nontrivial character $\lambda\in X(T)-\{1\}$ such that $\lambda_{|T'}=1$. Therefore we obtain $[f^*(\lambda)]=[f^*(1)]$ from the previous lemma. On the other hand, we know $1-\lambda\in\Lambda$. Therefore we obtain the corollary. ∎

3.2. Direct images revisited.

LEMMA 3.7. *Let* $i\colon X=V\times T/T'\to Y$ *be a closed equivariant embedding of a smooth affine* T*-variety* X *into a smooth algebraic* T*-variety* Y, *where* T *acts on* V *trivially. We assume the normal bundle sheaf* $\mathcal{N}_{X|Y}$ *to be a* T*-equivariant free* $\mathcal{O}_X$*-module. We fix a basis* $v=\{v_1,\dots,v_m\}$ *so that* $\mathcal{N}_{X|Y}=\mathcal{O}_Xv_1\oplus\cdots\oplus\mathcal{O}_Xv_m$ *as* T*-equivariant* $\mathcal{O}_X$*-modules and that the restriction* $\mathrm{res}(v_i)$ *of each* v_i *to* $V\times\{\bar e\}$ *is a* T'*-weight vector. Then we can define a functor*

$$\int_{i,v}\colon D^b(\mathcal{O}_X)\to D^b(\mathcal{O}_Y)$$

so that the following diagram commutes:

$$\begin{array}{ccc} D^b(\mathcal{D}_X) & \xrightarrow{\int_i} & D^b(\mathcal{D}_Y) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ D^b(\mathcal{O}_X) & \xrightarrow{\int_{i,v}} & D^b(\mathcal{O}_Y)\,. \end{array}$$

PROOF. By definition,

$$\mathcal{D}_{Y\leftarrow X}=\omega_X\otimes i^*(\mathcal{D}_Y\underset{\mathcal{O}_Y}{\otimes}\omega_Y^{-1})=\omega_X\otimes(\bigoplus_\alpha\mathcal{D}_Xv_1^{\alpha_1}\cdots v_m^{\alpha_m})\otimes i^*\omega_Y^{-1}\,,$$

where $\alpha=(\alpha_1,\dots,\alpha_m)$ is a multi-index. Hence if $\mathcal{M}$ is a $\mathcal{D}_X$-module, then

$$\mathcal{D}_{Y\leftarrow X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M}=\bigoplus_\alpha v_m^{\alpha_m}\cdots v_1^{\alpha_1}\mathcal{O}_X\underset{\mathcal{O}_X}{\otimes}\mathcal{M}\otimes(\omega_X\otimes i^*\omega_Y^{-1})$$

as $i^{-1}\mathcal{O}_Y$-modules. Hence we define $\int_{i,v}\colon D^b(\mathcal{O}_X)\to D^b(\mathcal{O}_Y)$ by

$$\int_{i,v} \mathscr{M} := i_*\Big(\bigoplus_{\alpha} v_m^{\alpha_m} \cdots v_1^{\alpha_1} \mathscr{O}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{M} \otimes (\omega_X \otimes i^*\omega_Y^{-1})\Big) .$$

Clearly this satisfies the requirement. ∎

LEMMA 3.8. *Let X, Y, i and v be the same as in Lemma 3.7. Then we can define a functor*

$$\int_{1\times i,v} : D^b(\mathscr{O}_{T\times X}) \to D^b(\mathscr{O}_{T\times Y})$$

so that the following diagram commutes:

$$\begin{array}{ccc} D^b(\mathscr{O}_T \times \mathscr{D}_X) & \xrightarrow{\int_{1\times i}} & D^b(\mathscr{O}_T \times \mathscr{D}_Y) \\ \Big\downarrow F & & \Big\downarrow F \\ D^b(\mathscr{O}_{T\times X}) & \xrightarrow{\int_{1\times i,v}} & D^b(\mathscr{O}_{T\times Y}) . \end{array}$$

PROOF. Define $\mathscr{O}^v_{Y\leftarrow X} := \omega_X \otimes \mathrm{Hom}_{\mathscr{O}_X}(i^*\omega_Y, \bigoplus_\alpha \mathscr{O}_X v_1^{\alpha_1} \cdots v_m^{\alpha_m})$. If we define $\int_{1\times i,v}$ by

$$\int_{1\times i,v} \mathscr{M} := \boldsymbol{R}(1\times i)_*(\mathscr{O}_T \boxtimes \mathscr{O}^v_{Y\leftarrow X} \underset{\mathscr{O}_{T\times X}}{\otimes} \mathscr{M}) = (1\times i)_*(\mathscr{O}_T \boxtimes \mathscr{O}^v_{Y\leftarrow X} \underset{\mathscr{O}_{T\times X}}{\otimes} \mathscr{M}) ,$$

then the statement is obvious from the proof of Lemma 3.7. ∎

LEMMA 3.9. *Keeping the situation and notation as in Lemma 3.7, we have an isomorphism*

$$p_Y^* \int_{i,v} \mathscr{M} \xrightarrow{\sim} \int_{1\times i,v} p_X^* \mathscr{M}$$

in $D^b(\mathscr{O}_{T\times Y})$ for $\mathscr{M} \in D^b(\mathscr{O}_X)$.

