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LINEAR GALE TRANSFORMS AND GELFAND-KAPRANOV-ZELEVINSKIJ DECOMPOSITIONS

Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

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Abstract. In the convex-geometric setting of what we call linear Gale transforms and convex polyhedral cone decompositions, we generalize and reformulate results on (1) the secondary polytope of a convex polytope considered by Gelfand, Kapranov and Zelevinskij in connection with the discriminants of projective toric varieties, as well as (2) the wall geometry of fans considered by Reid in connection with Mori's birational geometry in the particular case of projective toric varieties.

Introduction. Let Ξ be a finite subset of an r-dimensional vector space W over the field $\boldsymbol{R}$ of real numbers. As one of us already sketched in [14, §2.6], let us now outline our results in this paper *assuming, for simplicity, that* Ξ *spans* W *over* $\boldsymbol{R}$, *that* Ξ *does not contain* 0 *and that each* $\xi\in\Xi$ *is not a positive scalar multiple of any other element in* $\Xi\setminus\{\xi\}$.

Among the pairs (V, f) of an $\boldsymbol{R}$-vector space V and a map $f: \Xi\to V$ such that $f(\Xi)$ spans V over $\boldsymbol{R}$ and that

$$\sum_{\xi\in\Xi}\xi\otimes f(\xi)=0 \qquad \text{in} \quad W\otimes_{\boldsymbol{R}}V,$$

there exists a pair $(G(W, \Xi), g)$, called the *linear Gale transform* of (W, Ξ), satisfying the following *universality*: For each (V, f) there exists a unique $\boldsymbol{R}$-linear map h: $G(W, \Xi)\to V$ such that $f=h\circ g$.

Convex-geometric and combinatorial properties for Ξ turn out to be reflected in those for $g(\Xi)$ in an interesting way. See Shephard [16] and McMullen [11], where the appellations *linear representation* and *linear transform* are used. Consider, for instance, convex polyhedral cones

$$W_{\geq 0}(\Xi):=\sum_{\xi\in\Xi}\boldsymbol{R}_{\geq 0}\xi \quad \text{and} \quad G_{\geq 0}(W, \Xi):=\sum_{\xi\in\Xi}\boldsymbol{R}_{\geq 0}g(\xi)$$

in W and $G(W, \Xi)$, respectively, spanned by Ξ and $g(\Xi)$ over the additive semigroup

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$\boldsymbol{R}_{\geq 0}$ of nonnegative real numbers. As we show in Proposition 1.4, $W_{\geq 0}(\Xi)$ is *strongly convex*, i.e., does not contain any nonzero $\boldsymbol{R}$-linear subspace if and only if $G_{\geq 0}(W, \Xi) = G(W, \Xi)$.

The Gelfand-Kapranov-Zelevinskij decompositions we are about to consider in this paper fit in nicely with linear Gale transforms and provide another important reflection of interest.

A *convex polyhedral cone decomposition* for W is a finite collection Π of convex polyhedral cones in W such that the faces of any $\sigma \in \Pi$ belong to Π and that $\sigma \cap \sigma'$ for any pair $\sigma, \sigma' \in \Pi$ is a face of both σ and σ'. Π is said to be *simplicial* (resp. *nondegenerate*) if every $\sigma \in \Pi$ is simplicial (resp. if $\{0\}$ belongs to Π). We call $|\Pi| := \bigcup_{\sigma \in \Pi} \sigma$ the *support* of Π.

Π is said to be *quasi-projective* if there exists a function $\eta : |\Pi| \to \boldsymbol{R}$ which is *piecewise linear* and *strictly convex* with respect to Π, that is, a globally linear functional $z_\sigma : W \to \boldsymbol{R}$ exists for each $\sigma \in \Pi$ such that

$$\eta(w) \geq z_\sigma(w) \qquad \text{for all} \quad w \in |\Pi|$$

and that the equality holds *if and only if* $w \in \sigma$.

As we see in Theorem 2.3, the Kleiman-Nakai criterion in algebraic geometry has a variant in our context: When $|\Pi|$ is a convex polyhedral cone, we can describe the cone $\mathrm{CPL}(\Pi)$ of *convex* functions $\eta : |\Pi| \to \boldsymbol{R}$ piecewise linear with respect to Π in terms of the *internal walls* in Π. We then get a criterion for Π to be quasi-projective.

A (possibly degenerate) convex polyhedral cone decomposition Π for W is said to be *admissible* for (W, Ξ) if (i) Π is quasi-projective, (ii) $|\Pi| = W_{\geq 0}(\Xi)$ and (iii) each cone $\sigma \in \Pi$ is spanned over $\boldsymbol{R}_{\geq 0}$ by a subset of Ξ.

We show in Theorem 3.5 that each nondegenerate and admissible Π gives rise to a convex polyhedral cone $\mathrm{cpl}(\Pi)$ in $G(W, \Xi)$, which we call the *Gelfand-Kapranov-Zelevinskij cone* (the GKZ-cone, for short) associated to Π. We have $\dim \mathrm{cpl}(\Pi) = \dim G(W, \Xi)$ if and only if Π is simplicial. Then

$$\{\text{the faces of the GKZ-cones } \mathrm{cpl}(\Pi) \,|\, \Pi \text{ simplicial and admissible for } (W, \Xi)\}$$

turns out to be a *nondegenerate* convex polyhedral cone decomposition for $G(W, \Xi)$ with support $G_{\geq 0}(W, \Xi)$. We call it the *Gelfand-Kapranov-Zelevinskij decomposition* (the GKZ-decomposition, for short). $\mathrm{cpl}(\Pi)$ for nondegenerate but not simplicial Π's turn out to be contained in the above family. Moreover, degenerate Π's also give rise to similar cones which appear in the above family and turn out to be contained in the boundary of $G_{\geq 0}(W, \Xi)$.

The GKZ-decomposition enables us to compare different admissible convex polyhedral cone decompositions Π and Π' in terms of the relative position of the GKZ-cones $\mathrm{cpl}(\Pi)$ and $\mathrm{cpl}(\Pi')$ inside $G_{\geq 0}(W, \Xi)$.

When $W_{\geq 0}(\Xi)$ is strongly convex, i.e., does not contain any nonzero $\boldsymbol{R}$-linear subspace, the GKZ-decomposition is a conic variant of the *secondary polytope* for a convex polytope considered by Gelfand-Zelevinskij-Kapranov [4], [5], [6], [7], [8] and [9] in connection with the Newton polytope of the discriminant of a projective toric variety.

On the other hand, the case $W_{\geq 0}(\Xi)=W$ includes the birational geometry of projective toric varieties. In particular, it includes the wall geometry which Reid [15] obtained in connection with Mori's birational geometry in the particular case of projective toric varieties. Referring the reader to [13] for necessary terminology, let us consider a free $\boldsymbol{Z}$-module $N \cong \boldsymbol{Z}^r$ of rank r and the corresponding algebraic torus $T := N \otimes_{\boldsymbol{Z}} \boldsymbol{C}^\times$ of dimension r isomorphic to the product of r copies of the multiplicative group $\boldsymbol{C}^\times$ of nonzero complex numbers. Let $W := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ and assume Ξ to be contained in the lattice $N \subset W$.

A nondegenerate and admissible Π in this case is nothing but a projective fan such that the set $\Pi(1)$ of one-dimensional cones in Π is contained in the preassigned allowable set $\{\boldsymbol{R}_{\geq 0}\xi \mid \xi \in \Xi\}$. Thus the codimension-one orbits of the corresponding projective toric variety X_Π occur only among the *preassigned allowable* set $\{\mathrm{orb}(\boldsymbol{R}_{\geq 0}\xi) \mid \xi \in \Xi\}$. We know that X_Π has at most quotient singularities if and only if Π is simplicial. We can identify $G(W, \Xi)$ with the scalar extension to $\boldsymbol{R}$ of the group of linear equivalence classes of *preassigned allowable* T-invariant *Weil divisors* on X_Π, while $G_{\geq 0}(W, \Xi)$ is the cone in $G(W, \Xi)$ spanned by the linear equivalence classes of *preassigned allowable* T-invariant *effective Weil divisors*. The GKZ-cone $\mathrm{cpl}(\Pi)$ coincides essentially with the cone in $G(W, \Xi)$ spanned by the linear equivalence classes of *pseudo-ample divisors* on X_Π, but is so adjusted that all the preassigned allowable divisor classes are taken into account as well. The cones $\mathrm{cpl}(\Pi)$ for simplicial Π's are exactly those of maximal dimension and turn out to intersect along faces and fill up the cone $G_{\geq 0}(W, \Xi)$ of preassigned allowable effective divisor classes without overlap. For different Π and Π', the birational correspondence (blowing-up, blowing-down or flop, for instance) between X_Π and $X_{\Pi'}$ is reflected in the relative position of the cones $\mathrm{cpl}(\Pi)$ and $\mathrm{cpl}(\Pi')$ inside $G_{\geq 0}(W, \Xi)$.

It might be interesting to relate this result with the codimension-one characterization of toric varieties due to Fine [2] and [3].

As we show in Corollary 3.8, the case of strongly convex $W_{\geq 0}(\Xi)$ also has an interesting application: Suppose a convex polyhedral cone π in W is strongly convex, i.e., does not contain any nonzero $\boldsymbol{R}$-linear subspace. Then there exists a *simplicial* and *quasi-projective* polyhedral cone decomposition Π with $|\Pi| = \pi$ such that the set $\Pi(1)$ of one-dimensional cones in Π coincides with that of one-dimensional faces of π. If all the proper faces of π are simplicial, then Π does not subdivide any of the proper faces of π. In particular, we get another proof of the following known result (cf. Stanley [17] and Goodman-Pach [10]): A simplicial convex polytope Q has a triangulation *without additional vertices*, that is, one by means of simplices having vertices only in the set of vertices of Q. Moreover, any two such triangulations turn out to be obtainable from each other by a finite succession of elementary operations called *flops* (cf. Corollary 3.9).

As an important consequence of the result above for π, we get a *non-divisorial*, relatively projective and equivariant modification of an affine toric variety with a bad isolated singularity: Namely, suppose that π above is rational with respect to a $\boldsymbol{Z}$-lattice N in W, and that all the proper faces of π are simplicial. Then the corresponding affine toric variety U_π may have a bad isolated singularity at the unique T-fixed point $\mathrm{orb}(\pi)$. The above decomposition Π gives rise to an equivariant, relatively projective and birational morphism $f: X_\Pi \to U_\pi$ from the toric variety X_Π with at most quotient singularities such that f is isomorphic outside $\mathrm{orb}(\pi)$ and that the fiber $f^{-1}(\mathrm{orb}(\pi))$ is of codimension *at least two* in X_Π.

Details for more applications to toric varieties will be published elsewhere.

For necessary results on convex geometry, we refer the reader to [13, Appendix], for instance.

Thanks are due to Eiji Horikawa who kindly made available the English translation of Gelfand-Zelevinskij-Kapranov [9]. Thanks are also due to Takayuki Hibi, who provided the references for triangulations, without additional vertices, of simplicial convex polytopes.

As the writing of this paper was coming to a close, we received a preprint of Billera-Filliman-Sturmfels [1] which also notices the relevance of the Gale transforms to the secondary polytopes of Gelfand, Kapranov and Zelevinskij.

This paper is dedicated to the sixtieth birthday April 9, 1991 of Professor Heisuke Hironaka, whose Harvard thesis in 1960 noticed the importance of cones in the birational geometry of projective algebraic varieties, which led to the tremendous later developments in birational geometry due to S. Kleiman and S. Mori among others, thereby helping to enrich the content of this paper as well.

