## A MINIMAL FLABBY SHEAF AND AN ABELIAN GROUP

By

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In the cohomology theory of sheaves, we may use any flabby extension F of a sheaf S to define the sheaf cohomology group  $H^1(T, S)$ , where T is a topological space. Consequently, the abelian group consisting of all global sections of the quotient sheaf F/S gives us little information about the group  $H^1(T, S)$ in general.

In this paper we define a particular flabby extension  $M_s$  which is minimal in a certain sense, for a simple sheaf S. We shall show that the abelian group  $H^1(T, S)$  and the one consisting of all global sections of the quotient  $M_s/S$  have common free summands, i.e. a free abelian group F is a summand of the former iff an isomorphic one is a summand of the latter. We shall use the notations of [14] and [15] for sheaves and those of [8] for  $\Omega$ -sets to simplify the presentations. In §1 we define the flabby sheaf  $M_s$  and investigate its property as a flabby extension. In §2 we study the abelian group consisting of all global sections of a sheaf which appears in the process to define  $H^1(T, S)$  in use of  $M_s$ .

## §1. A minimal flabby sheaf.

DEFINITION 1. A complete Heyting algebra (cHa) is a complete lattice  $\Omega = (\Omega, \wedge, \vee)$  satisfying the infinite distributive law:  $p \wedge \bigvee_{i \in I} q_i = \bigvee_{i \in I} p \wedge q_i$  for all  $p \in \Omega$  and all systems  $\{q; i \in I\} \subseteq \Omega$ .

We denote the least element of  $\Omega$  by 0 and the greatest by 1.  $p \Rightarrow q = \bigvee \{x; p \land x \leq q\}$  for  $p \in \Omega$ , and  $p \land q \Rightarrow r$  and  $p \lor q \Rightarrow r$  mean  $p \land (q \Rightarrow r)$  and  $p \lor (q \Rightarrow r)$  respectively.

An element p of Q is called dense under q, if  $p \le q$  and  $q \land p \Rightarrow 0=0$ . In the case q=1, p is called dense.

 $R: \Omega \rightarrow \Omega$  is defined by:  $R(p) = (p \Rightarrow 0) \Rightarrow 0$ . An element p of is called regular if R(p) = p.

 $R(\Omega)$  is the complete Boolean algebra (cBa) which consists of all the regular elements of  $\Omega$ .

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For a topological space T, O(T) is the cHa which consists of all the open subsets of T.

The category of  $\Omega$ -sets with  $\Omega$ -set morphisms is equivalent to that of sheaves over  $\Omega$  with sheaf morphisms [8]. We remark that a Boolean extension of the set theoretical universe by a cBa B is the family of all B-sets.

A sheaf over a  $cHa \ \Omega$  is a simple generalization of a sheaf over a topological space. It is sufficient to notice that in the definition of a presheaf there appears no element of T and we only need open subsets of T.

DEFINITION 2. For a sheaf  $S=(S, \rho)$  over  $\Omega$ , we denote all the sections of S by |S|. For  $s \in |S|$ , Es is the element of  $\Omega$  such that  $s \in S(Es)$ , i.e. s is an Es-section of S.

Let s and t be elements in |S|: [s=t] is the element of  $\Omega$  such that [s=t]=  $\bigvee \{p; \rho_p^{Es}(s) = \rho_p^{Et}(t) \text{ for } p \in \Omega\}$ ; t is an extension of s if  $Es \leq Et$  and  $\rho_{Es}^{Et}(t) = s$ ; s is dense under p if s is dense under p; s is simply called dense if s is dense under 1; s is maximal under p if  $Es \leq p$  and there exists no proper extension of s under p, i.e.,  $\rho_{Es}^{Et}(t) = s$  implies  $Et \wedge p = Es$  for any  $t \in |S|$ ; s is called maximal if it is maximal under 1.

In this paper a sheaf S is always an abelian sheaf, i.e., S(p) is an abelian group for each  $p \in \Omega$ . Hence there is at least one global section for every sheaf. Consequently, if s is a maximal section of a sheaf S then s is dense.