PROOF. This is clear from the definitions of $\int_{i,v}$ and $\int_{1\times i,v}$. ∎

LEMMA 3.10. *Keeping the situation and notation as in Lemma 3.7, we have an isomorphism*

$$(\bar{q}_Y)^* \int_{1\times i,v} \mathscr{M} \xrightarrow{\sim} \int_{1\times i,v} (\bar{q}_X)^* \mathscr{M}$$

in $D^b(\mathscr{O}_{T\times Y})$ for $\mathscr{M} \in D^b(\mathscr{O}_{T\times X})$, where the automorphism $\bar{q}_Y$ is defined by $T\times Y \ni (t, y) \mapsto (t, ty) \in T\times Y$ and $(\bar{q}_X)^*$ is its restriction to $T\times X$.

PROOF. By definition, we have

$$(\bar{q}_Y)^*\int_{1\times i,v}\mathscr{M}\xrightarrow{\sim}(\bar{q}_Y)^*(1\times i)_*(\mathcal{O}_T\boxtimes(\mathcal{O}^v_{Y\leftarrow X}\otimes\mathscr{M}))$$
$$\xrightarrow{\sim}(\bar{q}_{Y*})^{-1}(1\times i)_*(\mathcal{O}_T\boxtimes(\mathcal{O}^v_{Y\leftarrow X}\otimes\mathscr{M}))$$
$$\xrightarrow{\sim}((1\times i)\circ(\bar{q}_{X*})^{-1}(\mathcal{O}_T\boxtimes(\mathcal{O}^v_{Y\leftarrow X}\otimes\mathscr{M}))\,.$$

Since we have taken weight vectors $v_1, \ldots, v_m$ and $\mathcal{O}^v_{Y\leftarrow X}$ is a free $\mathcal{O}_X$-module, we have an isomorphism

$$(\bar{q}_{X*})^{-1}(\mathcal{O}_T\boxtimes(\mathcal{O}^v_{Y\leftarrow X}\otimes\mathscr{M}))\xrightarrow{\sim}\mathcal{O}_T\boxtimes(\mathcal{O}^v_{Y\leftarrow X}\otimes(\bar{q}_{X*})^{-1}\mathscr{M})\,.$$

Therefore we conclude

$$(\bar{q}_Y)^*\int_{1\times i,v}\mathscr{M}\xrightarrow{\sim}(1\times i)_*(\mathcal{O}_T\boxtimes(\mathcal{O}^v_{Y\leftarrow X}\otimes(\bar{q}_{X*})^{-1}\mathscr{M}))$$
$$\xrightarrow{\sim}(1\times i)_*(\mathcal{O}_T\boxtimes(\mathcal{O}^v_{Y\leftarrow X}\otimes(\bar{q}_X)^*\mathscr{M}))\xrightarrow{\sim}\int_{1\times i,v}(\bar{q}_X)^*\mathscr{M}\,.$$

■

PROPOSITION 3.11. *Keeping the situation and notation as in Lemma* 3.7, *the functor* $\int_{i,v}: D^b(\mathcal{O}_X)\to D^b(\mathcal{O}_Y)$ *induces* $\int_{i,v}: D^b_T(\mathcal{O}_X)\to D^b_T(\mathcal{O}_Y)$.

PROOF. For an equivariant complex $\mathscr{M}\in D^b_T(\mathcal{O}_X)$, its equivariance datum $\alpha: q_X^*\mathscr{M}\xrightarrow{\sim}p_X^*\mathscr{M}$ is an isomorphism in $D^b(\mathcal{O}_{T\times X})$. This α induces an isomorphism

$$\int_{1\times i,v}\alpha:\int_{1\times i,v}q_X^*\mathscr{M}\xrightarrow{\sim}\int_{1\times i,v}p_X^*\mathscr{M}$$

in $D^b(\mathcal{O}_{T\times Y})$. By the base change theorems (Lemmas 3.9 and 3.10), we finally obtain an isomorphism

$$q_Y^*\int_{i,v}\mathscr{M}\xrightarrow{\sim}p_Y^*\int_{i,v}\mathscr{M}$$

in $D^b(\mathcal{O}_{T\times Y})$ because $q_Y=p_Y\circ\bar{q}_Y$ and $q_X=p_X\circ\bar{q}_X$. ■

COROLLARY 3.12. *Keeping the situation and notation as in Lemma* 3.7, *the functor* $\int_i$ *induces a map* $\tilde{K}_T(X)\to\tilde{K}_T(Y)$, *which is denoted by* $\int_i$ *again.*

PROOF. This follows easily from Lemma 3.7 and Proposition 3.11. ■

PROPOSITION 3.13. *Keeping the situation and notation as in Lemma* 3.7, *the map* $\int_i: \tilde{K}_T(X)\to\tilde{K}_T(Y)$ *is an* $R(T)$*-morphism.*

PROOF. It is not hard to see that

$$\alpha_{\int_i \mathscr{M}} = \int_{1 \times i} \alpha_{\mathscr{M}}$$

for a weakly T-equivariant $\mathscr{D}_X$-module $\mathscr{M}$. Let f (resp. k) denote the map from X (resp. Y) to $\{\mathrm{pt}\}$. Then using the projection formula (cf. [6]), we obtain for $\lambda \in X(T)$

$$\alpha_{\int_i f^*(\lambda) \otimes \mathscr{M}} = \alpha_{k^*(\lambda)} \otimes \alpha_{\int_i \mathscr{M}} \, .$$

Thus $\int_i$ is an $R(T)$-morphism. ■

3.3. Localization theorem. We have a good stratification of X described in Proposition 2.6. We fix such a stratification

$$X = \coprod_{i=0}^{n} X_i$$

once and for all. Let $j_i \colon X_i \to X$ be the inclusion.