1. Linear Gale transforms. Throughout this paper, we fix a finite dimensional vector space W over the field $\boldsymbol{R}$ of real numbers with $r := \dim W$, and denote by $W^* := \mathrm{Hom}_{\boldsymbol{R}}(W, \boldsymbol{R})$ its dual space with the canonical $\boldsymbol{R}$-bilinear pairing $\langle\ ,\ \rangle : W^* \times W \to \boldsymbol{R}$.

Let Ξ be a finite subset of W spanning W over $\boldsymbol{R}$. Introduce an $\boldsymbol{R}$-vector space W_1 with a basis $\{e_\xi \mid \xi \in \Xi\}$ which is in bijective correspondence with Ξ. By sending e_ξ to $\xi \in W$, we get a surjective linear map $W_1 \to W$. Let $W_1^* := \mathrm{Hom}_{\boldsymbol{R}}(W_1, \boldsymbol{R})$ be the dual space with the dual basis $\{e_\xi^* \mid \xi \in \Xi\}$. Then we have the dual injective linear map $W^* \to W_1^*$ which sends $z \in W^*$ to $\sum_{\xi \in \Xi} \langle z, \xi \rangle e_\xi^*$.

DEFINITION. We denote the cokernel of the injective linear map above by

$$G(W, \Xi) := W_1^* / W^* ,$$

which is an $\boldsymbol{R}$-vector space of dimension $^\#\Xi - \dim W$. For each ξ, we denote by $g(\xi) \in G(W, \Xi)$ the image of $e_\xi^* \in W_1^*$. We denote $g(\Xi) := \{g(\xi) \mid \xi \in \Xi\}$ and call $(G(W, \Xi), g(\Xi))$ the *linear Gale transform* of (W, Ξ).

By definition, the defining relations among the elements of $g(\Xi)$ are

$$\sum_{\xi\in\Xi}\langle z,\xi\rangle g(\xi)=0 \qquad \text{for all} \quad z\in W^*,$$

which we may express more symmetrically as

$$\sum_{\xi\in\Xi}\xi\otimes g(\xi)=0 \qquad \text{in} \quad W\otimes_{\boldsymbol{R}}G(W,\Xi).$$

REMARK. McMullen [11] and Shephard [16] used the appellations *linear transform* and *linear representation*, while the term *Gale transform* was reserved for an affine version. We adopt our present terminology to avoid possible confusion in our context.

Instead of considering finite subsets Ξ of W, we could generalize our situation and formulate the linear Gale transform more symmetrically as follows: Consider a finite set Ξ, a finite dimensional $\boldsymbol{R}$-vector space W and a map $\iota:\Xi\to W$ whose image spans W over $\boldsymbol{R}$. Then the linear Gale transform $g:\Xi\to G$ of $\iota:\Xi\to W$ is defined to be the *universal* one among all the maps $f:\Xi\to V$ to $\boldsymbol{R}$-vector spaces V such that the image $f(\Xi)$ spans V over $\boldsymbol{R}$ and that

$$\sum_{\xi\in\Xi}\iota(\xi)\otimes f(\xi)=0 \qquad \text{in} \quad W\otimes_{\boldsymbol{R}}V.$$

The universality means that for each such $f:\Xi\to V$, there exists a unique $\boldsymbol{R}$-linear map $h: G\to V$ with $f=h\circ g$. It is then easy to see by symmetry that $\iota:\Xi\to W$ in turn is the linear Gale transform of $g:\Xi\to G$.

In the rest of this paper, however, we restrict ourselves to the case where ι is injective. Clearly, the linear Gale transform in this generalized sense then coincides with the earlier one given in the definition above.

Recall that a subset σ is said to be a *convex polyhedral cone* in W if there exist $w_1,\cdots,w_s\in W$ such that σ is the set of nonnegative linear combinations of $w_1,\cdots,w_s$, namely,

$$\sigma=\boldsymbol{R}_{\geq 0}w_1+\boldsymbol{R}_{\geq 0}w_2+\cdots+\boldsymbol{R}_{\geq 0}w_s,$$

where $\boldsymbol{R}_{\geq 0}$ is the additive semigroup of nonnegative real numbers. σ is said to be *simplicial* if $\{w_1,\cdots,w_s\}$ can be chosen to be $\boldsymbol{R}$-linearly independent. σ is said to be *strongly convex* if $\sigma\cap(-\sigma)=\{0\}$, that is, σ does not contain any nonzero $\boldsymbol{R}$-linear subspace of W. The *dual cone* for σ is defined to be

$$\sigma^\vee:=\{z\in W^*\mid\langle z,w\rangle\geq 0 \text{ for all } w\in\sigma\},$$

which is a convex polyhedral cone in the dual space W^*. A subset $\tau\subset\sigma$ is said to be a *face* of a convex polyhedral cone σ and denoted $\tau\prec\sigma$ if there exists $z\in\sigma^\vee$ such that $\tau=\sigma\cap\{z\}^\perp:=\{w\in\sigma\mid\langle z,w\rangle=0\}$.

We now derive immediate consequences of the definition, which we need in Section 3.

We refer the reader to [11] and [16] for more.

PROPOSITION 1.1. *Let ξ be a vector belonging to Ξ.*

(1) *$\xi \neq 0$ if and only if $g(\xi)$ is in the $\boldsymbol{R}$-linear subspace of $G(W, \Xi)$ spanned by $\{g(\xi') \mid \xi' \in \Xi, \xi' \neq \xi\}$.*

(2) *$g(\xi) \neq 0$ if and only if ξ is in the $\boldsymbol{R}$-linear subspace of W spanned by $\Xi \setminus \{\xi\}$.*

(3) *A subset Ω of Ξ is an $\boldsymbol{R}$-basis for W if and only if $\{g(\xi) \mid \xi \in \Xi \setminus \Omega\}$ is an $\boldsymbol{R}$-basis for $G(W, \Xi)$.*

PROOF. (1) If $\xi \neq 0$, there exists $z \in W^*$ such that $\langle z, \xi \rangle = -1$. Hence

$$g(\xi) = \sum_{\xi' \in \Xi \setminus \{\xi\}} \langle z, \xi' \rangle g(\xi') .$$

Conversely, suppose $g(\xi) = \sum_{\xi' \in \Xi \setminus \{\xi\}} c_{\xi'} g(\xi')$. Then by the definition of $G(W, \Xi)$, there exists $z \in W^*$ such that $\langle z, \xi \rangle = -1$ and $\langle z, \xi' \rangle = c_{\xi'}$ for all $\xi' \in \Xi \setminus \{\xi\}$. In particular, $\xi \neq 0$.

(2) can be proved symmetrically.

As for (3), suppose Ω is a basis for W. Then each $\xi \in \Xi \setminus \Omega$ can be written as $\xi = \sum_{\omega \in \Omega} a_{\xi,\omega} \omega$ for $a_{\xi,\omega} \in \boldsymbol{R}$, hence

$$0 = \sum_{\omega \in \Omega} \omega \otimes \left(g(\omega) + \sum_{\xi \in \Xi \setminus \Omega} a_{\xi,\omega} g(\xi) \right) .$$

Since Ω is supposed to be a basis for W, we have $g(\omega) + \sum_{\xi \in \Xi \setminus \Omega} a_{\xi,\omega} g(\xi) = 0$ for all $\omega \in \Omega$. Thus $\{g(\xi) \mid \xi \in \Xi \setminus \Omega\}$ spans $G(W, \Xi)$ over $\boldsymbol{R}$. Since its cardinality is $^\sharp\Xi - r = \dim G(W, \Xi)$, it is a basis for $G(W, \Xi)$. The converse can be proved symmetrically. q.e.d.

DEFINITION. For each pair (W, Ξ), we define convex polyhedral cones in W and $G(W, \Xi)$, respectively, by

$$W_{\geq 0}(\Xi) := \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} \xi \quad \text{and} \quad G_{\geq 0}(W, \Xi) := \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} g(\xi) .$$

If $W_{\geq 0}(\Xi) = W$ (resp. $G_{\geq 0}(W, \Xi) = G(W, \Xi)$), then we say that Ξ (resp. $g(\Xi)$) *positively spans* W (resp. $G(W, \Xi)$).

LEMMA 1.2. (1) *We have $W_{\geq 0}(\Xi) \neq W$ if and only if $\sum_{\xi \in \Xi} a_\xi g(\xi) = 0$ for a subset $\{a_\xi \mid \xi \in \Xi\} \subset \boldsymbol{R}_{\geq 0}$ not all zero.*

(2) *A subset $\Omega \subset \Xi$ is a* facial subset, *that is, $\Omega = \Xi \cap F$ for a face $F \prec W_{\geq 0}(\Xi)$, if and only if $\sum_{\xi \in \Xi \setminus \Omega} a_\xi g(\xi) = 0$ for positive real numbers a_ξ for $\xi \in \Xi \setminus \Omega$.*

PROOF. (1) Clearly, $W_{\geq 0}(\Xi) \neq W$ holds if and only if there exists a nonzero $z \in W^*$ such that $\langle z, \xi \rangle \geq 0$ for all $\xi \in \Xi$. If such a z exists, then $\sum_{\xi \in \Xi} \langle z, \xi \rangle g(\xi) = 0$ with $\langle z, \xi \rangle \geq 0$ not all zero, since Ξ spans W over $\boldsymbol{R}$ and $z \neq 0$. Conversely, suppose $\sum_{\xi \in \Xi} a_\xi g(\xi) = 0$ with $a_\xi \in \boldsymbol{R}_{\geq 0}$ not all zero. By the definition of $G(W, \Xi)$, there exists $z \in W^*$ such that $\langle z, \xi \rangle = a_\xi \geq 0$ for all $\xi \in \Xi$. We have $z \neq 0$, since a_ξ's are not all zero by assumption.

(2) By definition, Ω is a facial subset of Ξ if and only if there exists $z \in W^*$ such that $\langle z, \omega \rangle = 0$ for all $\omega \in \Omega$ and that $\langle z, \xi \rangle > 0$ for all $\xi \in \Xi \setminus \Omega$. The rest of the proof is similar to that for (1). q.e.d.

PROPOSITION 1.3. (1) *Suppose* $g(\xi) \neq 0$ *for all* $\xi \in \Xi$. *If* $G_{\geq 0}(W, \Xi)$ *is strongly convex, then* $W_{\geq 0}(\Xi) = W$.

(2) *If* $W_{\geq 0}(\Xi) = W$, *then* $G_{\geq 0}(W, \Xi)$ *is strongly convex.*

PROOF. (1) If $W_{\geq 0}(\Xi) \neq W$, then by Lemma 1.2, (1), there exist $\{a_\xi \mid \xi \in \Xi\} \subset \boldsymbol{R}_{\geq 0}$ such that $\sum_{\xi \in \Xi} a_\xi g(\xi) = 0$ with $a_{\xi_0} \neq 0$ for a $\xi_0 \in \Xi$. Hence

$$-a_{\xi_0} g(\xi_0) = \sum_{\xi \in \Xi \setminus \{\xi_0\}} a_\xi g(\xi)$$

is contained in

$$G_{\geq 0}(W, \Xi) \cap (-G_{\geq 0}(W, \Xi))$$

and is nonzero since $g(\xi_0) \neq 0$ by assumption. Thus $G_{\geq 0}(W, \Xi)$ is not strongly convex.