DEFINITION 3. A simple sheaf is a sheaf S such that for each dense section s of S there is a unique maximal section t of S which is an extension of s, i.e., t is an extension of t' for any extension t' of s.

It is easy to see the following.

A constant sheaf is a simple sheaf. The sheaf of germs of continuous functions whose ranges are a Hausdorff space is a simple sheaf.

LEMMA 1. Let S be a simple sheaf. If s is maximal, then  $\rho_{E_s \wedge p}^{E_s}(s)$  is maximal under p for each  $p \in \Omega$ . If s is dense under p, then there exists a unique maximal extension s' of s under p. Cousequently, if s is maximal under p, then  $\rho_{E_s \wedge q}^{E_s}(s)$  is maximal under q for each  $q \leq p$ .

PROOF. Let  $Et \le p$  and  $Es \land p \le [s=t]$ . Then,  $\rho_{Es \land p \ge 0}^{Es}(s)$  and t are compatible and hence let t' be a common extension of them. Since Es is dense and S is simple, there exists a unique maximal extension  $\tilde{t}$  of t'. Since  $Es \land (p \lor p \Rightarrow 0)$ 

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is dense and s and  $\bar{t}$  are extensions of  $\rho_{Es \wedge (p \vee p \Rightarrow 0)}^{Es}(s)$ ,  $\bar{t}=s$ . Hence  $Et=Es \wedge p$ . Now the first assertion of the lemma has been proved.

Let 0 be the zero element of the abelian group S(1). Since s is dense under p, a common extension of s and  $\rho_{p \to 0}^1(0)$  is a dense section and hence there exists a unique maximal extension  $\bar{s}$  of them. By the first assertion of the lemma,  $\rho_{E\bar{s}\wedge p}^{E\bar{s}}(\bar{s})$  is maximal under p. Let  $\rho_{E\bar{s}}^{Et}(t)=s$  and  $Et \leq p$ . Then, the maximal section which extends t and  $\rho_{p \to 0}^1(0)$  must be  $\bar{s}$ . Hence,  $Et \leq E\bar{s} \wedge p$ . The second assertion has been proved.

Think of the cHa  $\{q; q \le p \& q \in \Omega\}$   $(=\Omega_p)$  and the restriction of S to  $\Omega_p$ . By the second assertion, the restriction of S is also a simple sheaf. Now the third assertion is followed from the first.

In the following  $s_p$  means the pair (s, p) where s is a section and  $p \in \Omega$ .

DEFINITION 4. For a simple sheaf  $S=(S, \rho, +)$ , let  $(M_S, \rho', +')$  be the following:

(1)  $M_{S}(p) = \{s_{p}; s \text{ is a section of } S \text{ which is maximal under } p\};$ 

(2)  $\rho'_{q}^{p}(s_{p}) = (\rho_{q \wedge Es}^{Es}(s))_{q};$ 

(3) For  $s_p$ ,  $t_p \in M_s(p)$ ,  $s_p + p_p t_p = u_p$ , where u is the maximal extension of  $\rho_r^{Es}(s) + \rho_r^{Et}(t)$  under  $p(r = Es \wedge Et)$ .

 $M_S$  turns out to be a flabby sheaf which extends S for a simple sheaf S. If a flabby sheaf F is an extension of S, any maximal section of S can be extended to a global section. If F is the canonical flabby extension of S, it has many global sections which extend a maximal section of S in general. Since  $M_S$  has only one global section extends a maximal section of S,  $M_S$  is made tightly. According to the terminology of [8],  $M_S$  turns out to be the direct image  $R_* \cdot R(S)$  of the inverse image R(S), where  $R: \Omega \to R(\Omega)$ . Under this point of view, the following two lemmas are rather trivial.

We denote S(1),  $\phi(1)$  and  $\phi(Es)(s)$  by  $\hat{S}$ ,  $\hat{\phi}$  and  $\phi(s)$  respectively, where  $\phi$  is a sheaf homomorphism. In the following, S always stands for a simple sheaf.