THEOREM 3.14. *We have an isomorphism of $R(T)$-modules*

$$\tilde{K}_T(X) \xrightarrow{\sim} \bigoplus_{i=0}^{n} \tilde{K}_T(X_i) \, .$$

PROOF. Since X_n is a closed smooth subvariety of X, we have a triangle

$$\begin{tikzcd}[column sep=small] & \mathscr{M} \arrow[dr] & \\ \int_{j_n} j_n^! \mathscr{M} \arrow[ur] & & \int_{\bar{j}_n} \bar{j}_n^! \mathscr{M} \arrow[ll, "+1"'] \end{tikzcd}$$

in $D^b(\mathscr{D}_X)$ for an object $\mathscr{M}$ of $D^b_{T,h}(\mathscr{D}_X)$, where $\bar{j}_n \colon X - X_n \to X$ is the inclusion. Define a map $f_n \colon \tilde{K}_T(X) \to \tilde{K}_T(X_n) \oplus \tilde{K}_T(X - X_n)$ by $f_n([\mathscr{M}]) = j_n^![\mathscr{M}] \oplus \bar{j}_n^![\mathscr{M}]$. Define a map $g_n \colon \tilde{K}_T(X_n) \oplus \tilde{K}_T(X - X_n) \to \tilde{K}_T(X)$ by $g_n([\mathscr{N}_1] \oplus [\mathscr{N}_2]) = \int_{j_n} [\mathscr{N}_1] + \int_{\bar{j}_n} [\mathscr{N}_2]$. Then we have $g_n \circ f_n([\mathscr{M}]) = \int_{j_n} j_n^![\mathscr{M}] + \int_{\bar{j}_n} \bar{j}_n^![\mathscr{M}] = [\mathscr{M}]$. On the other hand, we also have $f_n \circ g_n([\mathscr{N}_1] \oplus [\mathscr{N}_2]) = [\mathscr{N}_1] \oplus [\mathscr{N}_2]$. Therefore we obtain an isomorphism $\tilde{K}_T(X) \xrightarrow{\sim} \tilde{K}_T(X_n) \oplus \tilde{K}_T(X - X_n)$. Now X_{n-1} is a closed smooth subvariety of $X - X_n$. Therefore repeat the above process to obtain the conclusion. ■

COROLLARY 3.15. *We have an isomorphism of $R(T)_A$-modules $\tilde{K}_T(X)_A \xrightarrow{\sim} \tilde{K}_T(X^T)_A$.*

PROOF. Clear from Corollary 3.6 and Theorem 3.14. ■

Iversen [9] has proved that X^T is a closed smooth algebraic variety. Therefore Corollary 3.15 can be restated as follows:

COROLLARY 3.16. *The homomorphism of $R(T)_A$-modules*

$$(\boldsymbol{R}\Gamma_{X^T})_\Lambda : \tilde{K}_T(X)_\Lambda \to \tilde{K}_T(X)_\Lambda$$

is the identity.

4. Characters. In this section, we try to define *formal character morphisms* in order to apply the localization theorem in Section 3 to obtaining character formulas. If the variety X has infinitely many T-fixed points, then the trivial action of a one-parameter multiplicative group is somehow involved. Therefore, we cannot define a formal character morphism in this case. Thus the condition that X has only finitely many T-fixed points is mandatory. Unfortunately, we have to add another condition to obtain a character morphism for the time being, namely the so-called *positivity condition*. For instance, the case where X is a partial flag variety G/P and T is a maximal torus of G, satisfies the positivity condition.

4.1. Affine space case. Let $A = \boldsymbol{C}[t_1, \dots, t_m]$ be a T-equivariant polynomial ring, namely, the coordinate ring of an m-dimensional vector space on which T acts linearly. We assume the weight of t_i to be $\mu_i \neq 1$. Let A_n be the space of homogeneous polynomials of degree n and A^n be the space of polynomials of degree less than or equal to n.

An A-module M is said to be T-equivariant when M has a locally finite linear action of T which is compatible with the T-module structure of A. A graded $A = \bigoplus_{n \in \boldsymbol{Z}} A_n$-module $M = \bigoplus_{m \in \boldsymbol{Z}} M_m$ is said to be T-equivariant when M is a T-equivariant A-module and each M_m is a T-submodule of M. Let $M = \bigoplus_{m \in \boldsymbol{Z}} M_m$ be a finitely generated T-equivariant graded A-module. We see easily that $M_m = 0$ for $m \ll 0$. We define the Poincaré series $P(M, q)$ of M by

$$P(M, q) = \sum_{m \in \boldsymbol{Z}} \mathrm{ch}(M_m) q^m \in R(T)[[q]][q^{-1}],$$

where $\mathrm{ch}(M_m) = \sum_{\chi \in X(T)} (\dim(M_m)_\chi)[\chi] \in R(T)$. Let Λ_q denote the multiplicatively closed subset of $R(T)[q^{-1}, q]$ generated by $1 - \chi q$ for nontrivial characters χ of T. We consider $P(M, q) \in R(T)[[q]][q^{-1}]_{\Lambda_q}$. We then obtain:

LEMMA 4.1. *$P(M, q) \in R(T)[q^{-1}, q]_{\Lambda_q}$. More precisely, there exists a Laurent polynomial $f(q) \in R(T)[q^{-1}, q]$ such that*

$$P(M, q) = \frac{f(q)}{\prod_{i=1}^{m} (1 - \mu_i q)}.$$

PROOF. The proof is standard (cf. [12]). ■

DEFINITION. For a finitely generated T-equivariant graded A-module M, we define its character $\mathrm{ch}(M)$ by $\mathrm{ch}(M) := P(M, 1) \in R(T)_\Lambda$.