(2) Suppose $G_{\geq 0}(W, \Xi)$ is not strongly convex. Then there exists a nonzero element

$$\sum_{\xi \in \Xi} a_\xi g(\xi) = \sum_{\xi \in \Xi} (-b_\xi) g(\xi)$$

in $G_{\geq 0}(W, \Xi) \cap (-G_{\geq 0}(W, \Xi))$ with $a_\xi, b_\xi \geq 0$ for all $\xi \in \Xi$. Thus $\sum_{\xi \in \Xi} (a_\xi + b_\xi) g(\xi) = 0$ with $a_\xi + b_\xi$ nonnegative and not all zero. As in the proof of Lemma 1.2, (1), there exists a nonzero $z \in W^*$ such that $\langle z, \xi \rangle = a_\xi + b_\xi \geq 0$ for all $\xi \in \Xi$. Hence $W_{\geq 0}(\Xi) \neq W$. q.e.d.

By symmetry, we have:

PROPOSITION 1.4. (1) *Suppose* Ξ *does not contain* 0. *If* $W_{\geq 0}(\Xi)$ *is strongly convex, then* $G_{\geq 0}(W, \Xi) = G(W, \Xi)$.

(2) *If* $G_{\geq 0}(W, \Xi) = G(W, \Xi)$, *then* $W_{\geq 0}(\Xi)$ *is strongly convex.*

2. Convex polyhedral cone decompositions.

DEFINITION. A *convex polyhedral cone decomposition* for W is a finite collection Π of convex polyhedral cones in W such that (i) the faces of any $\sigma \in \Pi$ belong to Π and (ii) the set-theoretical intersection $\sigma \cap \sigma'$ of any pair $\sigma, \sigma' \in \Pi$ is a face of both σ and σ'. Π is said to be *simplicial* if every $\sigma \in \Pi$ is simplicial. Π is said to be *nondegenerate* if every $\sigma \in \Pi$ is strongly convex. The *support* of Π is defined to be $|\Pi| := \bigcup_{\sigma \in \Pi} \sigma$. We say Π to be *complete* if $|\Pi| = W$. For each $0 \leq j \leq r$, we denote by $\Pi(j)$ the set of j-dimensional cones in Π.

Note that *simplicial* implies *nondegenerate*. A convex polyhedral cone decomposition Π is easily seen to be nondegenerate if and only if $\{0\}$ belongs to Π. More generally, a convex polyhedral cone decomposition Π possesses a unique $\boldsymbol{R}$-linear subspace $W_0 \subset W$ such that $W_0 \prec \sigma$ for all $\sigma \in \Pi$. The induced convex polyhedral cone decomposition

$$\Pi/W_0 := \{\sigma/W_0 \mid \sigma \in \Pi\}$$

for the quotient vector space W/W_0 is nondegenerate, and Π consists of the inverse images of the cones in Π/W_0. Clearly, $\Pi(j)$ is empty for $j < \dim W_0$.

In this paper, we restrict ourselves to convex polyhedral cone decompositions Π *with support* $|\Pi|$ *spanning* W *over* $\boldsymbol{R}$.

We denote by $\mathrm{PL}(\Pi)$ the finite-dimensional $\boldsymbol{R}$-vector space of functions $\eta : |\Pi| \to \boldsymbol{R}$ which are *piecewise linear with respect to* Π, that is, there exists $z_\sigma \in W^*$ for each $\sigma \in \Pi$ such that

$$\eta(w) = \langle z_\sigma, w \rangle \qquad \text{for all} \quad w \in \sigma .$$

We denote by $\mathrm{PL}_{\geq 0}(\Pi) \subset \mathrm{PL}(\Pi)$ the subset consisting of η with $\eta(w) \geq 0$ for all $w \in |\Pi|$.

LEMMA 2.1. *Π is simplicial if and only if Π is nondegenerate and* $\dim \mathrm{PL}(\Pi)$ *coincides with the cardinality of the set $\Pi(1)$ of one-dimensional cones in Π.*

PROOF. If Π is degenerate, then, clearly, Π is not simplicial.

We thus suppose Π to be nondegenerate, hence $\Pi(1)$ is nonempty. For each $\rho \in \Pi(1)$, we may regard the one-dimensional $\boldsymbol{R}$-vector space $W^*/\rho^\perp$ as the $\boldsymbol{R}$-vector space of linear maps $\rho \to \boldsymbol{R}$. By restriction, we get an $\boldsymbol{R}$-linear map

$$\alpha : \mathrm{PL}(\Pi) \to \bigoplus_{\rho \in \Pi(1)} (W^*/\rho^\perp) ,$$

which is clearly injective. For each $\sigma \in \Pi$, the $\boldsymbol{R}$-vector space $W^*/\sigma^\perp$ of linear maps $\sigma \to \boldsymbol{R}$ can also be embedded by restriction as

$$W^*/\sigma^\perp \to \bigoplus_{\rho \in \Pi(1),\, \rho \prec \sigma} W^*/\rho^\perp ,$$

which is bijective if and only if σ is simplicial. The rest of the proof is clear. q.e.d.

A function $\eta \in \mathrm{PL}(\Pi)$ is said to be *convex* if $\eta(w + w') \leq \eta(w) + \eta(w')$ for all w, $w' \in |\Pi|$. We denote by $\mathrm{CPL}(\Pi)$ the convex polyhedral cone in $\mathrm{PL}(\Pi)$ consisting of the convex functions which are piecewise linear with respect to Π. Clearly, a function $\eta : |\Pi| \to \boldsymbol{R}$ belongs to $\mathrm{CPL}(\Pi)$ if and only if there exists $z_\sigma \in W^*$ for each $\sigma \in \Pi$ such that

$$\eta(w) \geq \langle z_\sigma, w \rangle \qquad \text{for all} \quad w \in |\Pi|$$

and that the equality holds if $w \in \sigma$.

Since $|\Pi|$ is assumed to span W over $\boldsymbol{R}$, the natural linear map

$$W^* \to \mathrm{PL}(\Pi)$$

is injective with the image contained in $\mathrm{CPL}(\Pi)$. From now on we always identify W^* with its image in $\mathrm{CPL}(\Pi)$.

LEMMA 2.2. *We have* $\mathrm{CPL}(\Pi) \subset W^* + \mathrm{PL}_{\geq 0}(\Pi)$. *If* Π *contains an r-dimensional cone, then*

$$\mathrm{CPL}(\Pi) \cap (-\mathrm{CPL}(\Pi)) = W^* .$$

PROOF. Let $\eta \in \mathrm{CPL}(\Pi)$ and $\sigma \in \Pi$. By definition, there exists $z_\sigma \in W^*$ such that

$$\eta(w) \geq \langle z_\sigma, w \rangle \qquad \text{for all} \quad w \in |\Pi|$$

with the equality holding if $w \in \sigma$. Hence $\eta - z_\sigma$ belongs to $\mathrm{PL}_{\geq 0}(\Pi)$, and the first inclusion holds.

Let us now choose σ to be in $\Pi(r)$, which is nonempty by assumption, and suppose further that η belongs to $-\mathrm{CPL}(\Pi)$ as well. Hence there exists $z'_\sigma \in W^*$ such that

$$-\eta(w) \geq \langle z'_\sigma, w \rangle \qquad \text{for all} \quad w \in |\Pi|$$

with the equality holding if $w \in \sigma$. Thus

$$\langle -z'_\sigma, w \rangle \geq \eta(w) \geq \langle z_\sigma, w \rangle \qquad \text{for all} \quad w \in |\Pi| .$$

Moreover, the equalities hold for w belonging to the r-dimensional σ. Hence we necessarily have $-z'_\sigma = z_\sigma$, and, consequently, the equalities hold for all $w \in |\Pi|$.

We are done, since W^* is contained in $\mathrm{CLP}(\Pi)$ as well as $-\mathrm{CPL}(\Pi)$. q.e.d.

DEFINITION. A convex polyhedral cone decomposition Π for W is said to be *quasi-projective* if there exists $\eta \in \mathrm{PL}(\Pi)$ which is *strictly convex with respect to* Π, that is, there exists $z_\sigma \in W^*$ for each $\sigma \in \Pi$ such that

$$\eta(w) \geq \langle z_\sigma, w \rangle \qquad \text{for all} \quad w \in |\Pi|$$

and that the equality holds *if and only if* $w \in \sigma$.

REMARK. The above terminology was motivated by the algebraic geometry of toric varieties, where equivariant ample line bundles correspond to strictly convex support functions. Consequently, complete and quasi-projective Π may be said to be *projective*. In the literature, such a Π is also said to be *strongly polytopal* in combinatorics or *regular* by Gelfand-Zelevinskij-Kapranov [9].

EXAMPLE. Here are examples of Π which are simplicial but not quasi-projective with $\dim W = 3$.

(1) Choose an $\boldsymbol{R}$-basis $\{w_1, w_2, w_3\}$ for W. Let

$$w_0 := w_1 + w_2 + w_3 , \quad w'_1 := w_0 + w_1 , \quad w'_2 := w_0 + w_2 , \quad w'_3 := w_0 + w_3 ,$$

and denote by Π the set of faces of the following nine simplicial cones of dimension three:

$$\boldsymbol{R}_{\geq 0}w_0+\boldsymbol{R}_{\geq 0}w'_1+\boldsymbol{R}_{\geq 0}w'_2\,,\quad \boldsymbol{R}_{\geq 0}w_0+\boldsymbol{R}_{\geq 0}w'_2+\boldsymbol{R}_{\geq 0}w'_3\,,\quad \boldsymbol{R}_{\geq 0}w_0+\boldsymbol{R}_{\geq 0}w'_3+\boldsymbol{R}_{\geq 0}w'_1\,,$$
$$\boldsymbol{R}_{\geq 0}w_1+\boldsymbol{R}_{\geq 0}w'_3+\boldsymbol{R}_{\geq 0}w'_1\,,\quad \boldsymbol{R}_{\geq 0}w_2+\boldsymbol{R}_{\geq 0}w'_1+\boldsymbol{R}_{\geq 0}w'_2\,,\quad \boldsymbol{R}_{\geq 0}w_3+\boldsymbol{R}_{\geq 0}w'_2+\boldsymbol{R}_{\geq 0}w'_3\,,$$
$$\boldsymbol{R}_{\geq 0}w_1+\boldsymbol{R}_{\geq 0}w_2+\boldsymbol{R}_{\geq 0}w'_1\,,\quad \boldsymbol{R}_{\geq 0}w_2+\boldsymbol{R}_{\geq 0}w_3+\boldsymbol{R}_{\geq 0}w'_2\,,\quad \boldsymbol{R}_{\geq 0}w_3+\boldsymbol{R}_{\geq 0}w_1+\boldsymbol{R}_{\geq 0}w'_3\,.$$

Then Π is a convex polyhedral cone decomposition for W with $|\Pi|=\boldsymbol{R}_{\geq 0}w_1+\boldsymbol{R}_{\geq 0}w_2+\boldsymbol{R}_{\geq 0}w_3$ simplicial. In this case, no $\eta\in\mathrm{CPL}(\Pi)$ can be strictly convex with respect to Π. Indeed, since $w_1+w'_2=w'_1+w_2$, we have $w'_2=w'_1+w_2-w_1$, the right hand side of which is a linear combination of three vectors contained in a common cone belonging to Π. Thus $\eta(w'_2)\geq\eta(w'_1)+\eta(w_2)-\eta(w_1)$, so that $\eta(w_1)+\eta(w'_2)\geq\eta(w'_1)+\eta(w_2)$. Similarly, we have $\eta(w_2)+\eta(w'_3)\geq\eta(w'_2)+\eta(w_3)$ and $\eta(w_3)+\eta(w'_1)\geq\eta(w'_3)+\eta(w_1)$. Adding these three inequalities, we see that they are necessarily equalities.