LEMMA 2.  $\rho'_p^{R(p)}: M_s(R(p)) \rightarrow M_s(p)$  is an isomorphism.

PROOF. Suppose that a section s of S is maximal under p, then it is de under R(p) and hence there is a unique extension  $\bar{s}$  of s that is maximal u R(p) by Lemma 1.

LEMMA 3.  $M_s$  is a flabby sheaf.

**PROOF.** By Lemma 1 it is a routine to show that  $M_s$  is a sheaf, so

the proof. We now prove its flabbiness. Let s be a section of S which is maximal under p and  $\bar{s}$  be a maximal section of S which extends s. Then,  $\rho'_p(\bar{s}_1) = (\rho_{p \wedge E\bar{s}}^{E\bar{s}}(\bar{s}))_p = s_p$  by Lemma 1 and hence any section of  $M_s$  can be extended to a global section.

THEOREM 1. For a simple sheaf  $S M_s$  is a flabby extension of S. More precisely, there is a monomorphism  $i_s: S \to M_s$  such that  $i_s(s) = s_{Es}$  for  $s \in |S|$ .

The proof is clear by Lemma 3 and Definition 4. Next we show that  $M_s$  is minimal in certain sense.

LEMMA 4. For  $x, y \in \hat{M}_S$ ,  $[x = y] \in R(\Omega)$ .

The proof is clear by Lemma 2.

LEMMA 5. For maximal sections s and t of S,  $[s=t]=Es \wedge Et \wedge [s_1=t_1]$ .

The proof is clear by Definition 4.

LEMMA 6. Let F be a flabby sheaf and T a sheaf. If a homomorphism  $h: \hat{F} \rightarrow \hat{T}$  satisfies  $[x=y] \leq [h(x)=h(y)]$  for  $x, y \in \hat{F}$ , then there exists a unique homomorphism  $\phi: F \rightarrow T$  such that  $\hat{\phi} = h$ .

PROOF. Let  $\phi(s) = \rho_{Es}^1(h(\bar{s}))$  for some global section  $\bar{s}$  which extends s. Then  $\phi$  is well-defined and satisfies the property.

DEFINITION 5. Let T and  $U (=U, \rho)$  be sheaves and  $\phi: T \rightarrow U$  a homomorphism. The reduction  $^{r}U (=(^{r}U, ^{r}\rho))$  of U with respect to  $\phi$  and T is the following system:

x∈<sup>r</sup>U(p) iff x∈U(p) and p≤R(∨{[[x=φ(y)]]; y∈|T|});
 rρ and r+ are the restrictions of ρ and + respectively.

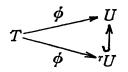
LEMMA 7.  $^{r}U$  is a subsheaf of U.

The proof is a routine.

LEMMA 8. If U is a flabby sheaf, then "U is also flabby.

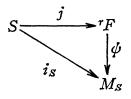
PROOF. Let s belong to  ${}^{r}U(p)$  and s' be a common extension of  ${}^{r}\rho_{p \Rightarrow 0}^{1}(0)$  and hen s' is a dense section of  ${}^{r}U$ . Hence  $R(\bigvee \{[s'=\phi(y)]; y \in |S|\})=1$ . is a global section  $\bar{s}$  of U such that  $\bar{s}$  extends s'. By the definition of s a global section of  ${}^{r}U$ .

## LEMMA 9. The following diagram commutes.



The proof is clear by the definition.

THEOREM 2. Let S be a simple sheaf and F a flabby extension of it, i.e.,  $0 \rightarrow S \xrightarrow{j} F$ . Then, there exists a unique epimorphism  $\psi$  such that the following diagram commutes and moreover  $\hat{\psi}: {}^{r}\hat{F} \rightarrow \hat{M}_{S}$  is surjective, where  ${}^{r}F$  is the reduction of F with respect to j and S.

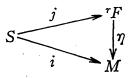


PROOF. Let t be a global section of  ${}^{r}F$  and  $p = \bigvee \{ [t=j(s)]; s \in |S| \}$ . Then p is dense. Hence, there is a dense section t' of S such that [t=j(t')] is dense. There uniquely exists a maximal section  $\tilde{t}'$  of S which extends t'.