DEFINITION. For a finitely generated T-equivariant A-module M, an increasing filtration $M=\bigcup_{k\in \boldsymbol{Z}}M^k$ is said to be *good* when the following are satisfied:

(i) Each M^k is a finite-dimensional T-module;

(ii) $M^k=0$ for $k\ll 0$;

(iii) $A^l M^k \subset M^{k+l}$ for all k, m;

(iv) $A^1 M^k = M^{k+1}$ for $k\gg 0$.

For any finitely generated T-equivariant A-module M, there exists a good filtration of M; for instance, for a system $\{x_1, \ldots, x_n\}$ of equivariant generators of M, set $M^k := A^k x_1 + \cdots + A^k x_n \subset M$. If $M=\bigcup_{k\in \boldsymbol{Z}}M^k$ is a good filtration, then $\mathrm{gr} M = \bigoplus_{k\in \boldsymbol{Z}} M_k$, with $M_k := M^k/M^{k-1}$, is a finitely generated T-equivariant graded A-module.

DEFINITION. Let M be a finitely generated T-equivariant A-module and $\{M^k\}$ be a good filtration of M. Then we define the character $\mathrm{ch}(M)$ of M (with respect to this good filtration) by $\mathrm{ch}(M) := \mathrm{ch}(\mathrm{gr}\, M) \in R(T)_A$.

LEMMA 4.2. *Suppose M is a finitely generated T-equivariant A-module. Then* $\mathrm{ch}(M)$ *is independent of the choice of its good filtration.*

PROOF. It is clear that shifting filtrations does not have any effect on $\mathrm{ch}(M)$. Thus we just have to consider two good filtrations $\{F^k M\}$ and $\{F'^k M\}$ of M such that $F^k M \subset F'^k M$ for any k and $F'^k M \subset F^{k+d} M$ for any k, where $d \in \boldsymbol{Z}_+$.

We use induction on d. First of all, we assume $d=1$. Then, for any $k\in \boldsymbol{Z}$, we have exact sequences of finite-dimensional T-modules

$$0 \longrightarrow F'^k M/F^k M \longrightarrow F^{k+1} M/F^k M \longrightarrow F^{k+1} M/F'^k M \longrightarrow 0$$

and

$$0 \longrightarrow F^{k+1} M/F'^k M \longrightarrow F'^{k+1} M/F'^k M \longrightarrow F'^{k+1} M/F^{k+1} M \longrightarrow 0 .$$

Hence we obtain exact sequences of finitely generated T-equivariant graded A-modules

$$0 \longrightarrow C \longrightarrow \mathrm{gr}_F M \longrightarrow B \longrightarrow 0$$

and

$$0 \longrightarrow B \longrightarrow \mathrm{gr}_{F'} M \longrightarrow C[-1] \longrightarrow 0 ,$$

where we put $C := \bigoplus_k F'^k M/F^k M$ and $B := \bigoplus_k F^{k+1} M/F'^k M$. Therefore we obtain $P(\mathrm{gr}_F M, q) = P(C, q) + P(B, q)$ and $P(\mathrm{gr}_{F'} M, q) = P(C, q)q^{-1} + P(B, q)$. Hence we conclude

$$\mathrm{ch}(\mathrm{gr}_F M) = \mathrm{ch}(C) + \mathrm{ch}(B) = \mathrm{ch}(\mathrm{gr}_{F'} M) .$$

Now, we assume $d \geq 1$. We define a new filtration F'' by

$$F''^k M := F^k M + F'^{k-1} M .$$

Then it is clear that $\{F''^k M\}$ is a good filtration as well. This new filtration satisfies

$$F^k M \subset F''^k M \subset F^{k+d-1} M$$

for any k. Therefore we obtain $\mathrm{ch}(\mathrm{gr}_F M) = \mathrm{ch}(\mathrm{gr}_{F''} M)$ by the induction hypothesis. On the other hand, we have $F'^{k-1} M \subset F''^k M \subset F'^k M$ for any k. Therefore we obtain $\mathrm{ch}(\mathrm{gr}_{F'} M) = \mathrm{ch}(\mathrm{gr}_{F''} M)$ as well. Thus we have completed the proof. ∎

4.2. Characters with positivity condition. Let X be a smooth affine T-variety. Assume all weights in $\boldsymbol{C}[X]/\boldsymbol{C}$ are positive with respect to some partial ordering.

LEMMA 4.3. (cf. [6]). *Suppose X is the above T-variety. Then we have the following*:

(i) *X has a unique T-fixed point x_0 and any closed T-subvariety of X contains x_0.*

(ii) *Let $\boldsymbol{m}$ be the maximal ideal at x_0. Assume that a finitely generated T-equivariant $\boldsymbol{C}[X]$-module M satisfies $M_{\boldsymbol{m}} = 0$. Then $M = 0$.*

(iii) *Any T-equivariant locally free $\mathcal{O}_X$-module of finite rank is actually free.*

PROOF. (i) Since T acts on X regularly, $\boldsymbol{C}[X]$ is decomposed into its weight spaces as $\boldsymbol{C}[X] = \bigoplus_{\lambda \in X(T)} \boldsymbol{C}[X]_\lambda$. We have fixed an ordering in $X(T)$. By assumption, $\boldsymbol{m} := \bigoplus_{\lambda > 0} \boldsymbol{C}[X]_\lambda$ is a maximal ideal of $\boldsymbol{C}[X]$. It is obvious that any T-stable proper ideal of $\boldsymbol{C}[X]$ is contained in $\boldsymbol{m}$.