(2) Let $w_0:=-(w_1+w_2+w_3)$ instead, and consider the set Π consisting of the faces of the nine simplicial cones in (1) as well as the tenth simplicial cone

$$\boldsymbol{R}_{\geq 0}w_1+\boldsymbol{R}_{\geq 0}w_2+\boldsymbol{R}_{\geq 0}w_3\,.$$

Then Π is complete but there exists no $\eta\in\mathrm{CPL}(\Pi)$ which is strictly convex with respect to Π for exactly the same reason as above. This is precisely the simplest fan which gives rise to a smooth non-projective toric variety of dimension three (cf. [13, pp. 84–85]).

We refer the reader to [12, pp. 68–74] for more examples.

To characterize quasi-projective Π, let us consider the dual space

$$\mathrm{PL}(\Pi)^*:=\mathrm{Hom}_{\boldsymbol{R}}(\mathrm{PL}(\Pi),\boldsymbol{R})$$

with the canonical $\boldsymbol{R}$-bilinear pairing denoted again by

$$\langle\ ,\ \rangle\colon \mathrm{PL}(\Pi)^*\times\mathrm{PL}(\Pi)\to\boldsymbol{R}\,.$$

Since $|\Pi|$ is assumed to span W over $\boldsymbol{R}$, we have a natural surjective linear map $\mathrm{PL}(\Pi)^*\to W$. We need to consider the dual polyhedral cone

$$\mathrm{CPL}(\Pi)^{\vee}:=\{l\in\mathrm{PL}(\Pi)^*\mid\langle l,\eta\rangle\geq 0\ \text{for all}\ \eta\in\mathrm{CPL}(\Pi)\}\,.$$

Beside assuming $|\Pi|$ to span W over $\boldsymbol{R}$, let us *further assume* that the support $|\Pi|$ is a *convex polyhedral cone*, although we do *not* assume it to be strongly convex. Thus a complete Π, for instance, meets our requirement.

Under this further assumption, Π obviously contains an r-dimensional cone. Hence by Lemma 2.2, we see that $\mathrm{CPL}(\Pi)^{\vee}$ spans, over $\boldsymbol{R}$, the kernel of the surjective linear map $\mathrm{PL}(\Pi)^*\to W$.

Recall that $r:=\dim W$. A polyhedral cone in $\Pi(r-1)$ is called a *wall* for Π. We say a wall $\tau\in\Pi(r-1)$ to be *internal* if τ is not contained in the boundary of the support

$|\Pi|$. It is the case if and only if there exist $\sigma, \sigma' \in \Pi(r)$ such that $\tau = \sigma \cap \sigma'$, since the r-dimensional convex polyhedral cone $|\Pi|$ is necessarily the union of cones in $\Pi(r)$. In particular, every wall is internal when Π is complete.

The following result, which was motivated by the toric version of the Kleiman-Nakai criterion in algebraic geometry, plays a crucial role later. It is essentially a local characterization of convexity and strict convexity of piecewise linear functions.

THEOREM 2.3 (Kleiman-Nakai criterion). *Let Π be a convex polyhedral cone decomposition for W such that $|\Pi|$ is a convex polyhedral cone spanning W over $\boldsymbol{R}$. Then for each internal wall $\tau \in \Pi(r-1)$, there exists a nonzero element $l_\tau \in \mathrm{PL}(\Pi)^*$ uniquely determined up to positive scalar multiple such that the following are satisfied:*

(1) *l_τ for internal walls $\tau \in \Pi(r-1)$ span, over $\boldsymbol{R}$, the kernel of the surjective linear map $\mathrm{PL}(\Pi)^* \to W$.*

(2) *The polyhedral cone $\mathrm{CPL}(\Pi)^\vee$ consists of nonnegative linear combinations of l_τ for internal walls $\tau \in \Pi(r-1)$, that is,*

$$\mathrm{CPL}(\Pi)^\vee = \sum_{\tau \text{ internal walls}} \boldsymbol{R}_{\geq 0} l_\tau \,.$$

Consequently, $\eta \in \mathrm{PL}(\Pi)$ belongs to $\mathrm{CPL}(\Pi)$ if and only if

$$\langle l_\tau, \eta \rangle \geq 0 \qquad \textit{for all internal walls} \quad \tau \in \Pi(r-1)\,.$$

(3) *$\eta \in \mathrm{PL}(\Pi)$ is strictly convex with respect to Π if and only if*

$$\langle l_\tau, \eta \rangle > 0 \qquad \textit{for all internal walls} \quad \tau \in \Pi(r-1)\,.$$

PROOF. (2) implies (1), since $\mathrm{CPL}(\Pi)^\vee$ spans the kernel of $\mathrm{PL}(\Pi)^* \to W$ as we noted above.

For each $\sigma \in \Pi(r)$ (resp. each internal wall τ), let us denote $\mathrm{PL}(\sigma) := W^*$ (resp. $\mathrm{PL}(\tau) := W^*/\tau^\perp$), which consists of the linear maps $\sigma \to \boldsymbol{R}$ (resp. $\tau \to \boldsymbol{R}$). Since $|\Pi|$ is assumed to be an r-dimensional convex polyhedral cone, we have $|\Pi| = \bigcup_{\sigma \in \Pi(r)} \sigma$ as we remarked above.

For each internal wall τ, there exist $\sigma(\tau), \sigma'(\tau) \in \Pi(r)$ such that $\tau = \sigma(\tau) \cap \sigma'(\tau)$. Define

$$\delta_\tau \colon \prod_{\sigma \in \Pi(r)} \mathrm{PL}(\sigma) \to W^*$$

to be the $\boldsymbol{R}$-linear map sending $\zeta = (z_\sigma)_{\sigma \in \Pi(r)}$ to $\delta_\tau(\zeta) := z_{\sigma(\tau)} - z_{\sigma'(\tau)}$, and denote by $\bar{\delta}_\tau \colon \prod_{\sigma \in \Pi(r)} \mathrm{PL}(\sigma) \to \mathrm{PL}(\tau)$ the composite of δ_τ with the projection $W^* \to W^*/\tau^\perp = \mathrm{PL}(\tau)$. Then $\mathrm{PL}(\Pi)$ can obviously be identified with the kernel of their product

$$\bar{\delta} := (\bar{\delta}_\tau)_{\tau \text{ internal walls}} \colon \prod_{\sigma \in \Pi(r)} \mathrm{PL}(\sigma) \to \prod_{\tau \text{ internal walls}} \mathrm{PL}(\tau)\,.$$

In other words, $\mathrm{PL}(\Pi)$ coincides with the subspace of $\prod_{\sigma \in \Pi(r)} \mathrm{PL}(\sigma)$ consisting of ζ with

$$\delta_\tau(\zeta) \in \tau^\perp \qquad \text{for all internal walls} \quad \tau\,.$$

The induced $\boldsymbol{R}$-linear map $\delta_\tau\colon \mathrm{PL}(\Pi) \to \tau^\perp$ is surjective. Furthermore, $\tau^\perp \cap \sigma(\tau)^\vee$ is one of the two half-lines inside the one-dimensional $\boldsymbol{R}$-vector space $\tau^\perp$.

We now claim that

$$\mathrm{CPL}(\Pi) = \left\{ \zeta \in \prod_{\sigma \in \Pi(r)} \mathrm{PL}(\sigma) \,\middle|\, \delta_\tau(\zeta) \in \tau^\perp \cap \sigma(\tau)^\vee \text{ for all internal walls } \tau \right\},$$

and that $\zeta \in \mathrm{CPL}(\Pi)$ is strictly convex with respect to Π if and only if $\delta_\tau(\zeta) \neq 0$ for all internal walls τ. Indeed, $\mathrm{CPL}(\Pi)$ is obviously contained in the set on the right hand side of the claimed equality. Suppose now that ζ is an element of the set on the right hand side. In particular, ζ is an element of $\mathrm{PL}(\Pi)$. The r-dimensional convex polyhedral cone $|\Pi|$ is the union of cones in $\Pi(r)$. By the defining property for ζ, we see that $\zeta\colon |\Pi| \to \boldsymbol{R}$ is locally convex, that is, for each $\sigma \in \Pi(r)$, the function ζ is convex on the union of $\{\sigma' \in \Pi(r) \,|\, \sigma' \cap \sigma \in \Pi(r-1)\}$. By definition, ζ is a globally convex function on $|\Pi|$ if and only if so is its restriction to every affine line segment. Namely, for any $w, w' \in |\Pi|$, the function

$$f(\lambda) := \zeta(\lambda w + (1-\lambda) w')\,, \qquad 0 \leq \lambda \leq 1$$

is convex in the variable λ. This is clearly the case, since a function in one variable which is locally convex is necessarily globally convex.

The assertion on the strict convexity can be proved similarly.

The $\boldsymbol{R}$-linear map

$$\delta_\tau^*\colon\ W/\boldsymbol{R}\tau \to \mathrm{PL}(\Pi)^*$$

dual to $\delta_\tau\colon \mathrm{PL}(\Pi) \to \tau^\perp$ is injective so that the image under it of the one-dimensional cone $(\sigma + \boldsymbol{R}\tau)/\boldsymbol{R}\tau$ is again one-dimensional and is of the form

$$\delta_\tau^*((\sigma + \boldsymbol{R}\tau)/\boldsymbol{R}\tau) = \boldsymbol{R}_{\geq 0} l_\tau$$

for a nonzero element l_τ determined uniquely up to positive scalar multiple. Obviously, we have

$$\mathrm{CPL}(\Pi)^\vee = \sum_{\tau \text{ internal walls}} \boldsymbol{R}_{\geq 0} l_\tau\,.$$

For $\zeta \in \mathrm{PL}(\Pi)$, we clearly have $\delta_\tau(\zeta) \in \tau^\perp \cap \sigma(\tau)^\vee$ (resp. $\delta_\tau(\zeta) \in \tau^\perp \cap \sigma(\tau)^\vee \setminus \{0\}$) if and only if $\langle l_\tau, \zeta \rangle \geq 0$ (resp. $\langle l_\tau, \zeta \rangle > 0$) for all internal walls τ. q.e.d.

Corollary 2.4. *The following are equivalent for a convex polyhedral cone decomposition Π for W such that $|\Pi|$ is a convex polyhedral cone spanning W over $\boldsymbol{R}$.*

(1) *Π is quasi-projective.*

(2) $\mathrm{CPL}(\Pi)$ *spans* $\mathrm{PL}(\Pi)$ *over* $\boldsymbol{R}$.

(3) $\mathrm{CPL}(\Pi)^\vee$ *is strongly convex.*

In this case, the interior of $\mathrm{CPL}(\Pi)$ *consists of those* $\eta \in \mathrm{PL}(\Pi)$ *which are strictly convex with respect to* Π.

PROOF. (2) and (3) are equivalent by the duality for convex polyhedral cones (see, for instance, [13, Proposition A.6]).

(1) implies (2). Indeed, suppose $\eta \in \mathrm{PL}(\Pi)$ is strictly convex with respect to Π. Then for any $\eta' \in \mathrm{PL}(\Pi)$ we can find a large enough positive real number c such that $\langle l_\tau, \eta' + c\eta \rangle > 0$ for all internal walls $\tau \in \Pi(r-1)$. Hence $\eta' + c\eta$ is strictly convex with respect to Π by Theorem 2.3.

If (2) is satisfied, then $\mathrm{CPL}(\Pi)$, as a subset of $\mathrm{PL}(\Pi)$, has an interior point η, which is necessarily strictly convex with respect to Π again by Theorem 2.3. q.e.d.

REMARK. When Π is complete, it can be shown to be quasi-projective (hence projective) if and only if there exists a nonzero w_ρ in each $\rho \in \Pi(1)$ such that the convex hull $\square$ of $\{w_\rho \mid \rho \in \Pi(1)\}$ has exactly $\{w_\rho \mid \rho \in \Pi(1)\}$ as its set of vertices and that Π consists of the convex polyhedral cones spanned by the faces of $\square$.

A convex polyhedral cone decomposition Π is said to be a *subdivision* (or *refinement*) of another Π' if $|\Pi| = |\Pi'|$ and if each $\sigma \in \Pi$ is contained in a $\sigma' \in \Pi'$. In this case, $\mathrm{PL}(\Pi')$ can be canonically identified with a subspace of $\mathrm{PL}(\Pi)$, and $\mathrm{CPL}(\Pi') = \mathrm{PL}(\Pi') \cap \mathrm{CPL}(\Pi)$.

THEOREM 2.5. *Let* Π *be a convex polyhedral cone decomposition such that* $|\Pi|$ *is a convex polyhedral cone spanning* W *over* $\boldsymbol{R}$. *Then the faces of the convex polyhedral cone* $\mathrm{CPL}(\Pi)$ *are exactly of the form* $\mathrm{CPL}(\Pi')$ *for* quasi-projective *convex polyhedral cone decompositions* Π' *such that* Π *is a subdivision of* Π'.

PROOF. Choose and fix an arbitrary $\eta \in \mathrm{CPL}(\Pi)$.

On the one hand, there exists a unique convex polyhedral cone decomposition Π' such that Π is a subdivision of Π' and that η is strictly convex with respect to Π'. In particular, Π' is quasi-projective. Moreover, $\mathrm{PL}(\Pi')$ can be regarded as a subspace of $\mathrm{PL}(\Pi)$ and η is contained in the relative interior of $\mathrm{CPL}(\Pi') = \mathrm{PL}(\Pi') \cap \mathrm{CPL}(\Pi)$.

On the other hand, let F be the unique face of $\mathrm{CPL}(\Pi)$ containing η in its relative interior. It suffices to show that $F = \mathrm{CPL}(\Pi')$.

The face of $\mathrm{CPL}(\Pi)^\vee$ dual to F by the Galois correspondence in [13, Proposition A.6] is

$$F^* := \mathrm{CPL}(\Pi)^\vee \cap F^\perp = \mathrm{CPL}(\Pi)^\vee \cap \{\eta\}^\perp .$$

Let

$$I := \{\text{internal walls } \tau \text{ with } \langle l_\tau, \eta \rangle = 0\} .$$

Then by Theorem 2.3, we see that

$$F^* = \sum_{\tau \in I} \boldsymbol{R}_{\geq 0} l_\tau \quad \text{and} \quad F = \{\zeta \in \mathrm{CPL}(\Pi) \mid \langle l_\tau, \zeta \rangle = 0, \forall \tau \in I\} .$$

Each $\sigma' \in \Pi'(r)$ is the union of the family $\{\sigma \in \Pi(r) \mid \sigma \subset \sigma'\}$. This family coincides with that of the closures of the connected components of $\sigma' \setminus \bigcup_{\tau \in I} \tau$. Thus $\zeta \in \mathrm{CPL}(\Pi)$ is contained in $\mathrm{CPL}(\Pi')$ if and only if ζ is linear across the walls $\{\tau \mid \tau \subset \sigma'\}$, that is, $\langle l_\tau, \zeta \rangle = 0$. q.e.d.

Applying Theorem 2.5 to the improper face $\mathrm{CPL}(\Pi)$ itself, we get:

COROLLARY 2.6. *Let Π be a convex polyhedral cone decomposition such that $|\Pi|$ is a convex polyhedral cone spanning W over $\boldsymbol{R}$. Then there exists a* quasi-projective *convex polyhedral cone decomposition Π' such that* $\mathrm{CPL}(\Pi) = \mathrm{CPL}(\Pi')$ *and that Π is a subdivision of Π'.*

REMARK. It is essential to allow degenerate Π' to occur in Theorem 2.5 as well, even when Π is nondegenerate. As we supplement this theorem in Proposition 3.10, the description of facets (i.e., codimension-one faces) of $\mathrm{CPL}(\Pi)$ includes the wall geometry which Reid [15] obtained in connection with Mori's birational geometry in the particular case of projective toric varieties. Reid deals with the case where Π is a projective fan. By Corollary 2.4, $\mathrm{CPL}(\Pi)^\vee$ is then strongly convex. The one-dimensional faces of $\mathrm{CPL}(\Pi)^\vee$ are called *extremal rays*, and correspond to the facets of $\mathrm{CPL}(\Pi)$ by duality (see, for instance, [13, Proposition A.6]). Contraction of an extremal ray R amounts to removal of the walls $\tau \in \Pi(r-1)$ satisfying $l_\tau \in R$ and gives rise to Π' which is a possibly degenerate fan. The corresponding equivariant morphism of toric varieties is Mori's contraction of extremal rational curves whose numerical equivalence classes are contained in R.

3. Gelfand-Kapranov-Zelevinskij decompositions. In this section, a finite subset $\Xi \subset W$, which spans W over $\boldsymbol{R}$ as in Section 1, is further assumed to satisfy the following conditions: *Ξ does not contain* 0, *and each $\xi \in \Xi$ is not a positive scalar multiple of any other element of $\Xi \setminus \{\xi\}$*. Thus $\boldsymbol{R}_{\geq 0}\xi$ for $\xi \in \Xi$ are mutually distinct one-dimensional cones in W.

DEFINTION. A (possibly degenerate) convex polyhedral cone decomposition Π for W is said to be *admissible* for (W, Ξ) if

(i) Π is quasi-projective,

(ii) $|\Pi| = W_{\geq 0}(\Xi)$ and

(iii) each cone $\sigma \in \Pi$ is spanned over $\boldsymbol{R}_{\geq 0}$ by a subset of Ξ.

By definition, an admissible Π is complete if and only if $W_{\geq 0}(\Xi) = W$.

REMARK. For the Gelfand-Kapranov-Zelevinskij decomposition considered later in this section, it is essential to allow *degenerate* Π to be admissible as well. However, if $W_{\geq 0}(\Xi)$ is strongly convex as in the case dealt with by Gelfand, Kapranov and Zelevinskij, then an admissible Π is necessarily nondegenerate.

DEFINITION. Let Π be a convex polyhedral cone decomposition admissible for (W, Ξ). A subset $S \subset \Xi$ is said to be a *spanning subset* for Π if for each $\sigma \in \Pi$, there exist $\xi_1, \cdots, \xi_p \in S$ such that

$$\sigma = \boldsymbol{R}_{\geq 0}\xi_1 + \cdots + \boldsymbol{R}_{\geq 0}\xi_p .$$

When Π is *nondegenerate*, we denote

$$\Xi(\Pi) := \{\xi \in \Xi \mid \boldsymbol{R}_{\geq 0}\xi \in \Pi(1)\} ,$$

where $\Pi(1)$ is the set of one-dimensional cones in Π.

If Π is nondegenerate, then $\Xi(\Pi)$ is the smallest spanning subset for Π. Indeed, we have $\Pi(1) = \{\boldsymbol{R}_{\geq 0}\xi \mid \xi \in \Xi(\Pi)\}$. Moreover, the one-dimensional faces of strongly convex $\sigma \in \Pi$ are of the form $\boldsymbol{R}_{\geq 0}\xi_1, \cdots, \boldsymbol{R}_{\geq 0}\xi_p$ for a subset $\{\xi_1, \cdots, \xi_p\} \subset \Xi(\Pi)$. Hence $\sigma = \boldsymbol{R}_{\geq 0}\xi_1 + \cdots + \boldsymbol{R}_{\geq 0}\xi_p$.

By Lemma 2.1 and Corollary 2.4, an admissible Π is simplicial if and only if it is nondegenerate and $\dim \mathrm{CPL}(\Pi) = \dim \mathrm{PL}(\Pi) = {}^{\#}\Xi(\Pi)$.

We have introduced in Section 1 a surjective linear map $W_1 = \bigoplus_{\xi \in \Xi} \boldsymbol{R}e_\xi \to W$ which sends e_ξ to ξ, as well as the dual injective linear map $W^* \to W_1^* = \bigoplus_{\xi \in \Xi} \boldsymbol{R}e_\xi^*$ which sends $z \in W^*$ to $\sum_{\xi \in \Xi} \langle z, \xi \rangle e_\xi^*$. We again identify W^* with its image under this map.

The key idea due to Gelfand, Kapranov and Zelevinskij of considering the following $\mathrm{CPL}^{\sim}(\Pi, S)$'s instead of $\mathrm{CPL}(\Pi)$'s enables us to compare different Π's in a quite convenient manner.

DEFINITION. Let Π be a convex polyhedral cone decomposition admissible for (W, Ξ) and S a spanning subset for Π. Denote by $\mathrm{PL}^{\sim}(\Pi, S)$ the linear subspace of W_1^* consisting of $x = \sum_{\xi \in \Xi} x_\xi e_\xi^* \in W_1^*$ for which there exists $\eta \in \mathrm{PL}(\Pi)$ such that $x_\xi = \eta(\xi)$ for all $\xi \in S$. We denote by $\mathrm{CPL}^{\sim}(\Pi, S) \subset \mathrm{PL}^{\sim}(\Pi, S)$ the convex polyhedral cone consisting of $x = \sum_{\xi \in \Xi} x_\xi e_\xi^* \in W_1^*$ such that there exists $\eta \in \mathrm{CPL}(\Pi)$ satisfying $x_\xi \geq \eta(\xi)$ for all $\xi \in \Xi$ and $x_\xi = \eta(\xi)$ for all $\xi \in S$.

For nondegenerate Π, we denote

$$\mathrm{CPL}^{\sim}(\Pi) := \mathrm{CPL}^{\sim}(\Pi, \Xi(\Pi)) .$$

PROPOSITION 3.1. *Let Π be a convex polyhedral cone decomposition admissible for (W, Ξ). Then for each spanning subset S for Π, we have*

$$\mathrm{CPL}^{\sim}(\Pi, S) \cap (-\mathrm{CPL}^{\sim}(\Pi, S)) = W^* .$$

We have $\dim \mathrm{CPL}^{\sim}(\Pi, S) = {}^{\#}\Xi$ *if and only if Π is simplicial and $S = \Xi(\Pi)$. The faces of* $\mathrm{CPL}^{\sim}(\Pi, S)$ *are exactly of the form* $\mathrm{CPL}^{\sim}(\Pi', S')$, *where Π' runs through the admissible convex polyhedral cone decompositions such that Π is a subdivision of Π', while S' runs through the spanning subsets for Π' such that $S' \supset S$.*

PROOF. If $x \in \mathrm{PL}^{\sim}(\Pi, S)$, there exists a unique $\eta \in \mathrm{PL}(\Pi)$ such that $x_\xi = \eta(\xi)$ for all $\xi \in S$. Sending $x \in \mathrm{PL}^{\sim}(\Pi, S)$ to

$$\left(\eta, \sum_{\xi \in \Xi \setminus S} (x_\xi - \eta(\xi)) e_\xi^*\right),$$

we have a linear isomorphism

$$\mathrm{PL}^{\sim}(\Pi, S) \xrightarrow{\sim} \mathrm{PL}(\Pi) \times \left(\bigoplus_{\xi \in \Xi \setminus S} \boldsymbol{R} e_\xi^*\right)$$

which sends $\mathrm{CPL}^{\sim}(\Pi, S)$ onto the product cone

$$\mathrm{CPL}(\Pi) \times \left(\sum_{\xi \in \Xi \setminus S} \boldsymbol{R}_{\geq 0} e_\xi^*\right).$$

The rest of the proof is an immediate consequence of Lemmas 2.1 and 2.2, as well as Theorem 2.5 and the following easy lemma. q.e.d.