(ii) Let v be a weight vector in M. Since $M_{\boldsymbol{m}} = 0$, there exists $f \in \boldsymbol{C}[X] - \boldsymbol{m}$ such that $fv = 0$. On the other hand $I := \mathrm{Ann}(v)$ is a T-stable ideal of $\boldsymbol{C}[X]$. Since $f \in I$, I has to be the whole ring $\boldsymbol{C}[X]$ from (i). Since M is generated by finitely many weight vectors, M is actually 0.

(iii) Let M be a T-equivariant locally free $\boldsymbol{C}[X]$-module of rank r. By assumption, there exists a set $\{v_1, \dots, v_r\}$ of weight vectors in $M_{\boldsymbol{m}}$ such that

$$M_{\boldsymbol{m}} = \bigoplus_{i=1}^{r} \boldsymbol{C}[X]_{\boldsymbol{m}} v_i .$$

Let $\bar{M}$ denote the canonical image of M in $M_{\boldsymbol{m}}$. Take $g \in \boldsymbol{C}[X] - \boldsymbol{m}$ so that $g \cdot v_i \in \bar{M}$ for any i. Then we have $M_{\boldsymbol{m}} \supset \bigoplus_i \boldsymbol{C}[X] v_i =: M'$ and $M' \supset gM' \subset \bar{M}$. Let $\tilde{M}$ denote the intersection of $\bar{M}$ with M'. Since we have $M' \supset \tilde{M} \supset gM'$, we obtain an isomorphism $\tilde{M}_{\boldsymbol{m}} \xrightarrow{\sim} M'_{\boldsymbol{m}}$. Thus we obtain an isomorphism $\tilde{M} \xrightarrow{\sim} M'$ from (ii). Similarly we obtain isomorphisms $\tilde{M} \xrightarrow{\sim} \bar{M}$ and $M \xrightarrow{\sim} \bar{M}$. Therefore we conclude $M \xrightarrow{\sim} M'$. ∎

NOTATIONS. (1) Let $x_0 \in X$ be the unique fixed point in X. Assume the weights in $T_{x_0} X$ to be $\lambda_1, \dots, \lambda_n$. Then we define

$$\Delta := \frac{1}{(1-\lambda_1) \cdots (1-\lambda_n)} \in R(T)_\Delta .$$

(2) For a T-equivariant $\boldsymbol{C}[X]$-module M and $\lambda \in X(T)$, let $M[\lambda] = M \otimes_{\boldsymbol{C}} \boldsymbol{C}_\lambda$ denote the translation of M by λ, where $\boldsymbol{C}_\lambda$ is the one-dimensional T-module with weight λ.

DEFINITION. Let M be a T-equivariant $\boldsymbol{C}[X]$-module. A T-equivariant free $\boldsymbol{C}[X]$-module resolution $\{F^i\}$ of M is said to be *admissible* if the following are satisfied:

(i) $F^n=0$ for $n\gg 0$;

(ii) $F^i \xrightarrow{\sim} \bigoplus_{j=1}^{s_i} D(X)[\mu_{ij}]$ for any i, some $\mu_{ij}\in X(T)$, and $s_i\in \boldsymbol{Z}_+$,

where $D(X)$ is the ring of global differential operators.

Let $\boldsymbol{m}$ be the maximal ideal at x_0. Take weight vectors $f_1,\dots,f_m\in \boldsymbol{m}$ so that $R=\boldsymbol{C}[X]$ is generated by $f_1,\dots,f_m$ as a $\boldsymbol{C}$-algebra. Define the T-submodule R^k of R by

$$R^k:=\left\{\sum_{|\alpha|\le k} c_\alpha f_1^{\alpha_1}\cdots f_m^{\alpha_m}\in R \,\middle|\, c_\alpha\in \boldsymbol{C}\right\},$$

where $\alpha=(\alpha_1,\dots,\alpha_m)$ is a multi-index and $|\alpha|=\sum_{i=1}^m \alpha_i$. Clearly $\{R^k\}$ defines a T-stable filtration of R.

Let μ_i $(i=1,\dots,m)$ be the weight of f_i $(i=1,\dots,m)$. If we define a T-equivariant polynomial ring $A:=\boldsymbol{C}[t_1,\dots,t_m]$ with weight μ_i of t_i, then R is clearly a finitely generated T-equivariant A-module and $\{R^k\}$ is a good filtration of R. By what we saw in the previous section we may define the character $\mathrm{ch}(R)\in R(T)_A$.

DEFINITION. For a complex $\{F^i\}$ of T-equivariant R-modules with two properties

(i) $F^i=0$ for $i\gg 0$ or $i\ll 0$

(ii) $F^i \xrightarrow{\sim} \bigoplus_{j=1}^{s_i} D(X)[\mu_{ij}]$ for any i,

we define its character $\mathrm{ch}(\{F^i\})$ by

$$\mathrm{ch}(\{F^i\}):=\sum_{i,j}(-1)^i \Delta\,\mathrm{ch}(R)[\mu_{ij}]\in R(T)_A\,.$$

LEMMA 4.4. *Suppose $\{F^i\}$ is an admissible free resolution of* 0. *Then we obtain*

$$\mathrm{ch}(\{F^i\})=0\,.$$

PROOF. Suppose $F^k=0$ for $k\ge d+1$, and $k\le -1$. We use induction on d. First of all, we assume $d=1$, i.e., $F^0\simeq F^1$. Let $F^0=\bigoplus_{j=1}^{s} D(X)[\mu_j]$ and $F^1=\bigoplus_{j=1}^{t} D(X)[\nu_j]$.

Claim: $s=t$ and $\{\mu_j\}=\{\nu_j\}$ counting multiplicities. (This claim is not true without the positivity condition. If this claim holds, then the statement of the lemma follows immediately.)

Indeed, we see that the weights of the R-module basis of F^0 are $\{\mu_j-\sum_{i=1}^n \alpha_i\lambda_i \mid \alpha=(\alpha_1,\dots,\alpha_n)\in \boldsymbol{Z}_+^n\}$. From the isomorphism we have

$$^\#\{(\mu_j,\alpha)\mid \lambda=\mu_j-\sum \alpha_i\lambda_i\}={}^\#\{(\nu_j,\beta)\mid \lambda=\nu_j-\sum \beta_i\lambda_i\}$$

for each $\lambda\in X(T)$. Especially, maximal weights among $\{\mu_j\}$ and among $\{\nu_j\}$ coincide counting multiplicities. Therefore we inductively obtain $s=t$ and $\{\mu_j\}=\{\nu_j\}$ counting multiplicities.

Let $m\ge 1$. We have an exact sequence

$$0 \longrightarrow F^m \longrightarrow F^{m-1} \longrightarrow \cdots \longrightarrow F^1 \xrightarrow{\psi} F^0 \longrightarrow 0 .$$

Then the sequences

$$0 \longrightarrow F^m \longrightarrow F^{m-1} \longrightarrow \cdots \longrightarrow F^2 \longrightarrow \operatorname{Ker}\psi \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Ker}\psi \longrightarrow F^1 \longrightarrow F^0 \longrightarrow 0$$

are exact. Since F^0 is free, we have an isomorphism $F^1 \xrightarrow{\sim} F^0 \oplus \operatorname{Ker}\psi$. In other words, we have an exact sequence $0 \to F^0 \to F^1 \to \operatorname{Ker}\psi \to 0$. Then we may lift the map $F^1 \to \operatorname{Ker}\psi$ to a map $F^1 \to F^2$. We obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^0 \arrow[r] \arrow[d] & F^1 \arrow[r] \arrow[d] & \operatorname{Ker}\psi \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & F^3 \arrow[r] & F^2 \arrow[r] & \operatorname{Ker}\psi \arrow[r] & 0 .
\end{tikzcd}$$

Therefore we obtain an exact sequence

$$0 \longrightarrow F^m \longrightarrow \cdots \longrightarrow F^5 \longrightarrow F^0 \oplus F^4 \longrightarrow F^1 \oplus F^3 \longrightarrow F^2 \longrightarrow 0 .$$

By induction we have proved the lemma. ∎

LEMMA 4.5. *Suppose M has two admissible free resolutions $\{F^i\}$ and $\{G^i\}$. Then we have*

$$\operatorname{ch}(\{F^i\}) = \operatorname{ch}(\{G^i\}) .$$

PROOF. We have a commutative diagram

$$\begin{tikzcd}
\arrow[r] & F^2 \arrow[r, "d_F"] & F^1 \arrow[r, "d_F"] & F^0 \arrow[r] & M \arrow[r] & 0 \\
\arrow[r] & G^2 \arrow[r, "d_G"] \arrow[u, "\psi_2"'] & G^1 \arrow[r, "d_G"] \arrow[u, "\psi_1"'] & G^0 \arrow[r] \arrow[u, "\psi_0"'] & M \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

Consider a new complex (the mapping cone of ψ)

$$\cdots \longrightarrow F^2 \oplus G^1 \xrightarrow{\partial_1} F^1 \oplus G^0 \xrightarrow{\partial_0} F^0 \longrightarrow 0 ,$$

where $\partial_i(x, y) := (\psi_i(y) - d_F(x), d_G(y))$. This new sequence is exact. Therefore the statement follows from Lemma 4.4. ∎

DEFINITION. Let M be a T-equivariant $\boldsymbol{C}[X]$-module. If there exists an admissible free resolution $\{F^i\}$ of M, then we define the character $\operatorname{ch}(M)$ of M by $\operatorname{ch}(M) := \operatorname{ch}(\{F^i\})$.

LEMMA 4.6. *Let L and N be T-equivariant $\mathbf{C}[X]$-modules with character. Suppose that there exists an exact sequence of T-equivariant $\mathbf{C}[X]$-modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 .$$

Then M admits a character and we have an equality $\mathrm{ch}(M)=\mathrm{ch}(L)+\mathrm{ch}(N)$.

PROOF. Let $\{F^i\}$ (resp. $\{G^i\}$) be an admissible free resolution of L (resp. N). Then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F^0 \arrow[r] \arrow[d, "d_{F^0}"] & F^0 \oplus G^0 \arrow[r] & G^0 \arrow[r] \arrow[d, "d_{G^0}"] & 0 \\ 0 \arrow[r] & L \arrow[r, "\iota"] \arrow[d] & M \arrow[r, "p"] & N \arrow[r] \arrow[d] & 0 \\ & 0 & & 0 . & \end{tikzcd}$$