LEMMA 3.2. *Let C (resp. C') be a convex polyhedral cone in a finite dimensional $\boldsymbol{R}$-vector space V (resp. V'). Then the faces of the product cone $C \times C'$ in $V \times V'$ are exactly of the form $F \times F'$ for faces $F \prec C$ and $F' \prec C'$.*

PROPOSITION 3.3. *Inside the vector space W_1^*, we have*

$$W^* + \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} e_\xi^* = \bigcup_{\Pi, S} \mathrm{CPL}^{\sim}(\Pi, S),$$

where Π runs through the convex polyhedral cone decompositions admissible for (W, Ξ), while S runs through the spanning subsets for Π.

PROOF. The right hand side is obviously contained in the left hand side by Lemma 2.2. To prove the opposite inclusion, let $x = \sum_{\xi \in \Xi} x_\xi e_\xi^*$ be an element of the left hand side. In the product space $W \times \boldsymbol{R}$, consider the convex polyhedral cone

$$E(x) := \boldsymbol{R}_{\geq 0}(0, 1) + \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0}(\xi, x_\xi).$$

Since x belongs to the left hand side of the claimed equality, there obviously exists a function $\eta: W_{\geq 0}(\Xi) \to \boldsymbol{R}$ such that $E(x)$ coincides with the *epigraph*

$$\mathrm{epi}(\eta) := \{(w, c) \in W_{\geq 0}(\Xi) \times \boldsymbol{R} \mid c \geq \eta(w)\}$$

of η. Since $E(x)$ is a convex polyhedral cone, we see that η is convex and that the *graph* of η, denoted $\mathrm{graph}(\eta)$, is the union of the non-vertical faces $F \prec \mathrm{epi}(\eta)$ of the form $F = \mathrm{epi}(\eta) \cap \{(z, 1)\}^\perp$ for some $(z, 1) \in \mathrm{epi}(\eta)^\vee$. The first projection $\mathrm{pr}_1: W \times \boldsymbol{R} \to W$ maps the faces $F \prec \mathrm{epi}(\eta)$ with $F \subset \mathrm{graph}(\eta)$ isomorphically onto the convex polyhedral cones $\mathrm{pr}_1(F)$ which obviously form a convex polyhedral cone decomposition Π with support $W_{\geq 0}(\Xi)$. Clearly, each $\sigma \in \Pi$ is spanned over $\boldsymbol{R}_{\geq 0}$ by elements of $S := \{\xi \in \Xi \mid (\xi, x_\xi) \in \mathrm{graph}(\eta)\}$. By construction, η is strictly convex with respect to Π. Consequently, Π is admissible for (W, Ξ). Moreover, we have $x_\xi \geq \eta(\xi)$ for any $\xi \in \Xi$ with the equality holding if ξ belongs to S. Hence $x \in \mathrm{CPL}^{\sim}(\Pi, S)$. q.e.d.

REMARK. η and Π in the above proof are uniquely determined by x, and are denoted by η_x and Π_x, respectively. Π_x is nondegenerate if and only if $\mathrm{epi}(\eta_x)$ is strongly convex. This is the case if and only if the restriction of Π_x to the linear subspace $W_{\geq 0}(\Xi) \cap (-W_{\geq 0}(\Xi))$ is nondegenerate.

THEOREM 3.4. *Under the assumptions on Ξ at the beginning of this section,*

$$\{\mathrm{CPL}^{\sim}(\Pi, S) \mid \Pi \text{ admissible for } (W, \Xi) \text{ and } S \text{ spanning subsets for } \Pi\}$$

is a convex polyhedral cone decomposition for W_1^* with the smallest cone W^* and with support equal to $W^* + \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} e_\xi^*$.

PROOF. In view of Propositions 3.1 and 3.3, we only need to show the following: For any pair Π, Π' of convex polyhedral cone decompositions admissible for (W, Ξ) and for any spanning subsets $S, S' \subset \Xi$ for Π and Π', respectively, we have

$$\mathrm{CPL}^{\sim}(\Pi, S) \cap \mathrm{CPL}^{\sim}(\Pi', S') = \mathrm{CPL}^{\sim}(\Pi'', S \cup S')$$

for another convex polyhedral cone decomposition Π'' admissible for (W, Ξ) such that Π and Π' are subdivisions of Π''.

(1) We first show that $\mathrm{CPL}^{\sim}(\Pi, S) \cap \mathrm{CPL}^{\sim}(\Pi', S')$ is a union of faces of $\mathrm{CPL}^{\sim}(\Pi, S)$, hence is a union of faces of $\mathrm{CPL}^{\sim}(\Pi', S')$ as well by symmetry. Indeed, let $\tilde{F}$ be a face of $\mathrm{CPL}^{\sim}(\Pi, S)$ such that an element x in the relative interior of $\tilde{F}$ also belongs to $\mathrm{CPL}^{\sim}(\Pi', S')$. Then in the notation in the remark above, η_x is strictly convex with respect to Π_x of which Π and Π' are subdivisions. Consequently, $\tilde{F}$ is contained entirely in $\mathrm{CPL}^{\sim}(\Pi, S) \cap \mathrm{CPL}^{\sim}(\Pi', S')$ in view of Theorem 2.5 and Proposition 3.1.

(2) We see that $\mathrm{CPL}^{\sim}(\Pi, S) \cap \mathrm{CPL}^{\sim}(\Pi', S')$ is indeed a face of both $\mathrm{CPL}^{\sim}(\Pi, S)$ and $\mathrm{CPL}^{\sim}(\Pi', S')$, since it is convex and, by (1), a union of faces of $\mathrm{CPL}^{\sim}(\Pi, S)$ as well as $\mathrm{CPL}^{\sim}(\Pi', S')$.

The rest of the proof is clear. q.e.d.

REMARK. As a consequence of Corollary 3.11 given later, $\mathrm{CPL}^{\sim}(\Pi, S)$ for degenerate Π's turn out to be contained in the boundary of the support $W^* + \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} e_\xi^*$.

The convex polyhedral cone decomposition in Theorem 3.4 is degenerate unless $W = \{0\}$, since W^* is the smallest cone appearing in it. The linear Gale transform in Section 1 is exactly what we need to obtain a nondegenerate convex polyhedral cone decomposition. Recall that

$$G(W, \Xi) := W_1^* / W^* = \sum_{\xi \in \Xi} \boldsymbol{R} g(\xi),$$

which is an $\boldsymbol{R}$-vector space of dimension $^\#\Xi - \dim W$, and

$$G_{\geq 0}(W, \Xi) := \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} g(\xi) .$$

DEFINITION. Let Π be a convex polyhedral cone decomposition admissible for (W, Ξ) and let S be a spanning subset for Π. We denote the image in $G(W, \Xi)$ of $\mathrm{CPL}^{\sim}(\Pi, S) \subset W_1^*$ by $\mathrm{cpl}(\Pi, S)$, and call it the *Gelfand-Kapranov-Zelevinskij* cone (the GKZ-cone, for short) associated to Π and S. In particular, we denote $\mathrm{cpl}(\Pi) := \mathrm{cpl}(\Pi, \Xi(\Pi))$ for nondegenerate Π and call it the GKZ-cone associated to Π.

By Proposition 3.1, we see that the faces of the GKZ-cone $\mathrm{cpl}(\Pi, S)$ are exactly of the form $\mathrm{cpl}(\Pi', S')$, where Π' runs through the admissible convex polyhedral cone decompositions such that Π is a subdivision of Π', while S' runs through the spanning subsets for Π' with $S' \supset S$.

Taking the image of the convex polyhedral cone decomposition in Theorem 3.4, we have the following main result in view of Proposition 3.1:

THEOREM 3.5. *Suppose that a finite subset $\Xi \subset W$ spanning W over $\boldsymbol{R}$ does not contain* 0 *and that each $\xi \in \Xi$ is not a positive scalar multiple of any other element of $\Xi \setminus \{\xi\}$. Then the collection*

$$\{\mathrm{cpl}(\Pi, S) \mid \Pi \textit{ admissible for } (W, \Xi) \textit{ and } S \textit{ spanning subsets for } \Pi\}$$

of the GKZ-*cones gives rise to a* nondegenerate *convex polyhedral cone decomposition for $G(W, \Xi)$ with support equal to $G_{\geq 0}(W, \Xi)$. We call it the* Gelfand-Kapranov-Zelevinskij decomposition (*the* GKZ-*decomposition, for short*) *for* (W, Ξ). *We have* $\dim \mathrm{cpl}(\Pi, S) = \dim G(W, \Xi)$ *if and only if Π is simplicial and $S = \Xi(\Pi)$.*

REMARK. The GKZ-decomposition might also be called the *secondary fan* as in [1]. As a consequence of Corollary 3.11 given later, $\mathrm{cpl}(\Pi, S)$ for degenerate Π's turn out to be contained in the boundary of the support $G_{\geq 0}(W, \Xi)$.

By Proposition 1.3, (2) and Theorem 3.5, we have the following which includes the case of projective toric varieties:

COROLLARY 3.6. *Suppose that a finite subset $\Xi \subset W$* positively *spanning W does not contain* 0 *and that each $\xi \in \Xi$ is not a positive scalar multiple of any other element of $\Xi \setminus \{\xi\}$. Then the faces of the* GKZ-*cones* $\mathrm{cpl}(\Pi)$, *with Π running through the* simplicial *convex polyhedral cone decompositions admissible for* (W, Ξ), *give rise to a nondegenerate convex polyhedral cone decomposition for $G(W, \Xi)$ having a* strongly convex *polyhedral cone $G_{\geq 0}(W, \Xi)$ as support.*

On the other hand, by Proposition 1.4, (1) and Theorem 3.5, we have a conic variant of a result obtained by Gelfand-Zelevinskij-Kapranov [9]:

COROLLARY 3.7. *Suppose that a finite subset $\Xi \subset W$ not containing* 0 *spans a* strongly convex *polyhedral cone $W_{\geq 0}(\Xi)$ of dimension equal to $r := \dim W$* and that each *$\xi \in \Xi$ is* *not a positive scalar multiple of any other element of* $\Xi \setminus \{\xi\}$. *Then the faces of the* GKZ-*cones* $\mathrm{cpl}(\Pi)$, *with* Π *running through the* simplicial *convex polyhedral cone decompositions admissible for* (W, Ξ), *give rise to a* complete *and nondegenerate polyhedral cone decomposition for* $G(W, \Xi)$.

REMARK. In fact, the GKZ-decomposition in Corollary 3.7 is *projective*. To show this, let us modify results in Gelfand-Zelevinskij-Kapranov [9] and first construct a function $\kappa\colon W^* + \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} e_\xi^* \to \boldsymbol{R}$ which is piecewise linear and strictly convex with respect to the cone decomposition for W_1^* consisting of the faces of $\mathrm{CPL}^\sim(\Pi)$ for simplicial Π's. A translate of it by a globally linear function then induces a function $\bar{\kappa}\colon G(W, \Xi) = G_{\geq 0}(W, \Xi) \to \boldsymbol{R}$ which is piecewise linear and strictly convex with respect to the GKZ-decomposition.