Since G^0 is free, there exists a morphism $g_0\colon G^0 \to M$ such that $p \circ g_0 = d_{G^0}$. Define $\partial_0\colon F^0 \oplus G^0 \to M$ by $\partial_0 = \iota \circ d_{F^0} + g_0$. Then we can easily check ∂_0 to be surjective. From the snake lemma we obtain an exact sequence

$$0 \longrightarrow \operatorname{Ker} d_{F^0} \longrightarrow \operatorname{Ker} \partial_0 \longrightarrow \operatorname{Ker} d_{G^0} \longrightarrow 0 .$$

Hence we can construct inductively an admissible free resolution of M, with the i-th term $F^i \oplus G^i$. ■

In general, any coherent T-equivariant $\mathscr{D}_X$-module $\mathscr{M}$ has a T-equivariant resolution of finite length composed of modules of the form $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L}$, for a locally free T-equivariant $\mathscr{O}_X$-module $\mathscr{L}$ of finite rank, since $\mathscr{M}$ admits a global good T-equivariant filtration (cf. [10]). In our case, the above $\mathscr{L}$ is always free thanks to Lemma 4.3, (iii). Therefore any T-equivariant coherent $\mathscr{D}_X$-module has an admissible free resolution when X is a smooth affine T-variety satisfying the positivity condition. Thus we have defined an $R(T)$-module morphism

$$\mathrm{ch}\colon \tilde{K}_T(X) \to R(T)_\Lambda ,$$

where X is a smooth affine T-variety satisfying the positivity condition.

The definition of $\mathrm{ch}(\mathscr{M})$ for a T-equivariant $\mathscr{D}_X$-module $\mathscr{M}$ looks dependent on the choice of generating weight vector subsets of R. Here we show it to be independent of the choice.

From Section 3, we know $[\mathscr{M}] = \int_i [i^! \mathscr{M}]$ in $\tilde{K}_T(X)_\Lambda$, where $i\colon \{x_0\} \to X$ is the inclusion. Thus we have to show $\mathrm{ch}(\int_i C_\lambda)$ to be independent of the choice of generating weight vector subsets.

Let us look back to the definition of $\mathrm{ch}(\mathscr{D}_X[\mu])$. The module $\mathscr{D}_X[\mu]$ has a natural filtration $\{\mathscr{D}_X(p)[\mu]\}$. Thus we can define a Poincaré series

$$P(\mathscr{D}_X[\mu], q) := \sum_p \mathrm{ch}(\mathscr{D}_X(p)[\mu]/\mathscr{D}_X(p-1)[\mu])q^p \in R(T)_\Lambda[[q]][q^{-1}]_{\Lambda_q}\,.$$

Since the tangent sheaf is free, we obtain

$$\begin{aligned}P(\mathscr{D}_X[\mu], q) &= \sum_{p,|\alpha|=p} \mathrm{ch}(R)\lambda_1^{\alpha_1}\cdots\lambda_n^{\alpha_n}q^p[\mu] = \sum_{p,|\alpha|=p} \mathrm{ch}(R)(\lambda_{1q})^{\alpha_1}\cdots(\lambda_n q)^{\alpha_n}[\mu] \\ &= \frac{\mathrm{ch}(R)}{(1-\lambda_1 q)\cdots(1-\lambda_n q)}[\mu]\,.\end{aligned}$$

Thus we have the following:

LEMMA 4.7.

$$\mathrm{ch}(\mathscr{D}_X[\mu]) = P(\mathscr{D}_X[\mu], 1)\,.$$

REMARK. As usual, $P(\mathscr{D}_X[\mu], 1)$ does not depend on the choice of a T-equivariant good filtration of $\mathscr{D}_X[\mu]$.

LEMMA 4.8. *Let M be a finite-dimensional T-equivariant R-module. Then* $\mathrm{ch}(M)$ *does not depend on the choice of generating weight vector subsets of R.*

PROOF. It is clear that $\cdots \subset 0 \subset 0 \subset M \subset M \subset \cdots$ is a good filtration for any generating weight vector subset of R. Therefore $\mathrm{ch}(M)$ is the same as the character of M as a finite-dimensional T-module. ∎

Let M be the global section of $\int_i^0 C_\lambda$. Then M has a good filtration $\{M(p)\}$ such that each $M(p)$ is a finite-dimensional T-module. There exists an admissible resolution

$$0 \longrightarrow F^l \longrightarrow F^{l-1} \longrightarrow \cdots \longrightarrow F^0 \longrightarrow M \longrightarrow 0$$

which induces an exact sequence

$$0 \longrightarrow F^l(p) \longrightarrow F^{l-1}(p) \longrightarrow \cdots \longrightarrow F^0(p) \longrightarrow M(p) \longrightarrow 0\,,$$

for each p, where $\{F^i(p)\}$ is a good filtration for each i.

Let S and S' be two generating weight vector subsets of R. We denote by $\mathrm{ch}_S(\ )$ (resp. $\mathrm{ch}_{S'}(\ )$) the characters with respect to S (resp. S'). We also use similar notation for the Poincaré series. Then we have

$$\begin{aligned}\sum(-1)^i P_S(F^i, q) &= \sum_i\sum_p(-1)^i \mathrm{ch}_S(F^i(p)/F^i(p-1))q^p \\ &= \sum_p\left[\sum_i(-1)^i\mathrm{ch}_S(F^i(p)/F^i(p-1))\right]q^p = \sum_p \mathrm{ch}_S(M(p)/M(p-1))q^p \\ &= \sum_p \mathrm{ch}_{S'}(M(p)/M(p-1))q^p = \sum(-1)^i P_{S'}(F^i, q)\,.\end{aligned}$$

Thus we conclude

$$\mathrm{ch}_S(M)=\sum(-1)^i\mathrm{ch}_S(F^i)=\sum(-1)^iP_S(F^i,1)=\sum(-1)^iP_{S'}(F^i,1)=\mathrm{ch}_{S'}(M)\,.$$

We have thus proved:

PROPOSITION 4.9. *The definition of characters for T-equivariant $\mathscr{D}_X$-modules is independent of the choice of generating weight vector subsets of R.*

Suppose X has a finite open covering $\bigcup_{i=1}^n U_i$ and assume each U_i to be a smooth affine T-variety satisfying the positivity condition. The ordering on $X(T)$ may depend on i.