Indeed, let H_- be a half-space of W containing the origin of W in its interior and that the intersection $K := H_- \cap W_{\geq 0}(\Xi)$ is an r-dimensional convex polytope. Such an H_- exists, since $W_{\geq 0}(\Xi)$ is assumed to be an r-dimensional strongly convex polyhedral cone. Introduce the appropriately normalized Lebesgue measure $d\mu$ on W and let

$$\kappa(x) := -\int_K \eta_x d\mu \qquad \text{for} \quad x \in W^* + \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} e_\xi^* ,$$

where $\eta_x\colon W_{\geq 0}(\Xi) \to \boldsymbol{R}$ is the function piecewise linear and strictly convex with respect to Π_x as in Proposition 3.3 and the remark immediately after that. This κ turns out to be piecewise linear and strictly convex with respect to the convex polyhedral cone decomposition for W_1^* consisting of the faces of $\mathrm{CPL}^\sim(\Pi)$ for simplicial Π's admissible for (W, Ξ).

For instance, suppose $x \in \mathrm{CPL}^\sim(\Pi)$ so that Π is a subdivision of Π_x. Then

$$\kappa(x) = -\sum_{\sigma \in \Pi(r)} \int_{K \cap \sigma} \eta_x d\mu$$

with the summand $\int_{K \cap \sigma} \eta_x d\mu$ linear in the variables $\{x_\xi \mid \xi \in \Xi, \boldsymbol{R}_{\geq 0}\xi \prec \sigma\}$, since the restriction of η_x to σ is linear. Hence there exists $y(\Pi) \in W_1$ such that

$$\kappa(x) = \langle x, y(\Pi) \rangle \qquad \text{for} \quad x \in \mathrm{CPL}^\sim(\Pi) .$$

If x happens to belong to $W^* \subset W_1^*$ so that there exists a unique $z \in W^*$ with $x_\xi = \langle z, \xi \rangle$ for all $\xi \in \Xi$, then $\kappa(x)$ is linear in z. Denote by $\pi\colon W_1 \to W$ the surjective linear map appearing at the beginning of Section 1. Hence there exists $y_0 \in W_1$ such that

$$\kappa(x) = \langle x, y_0 \rangle = \langle z, \pi(y_0) \rangle \qquad \text{for} \quad x \in W^* .$$

Consequently, $\kappa - y_0$ vanishes on W^* and induces a function $\bar{\kappa}\colon G(W, \Xi) = G_{\geq 0}(W, \Xi) \to \boldsymbol{R}$ piecewise linear and strictly convex with respect to the GKZ-decomposition. Moreover, $\{y(\Pi) \mid \Pi$ simplicial and admissible for $(W, \Xi)\}$ is contained in the affine subspace $\pi^{-1}(\pi(y_0)) \subset W_1$.

As we stated in [13, Theorem A.18 and Corollary A.19] (however, with *upper convex* and inf there replaced by *convex* and sup, respectively), a convex polytope P in a finite dimensional $\boldsymbol{R}$-vector space V is in one-to-one correspondence with a function $h\colon V^* \to \boldsymbol{R}$ which is piecewise linear and strictly convex with respect to a complete and nondegenerate convex polyhedral cone decomposition for the dual space V^*: Namely, the *support function* h of P is defined by

$$h(u) = \sup\{\langle u, v\rangle \mid v \in P\} \qquad \text{for} \quad u \in V^*\,,$$

while P is described in terms of h by

$$P = \{v \in V \mid \langle u, v\rangle \leq h(u),\ \forall u \in V^*\}\,.$$

The cones of *outer normals* for P at various points of P comprise the complete and nondegenerate convex polyhedral cone decomposition for V^*, with respect to which h is piecewise linear and strictly convex.

As in Gelfand-Zelevinskij-Kapranov [9], consider a finite subset A of a finite dimensional $\boldsymbol{R}$-vector space W'. Denote by $\square$ the convex polytope in W' obtained as the convex hull of A. Then let $W := W' \times \boldsymbol{R}$ and $\Xi := \{(\alpha, 1) \mid \alpha \in A\}$. Hence

$$W_{\geq 0}(\Xi) = \boldsymbol{R}_{\geq 0}(\square \times \{1\})\,.$$

By Corollary 3.7, the faces of the GKZ-cones $\mathrm{cpl}(\Pi)$ for simplicial Π's give rise to a complete and nondegenerate polyhedral cone decomposition for $G(W, \Xi)$.

The *secondary polytope* constructed by [9] is, up to scale and symmetry with respect to the origin, the convex polytope obtained as the convex hull in W_1 of $\{y(\Pi) \mid \Pi$ simplicial and admissible for $(W, \Xi)\}$ with $y(\Pi)$ appearing above, where we let

$$H_- := \{(w', c) \in W := W' \times \boldsymbol{R} \mid c \leq 1\}\,.$$

The translate by $-y_0$ of this convex polytope is contained in the kernel $\pi^{-1}(0)$ of $\pi\colon W_1 \to W$. The GKZ-cones are nothing but its cones of outer normals in the space $G(W, \Xi)$ dual to $\pi^{-1}(0)$.

Corollary 3.7 has the following important application which guarantees the existence of a simplicial subdivision of a strongly convex polyhedral cone without any additional one-dimensional cones:

COROLLARY 3.8. *Let π be a strongly convex polyhedral cone in W. Then there exists a* simplicial *and* quasi-projective *cone decomposition Π with $|\Pi| = \pi$ such that $\Pi(1)$ coincides with the set of one-dimensional faces of π. If all the proper faces of π are simplicial, then such a Π does not subdivide any of the proper faces of π. Moreover, any two such subdivisions can be obtained from each other by a finite succession of flops to be defined below.*

PROOF. Choose and fix a nonzero vector on each one-dimensional face of π. If we denote by Ξ the set of all these vectors, then the faces of π form a convex polyhedral cone decomposition which is nondegenerate and admissible for (W, Ξ). Since $W_{\geq 0}(\Xi)=\pi$ is strongly convex by assumption, we see by Proposition 1.4, (1) that $G_{\geq 0}(W, \Xi)=G(W, \Xi)$. Choose Π to be any one of the admissible convex polyhedral cone decompositions in Corollary 3.7 such that the corresponding GKZ-cones $\mathrm{cpl}(\Pi)$ comprising the complete GKZ-decomposition satisfy $\dim \mathrm{cpl}(\Pi)=\dim G(W, \Xi)$. Clearly, Π is simplicial with $\Xi(\Pi)=\Xi$, and the last assertion holds. The last assertion is an easy consequence of Theorem 3.12 below. q.e.d.

As a special case of Corollary 3.8, we get another proof as well as a strengthening of a result in Stanley [17] and Goodman-Pach [10]:

COROLLARY 3.9. *A simplicial convex polytope Q admits a triangulation without additional vertices, i.e., one by means of simplices having vertices only in the set of vertices of Q. Moreover, any two such triangulations can be obtained from each other by a finite succession of flops.*

Let us now follow Reid [15, §§2–3] and Gelfand-Zelevenskij-Kapranov [9] to analyze when maximal dimensional $\mathrm{CPL}^{\sim}(\Pi)$'s (hence $\mathrm{cpl}(\Pi)$'s), with Π simplicial and admissible, intersect along facets (i.e., codimension-one faces).

We first supplement Theorem 2.5 and Proposition 3.1.

PROPOSITION 3.10. *Suppose Π is a* simplicial *convex polyhedral cone decomposition admissible for (W, Ξ). Then each* facet *$F \prec \mathrm{CPL}(\Pi)$ is of the form $F=\mathrm{CPL}(\bar{\Pi})$ for a convex polyhedral cone decomposition $\bar{\Pi}$ admissible for (W, Ξ) such that Π is a subdivision of $\bar{\Pi}$ and that one of the following holds:*

(1) *$\bar{\Pi}$ is degenerate.*

(2) *$\bar{\Pi}$ is simplicial and Π is the* star subdivision *of $\bar{\Pi}$ with respect to a $\xi_1 \in \Xi \setminus \Xi(\bar{\Pi})$. Namely, let $\alpha \in \bar{\Pi}$ be the unique cone containing ξ_1 in its relative interior and let $\beta_1, \cdots, \beta_s$ be the facets of α with $s:=\dim \alpha$. Then Π consists of the faces of the cones belonging to the union of*

$$\bar{\Pi}(r) \setminus \{\sigma \in \bar{\Pi}(r) \mid \sigma \succ \alpha\}$$

and

$$\{\lambda+\beta_j+\boldsymbol{R}_{\geq 0}\xi_1 \mid 1 \leq j \leq s,\ \lambda \in \bar{\Pi}(r-s) \text{ with } \lambda+\alpha \in \bar{\Pi}(r) \text{ and } \lambda \cap \alpha=\{0\}\}\ .$$

(3) *$\bar{\Pi}$ is nondegenerate but not simplicial with $\Xi(\bar{\Pi})=\Xi(\Pi)$. There exists another simplicial convex polyhedral cone decomposition $\Pi^{\dagger}$ admissible for (W, Ξ) such that $\Pi^{\dagger}$ is a subdivision of $\bar{\Pi}$ with $\Xi(\Pi^{\dagger})=\Xi(\bar{\Pi})$ and that* $\mathrm{CPL}(\bar{\Pi})=\mathrm{CPL}(\Pi) \cap \mathrm{CPL}(\Pi^{\dagger})$ *is a facet of both* $\mathrm{CPL}(\Pi)$ *and* $\mathrm{CPL}(\Pi^{\dagger})$. *We call $\Pi^{\dagger}$ the* flop *of Π along $\bar{\Pi}$.*

PROOF. As in the proof of Theorem 2.5, consider

$$F^{*}:=\mathrm{CPL}(\Pi)^{\vee} \cap F^{\perp}$$

and

$$I := \{\text{internal walls } \tau \in \Pi \text{ with } \langle l_\tau, \zeta \rangle = 0,\ \forall \zeta \in F\}\,.$$

Since F is assumed to be a facet, F^* is a one-dimensional face of $\mathrm{CPL}(\Pi)^\vee$ (hence is an extremal ray in the context of Mori's birational geometry and Reid's wall geometry).

For $\tau \in I$, there exist $\xi_1, \cdots, \xi_r, \xi_{r+1} \in \Xi(\Pi)$ such that, in the simplified notation $\rho_j := \boldsymbol{R}_{\geq 0}\xi_j$ for $1 \leq j \leq r+1$, we have

$$\tau = \sum_{j=1}^{r-1} \rho_j \quad \text{and} \quad \tau = (\tau + \rho_r) \cap (\tau + \rho_{r+1}) \quad \text{with} \quad \tau + \rho_r, \tau + \rho_{r+1} \in \Pi(r)\,.$$

By rearranging the ξ_j's if necessary, we may assume that the linear relation, unique up to nonzero scalar multiple, among the ξ_j's is of the form

$$\sum_{j=1}^{r+1} a_j \xi_j = 0 \qquad \text{with} \quad \begin{cases} a_j < 0 & \text{for } 1 \leq j \leq p \\ a_j = 0 & \text{for } p+1 \leq j \leq q \\ a_j > 0 & \text{for } q+1 \leq j \leq r+1 \end{cases}$$

for $0 \leq p \leq q \leq r-1$. The facets of τ are

$$\omega_j := \rho_1 + \cdots + \overset{j}{\vee} + \cdots + \rho_{r-1}\ (\rho_j \text{ omitted}) \qquad \text{for} \quad 1 \leq j \leq r-1\,.$$

By Reid [15, Lemma 2.3 and Theorem 2.4], $P(\tau) := \tau + \rho_r + \rho_{r+1}$ is decomposed into a union of r-dimensional cones in two different ways

$$P(\tau) = (\tau + \rho_r) \cup (\tau + \rho_{r+1}) \cup \left(\bigcup_{j=q+1}^{r-1} (\omega_j + \rho_r + \rho_{r+1}) \right) = \bigcup_{i=1}^{p} (\omega_i + \rho_r + \rho_{r+1})$$

with $\pi := \sum_{i=1}^{p} \rho_i + \sum_{j=q+1}^{r+1} \rho_j$ a face of $P(\tau)$. Moreover, p, q and π are determined by I independently of any particular choice of $\tau \in I$.

By Reid [15, Theorem 2.4 and Corollary 2.10], we now get the following:

If $p = 0$, then π is an $(r-q)$-dimensional $\boldsymbol{R}$-subspace of W such that $P(\tau) = \tau + \pi$. In this case, the faces of the cones in

$$\bar{\Pi}(r) := \{\tau + \pi \mid \tau \in I\}$$

turn out to form a degenerate convex polyhedral cone decomposition $\bar{\Pi}$ admissible for (W, Ξ) with π as the smallest cone such that Π is a subdivision of $\bar{\Pi}$ and that $F = \mathrm{CPL}(\bar{\Pi})$. Thus we are in Case (1).

Suppose $p \neq 0$. Then there exists a nondegenerate convex polyhedral cone decomposition $\bar{\Pi}$ admissible for (W, Ξ) such that Π is a subdivision of $\bar{\Pi}$ and that $F = \mathrm{CPL}(\bar{\Pi})$. To describe $\bar{\Pi}$ in more detail, let us rewrite the linear relation among the minimal linearly dependent set $\{\xi_1, \cdots, \xi_p, \xi_{q+1}, \cdots, \xi_r, \xi_{r+1}\}$ more symmetrically as

$$(-a_1)\xi_1 + \cdots + (-a_p)\xi_p = a'_1\xi'_1 + \cdots + a'_{p'}\xi'_{p'} \quad \text{with } (-a_1), \cdots, (-a_p), a'_1, \cdots, a'_{p'} > 0\,,$$

where $p' := r + 1 - q \geq 2$ with

$$\xi'_1 := \xi_{r+1}, \xi'_2 := \xi_r, \cdots, \xi'_{p'} := \xi_{q+1}, \qquad a'_1 := a_{r+1}, a'_2 := a_r, \cdots, a'_{p'} := a_{q+1}.$$

Let us denote

$$\varepsilon := \rho_1 + \cdots + \rho_p, \qquad \varepsilon' := \rho'_1 + \cdots + \rho'_{p'}, \qquad \text{hence} \quad \pi = \varepsilon + \varepsilon',$$

where $\rho'_1 := \boldsymbol{R}_{\geq 0}\xi'_1, \rho'_2 := \boldsymbol{R}_{\geq 0}\xi'_2, \cdots, \rho'_{p'} := \boldsymbol{R}_{\geq 0}\xi'_{p'}$. Let us further denote

$$\varepsilon_i := \rho_1 + \cdots + \overset{i}{\vee} + \cdots + \rho_p \qquad 1 \leq i \leq p,$$
$$\varepsilon'_j := \rho'_1 + \cdots + \overset{j}{\vee} + \cdots + \rho'_{p'} \qquad 1 \leq j \leq p'.$$

We see that $\lambda_0 := \rho_{p+1} + \cdots + \rho_q \in \Pi(r+1-p-p')$ satisfies

$$\lambda_0 \cap (\varepsilon + \varepsilon') = \{0\}, \qquad \lambda_0 + \varepsilon + \varepsilon'_j \in \Pi(r), \qquad 1 \leq \forall j \leq p'.$$

In particular, $\tau + \rho_r = \lambda_0 + \varepsilon + \varepsilon'_1$, $\tau + \rho_{r+1} = \lambda_0 + \varepsilon + \varepsilon'_2$ and $\tau = \lambda_0 + \varepsilon + (\varepsilon'_1 \cap \varepsilon'_2)$. Hence λ_0 belongs to

$$\Lambda := \{\lambda \in \Pi(r+1-p-p') \mid \lambda \cap (\varepsilon + \varepsilon') = \{0\}, \lambda + \varepsilon + \varepsilon'_j \in \Pi(r), 1 \leq \forall j \leq p'\}.$$

We have

$$\bar{\Pi}(r) = (\Pi(r) \setminus \{\lambda + \varepsilon + \varepsilon'_j \mid \lambda \in \Lambda, 1 \leq j \leq p'\}) \coprod \{\lambda + \varepsilon + \varepsilon' \mid \lambda \in \Lambda\}.$$

If $p = 1$, then $\varepsilon + \varepsilon' = \varepsilon'$ is simplicial, hence so is $\bar{\Pi}$. Obviously, Π is the star subdivision of $\bar{\Pi}$ with respect to ξ_1, and we are in Case (2).

If $p \geq 2$, then $\bar{\Pi}$ is not simplicial. By Reid [15, Theorem 3.4], there exists a simplicial convex polyhedral cone decomposition $\Pi^\dagger$ admissible for (W, Ξ) satisfying

$$\Pi^\dagger(r) = (\Pi(r) \setminus \{\lambda + \varepsilon + \varepsilon'_j \mid \lambda \in \Lambda, 1 \leq j \leq p'\}) \coprod \{\lambda + \varepsilon_i + \varepsilon' \mid \lambda \in \Lambda, 1 \leq i \leq p\}.$$

By symmetry, $\mathrm{CPL}(\bar{\Pi})$ is a facet of $\mathrm{CPL}(\Pi^\dagger)$ as well. Thus we are in Case (3). q.e.d.

REMARK. The process of obtaining the flop $\Pi^\dagger$ from Π in (3) was called an *elementary transformation* by Reid [15, §3], who later introduced the new terminology *flip* in algebro-geometric context. An analogous but symmetric operation called a *flop* was also introduced in birational geometry. Since our operation is symmetric, we here adopt the latter terminology in our context. The process is analogous to what Gelfand-Zelevinskij-Kapranov [9] calls the *surgery* with respect to a *circuit*, i.e., a minimal dependent subset.

As the non-projective example in Section 2 shows, we also have the following possibility: for a face $\mathrm{CPL}(\bar{\Pi})$ of $\mathrm{CPL}(\Pi)$ of codimension *greater than one*, there can exist a flop $\Pi^\dagger$ of Π along $\bar{\Pi}$, which is not quasi-projective, hence is no longer admissible for (W, Ξ).

COROLLARY 3.11. *Suppose Π is a* simplicial *convex polyhedral cone decomposition admissible for (W, Ξ). Then each facet $\tilde{F} \prec \mathrm{CPL}^\sim(\Pi)$ is of one of the following forms*:

(a) *There exists a $\xi_0 \in \Xi \setminus \Xi(\Pi)$ such that*

$$\tilde{F}=\mathrm{CPL}^{\sim}(\Pi, \Xi(\Pi)\cup\{\xi_0\}) .$$

(b) *There exists a convex polyhedral cone decomposition* $\bar{\Pi}$ *admissible for* (W, Ξ) *such that* Π *is a subdivision of* $\bar{\Pi}$, *that* $\mathrm{CPL}(\bar{\Pi})$ *is a facet of* $\mathrm{CPL}(\Pi)$ *and that* $\tilde{F}=\mathrm{CPL}^{\sim}(\bar{\Pi}, \Xi(\Pi))$. *In this case, we have the following possibilities*:

(b1) $\bar{\Pi}$ *is degenerate.*

(b2) $\bar{\Pi}$ *is simplicial with* $\Xi(\Pi)\setminus\Xi(\bar{\Pi})=\{\xi_1\}$. *In this case,* Π *is the star subdivision of* $\bar{\Pi}$ *with respect to* ξ_1.

(b3) $\bar{\Pi}$ *is nondegenerate but not simplicial with* $\Xi(\bar{\Pi})=\Xi(\Pi)$. *There exists another simplicial polyhedral cone decomposition* $\Pi^{\dagger}$ *(called the flop of* Π *along* $\bar{\Pi}$*) admissible for* (W, Ξ) *such that* $\Pi^{\dagger}$ *is a subdivision of* $\bar{\Pi}$ *with* $\Xi(\Pi^{\dagger})=\Xi(\Pi)$ *and that* $\mathrm{CPL}^{\sim}(\bar{\Pi})=\mathrm{CPL}^{\sim}(\Pi)\cap\mathrm{CPL}^{\sim}(\Pi^{\dagger})$ *is a facet of* $\mathrm{CPL}^{\sim}(\Pi^{\dagger})$ *as well.*

PROOF. By the proof of Proposition 3.1, we have an isomorphism

$$\mathrm{CPL}^{\sim}(\Pi) \xrightarrow{\sim} \mathrm{CPL}(\Pi)\times\sum_{\xi\in\Xi\setminus\Xi(\Pi)} \boldsymbol{R}_{\geq 0}e_{\xi}^{*} .$$

In view of Lemma 3.2, a facet of the cone on the right hand side is either of the form

$$\mathrm{CPL}(\Pi)\times\sum_{\xi\in\Xi\setminus\Xi(\Pi),\,\xi\neq\xi_0} \boldsymbol{R}_{\geq 0}e_{\xi}^{*} \qquad \text{for a } \xi_0\in\Xi\setminus\Xi(\Pi)\,,$$

or of the form $F\times\sum_{\xi\in\Xi\setminus\Xi(\Pi)}\boldsymbol{R}_{\geq 0}e_{\xi}^{*}$ for a facet $F\prec\mathrm{CPL}(\Pi)$. Thus we are done by Proposition 3.10. q.e.d.

We are now ready to describe when maximal dimensional $\mathrm{CPL}^{\sim}(\Pi)$'s and $\mathrm{cpl}(\Pi)$'s, with Π simplicial and admissible, intersect along facets. The proof is an easy application of Corollary 3.11 above.

THEOREM 3.12 (Reid [15] and Gelfand-Zelevinskij-Kapranov [9]). *Suppose* Π *and* Π' *are simplicial convex polyhedral cone decompositions admissible for* (W, Ξ). *Then* $\mathrm{CPL}^{\sim}(\Pi)\cap\mathrm{CPL}^{\sim}(\Pi')$ *(resp.* $\mathrm{cpl}(\Pi)\cap\mathrm{cpl}(\Pi')$*) is a* facet *of both* $\mathrm{CPL}^{\sim}(\Pi)$ *and* $\mathrm{CPL}^{\sim}(\Pi')$ *(resp. both* $\mathrm{cpl}(\Pi)$ *and* $\mathrm{cpl}(\Pi')$*) if and only if one of the following holds*:

(i) Π' *is the star subdivision of* Π *with respect to a* $\xi_0\in\Xi\setminus\Xi(\Pi)$.

(ii) Π *is the star subdivision of* Π' *with respect to a* $\xi_1\in\Xi\setminus\Xi(\Pi')$.

(iii) Π *and* Π' *are flops of each other with respect to a nondegenerate but not simplicial convex polyhedral cone decomposition* $\bar{\Pi}$.

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