DEFINITION. For $[\mathscr{M}]\in\tilde{K}_T(X)$, we define its character $\mathrm{ch}([\mathscr{M}])$ by

$$\mathrm{ch}([\mathscr{M}]):=\sum_{i=1}^n \mathrm{ch}_{U_i}(\iota_i^*\boldsymbol{R}\Gamma_{X-U_{i-1}}\cdots\boldsymbol{R}\Gamma_{X-U_1}[\mathscr{M}])\,,$$

where $\iota_i\colon U_i\to X$ is the inclusion.

THEOREM 4.10. *Let $X=\bigcup_{i=1}^n U_i$ be a finite open covering of a smooth T-variety X. Assume that each U_i is a smooth affine T-variety satisfying the positivity condition. Let x_i be the unique fixed point in U_i. Then we obtain*

$$\mathrm{ch}([\mathscr{M}])=\mathrm{ch}([\boldsymbol{R}\Gamma_{X^T}(\mathscr{M})])=\sum_{i=1}^n\mathrm{ch}([\boldsymbol{R}\Gamma_{\{x_i\}}(\mathscr{M})])$$

for any $[\mathscr{M}]\in\tilde{K}_T(X)$.

REMARK. When X has an open covering satisfying the assumptions in the above theorem, such a covering is unique. We may check this fact easily using Lemma 4.3, (i).

Since we may see that the $\mathscr{O}$-module structure carries considerable information as far as holonomic $\mathscr{D}$-modules are concerned, we propose the following conjecture:

CONJECTURE. *If X has only finitely many T-fixed points, then we can define a formal character morphism*

$$\mathrm{ch}\colon \tilde{K}_T(X)\longrightarrow R(T)_\Lambda\,.$$

As a consequence we would obtain a character formula for holonomic $\mathscr{D}_X$-modules, i.e.,

$$\mathrm{ch}(\mathscr{M})=\sum_{w\in X^T}\mathrm{ch}(\boldsymbol{R}\Gamma_{\{w\}}(\mathscr{M}))\qquad\text{in } R(T)_\Lambda\,.$$

REFERENCES

[1] M. F. ATIYAH, Lectures on K-theory, mimeographed notes, Harvard Univ., 1964.
[2] M. F. ATIYAH AND R. BOTT, A Lefschetz fixed-point formula for elliptic complexes I, II: Applications,

Ann. of Math. 86 (1967), 374–407; 88 (1968), 451–491.
[3] J.-E. BJÖRK, Rings of Differential Operators, North-Holland Math. Library 21, North-Holland, Amsterdam, Oxford and New York, 1979.
[4] A. BOREL ET AL., Algebraic $\mathscr{D}$-modules, Perspectives in Math. 2, Academic Press, New York and London, 1986.
[5] W. BORHO AND J.-L. BRYLINSKI, Differential operators on homogeneous spaces II, Max-Plank-Institut für Mathematik, Bonn, 1986.
[6] J.-L. BRYLINSKI, Personal communication, 1988.
[7] A. GROTHENDIECK, Sur quelques points d'algèbre homologique, Tôhoku Math. J. 9 (1957), 119–221.
[8] R. HOTTA, Introduction to $\mathscr{D}$-modules, The Institute of Math. Sciences, Madras, 1987.
[9] B. IVERSEN, A fixed point formula for action of tori on algebraic varieties, Invent. Math. 16 (1972), 229–236.
[10] R. JOSHUA, Riemann Roch for equivariant $\mathscr{D}$-modules-I, Math. Z. 206 (1991), 131–144.
[11] D. T. LÊ AND Z. MEBKHOUT, Introduction to linear differential systems, Proc. Symp. Pure Math. 40 (2) (1983), 31–63.
[12] H. MATSUMURA, Commutative Ring Theory, Cambridge Studies in Advanced Math. 8, Cambridge Univ. Press, 1986.
[13] D. MUMFORD, Geometric Invariant Theory, Springer-Verlag, Berlin, Heidelberg and New York, 1973.
[14] T. ODA, Introduction to algebraic analysis on complex manifolds, in Algebraic and Analytic Varieties (S. Iitaka, ed.), Advanced Studies in Pure Math. 1, Kinokuniya, Tokyo and North-Holland, Amsterdam, New York, and Oxford, 1983, 29–48.
[15] F. PHAM, Singularités des systèmes de Gauss-Manin, Progress in Math. 2, Birkhäuser, Basel, Boston and Stuttgart, 1979.
[16] R. W. RICHARDSON JR., Principal orbit types for algebraic transformation spaces in characteristic zero, Invent. Math. 16 (1972), 6–14.
[17] M. SATO, M. KASHIWARA AND T. KAWAI, Microfunctions and pseudodifferential equations, Lecture Notes in Math. 287 (1973), Springer Verlag, Berlin, Heidelberg and New York, 265–529.
[18] H. SUMIHIRO, Equivariant completion, J. Math. Kyoto Univ. 14 (1974), 1–28.

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