Let t, u be global sections of  ${}^{r}F$  and  $\overline{t}'$ ,  $\overline{u}'$  the maximal sections of S defined in the above manner. By Lemma 4  $\llbracket(\overline{t}')_1 = (\overline{u}')_1 \rrbracket \in R(\Omega)$  and hence  $\llbracket t = u \rrbracket \leq \llbracket(\overline{t}')_1 = (\overline{u}')_1 \rrbracket$ . By Lemma 6 there exists a homomorphism  $\psi : {}^{r}F \to M_S$  such that  $\llbracket \psi(t) = (\overline{t}')_1 \rrbracket = 1$  for each global section t of  ${}^{r}F$ . Since  $\llbracket i_S(\overline{t}') = (\overline{t}')_1 \rrbracket$  and  $\llbracket j(t') = j(\overline{t}') \rrbracket$  are dense, the uniqueness of  $\psi$  is followed from Lemmas 4 and 6. Let sbe a maximal section of S and  $\overline{j(s)}$  a global section of F which extends j(s). Then  $\overline{j(s)}$  is a global section of  ${}^{r}F$ . Since  $Es \leq \llbracket j(s) = \overline{j(s)} \rrbracket$  and  $Es \leq \llbracket i_S(s) = s_1 \rrbracket$ and  $\llbracket \psi(\overline{j(s)}) = s_1 \rrbracket \in R(\Omega)$ ,  $\llbracket \psi(\overline{j(s)}) = s_1 \rrbracket = 1$ . Hence,  $\psi : \widehat{F} \to \widehat{M}_S$  is surjective.  $\psi$  is an epimorphism, since  $M_S$  is flabby.

Next we show that Theorem 2 characterizes  $M_s$  categorically.

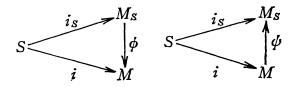
THEOREM 3. Let S be a simple sheaf. Suppose that a flabby extension M of  $S \ (0 \rightarrow S \xrightarrow{i} M)$  satisfies the following: if  $0 \rightarrow S \xrightarrow{j} F$  and F is flabby, there exists an epimorphism  $\eta: {}^{r}F \rightarrow M$  such that the following diagram commutes, where  ${}^{r}F$  is the reduction of F with respect to j and S.



Then, M and  $M_s$  are isomorphic.

PROOF. There exists an epimorphism  $\eta: {}^{r}M \to M$  such that  $Ex \leq [[i(x) = \eta \cdot i(x)]]$ for  $x \in |S|$ . Let x be a global section of M. Then there exists a global section y of  ${}^{r}M$  such that  $[[i(x) = \eta(y)]]$  is dense. For some dense section z of S, [[i(z) = y]] is dense, so x is a global section of  ${}^{r}M$ . Hence  $M = {}^{r}M$ .

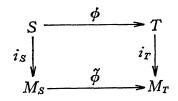
Since  $M_s = {}^r(M_s)$ , there is an epimorphism  $\phi: M_s \to M$  making the diagram (1) commutative. On the other hand there is an epimorphism  $\psi: M \to M_s$  making the diagram (2) commutative by Theorem 2.



Let s and t be maximal sections of S. Since  $Es \leq [[i(s)=\phi(s_1)]]$ ,  $Es \leq [[\psi \cdot \phi(s_1)=s_1]]$ and hence  $\mathbf{1}=[[\psi \cdot \phi(s_1)=s_1]]$ . By Lemmas 4 and 5,  $[[\phi(s_1)=\phi(t_1)]] \leq [[s_1=t_1]]$ . Therefore  $\phi$  is a monomorphism.

Next we show some functorial properties concerning  $M_s$ .

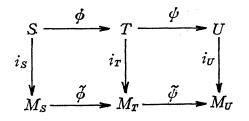
LEMMA 10. Let S and T be simple sheaves and  $\phi: S \rightarrow T$  a homomorphism. Then, there exists a unique homomorphism  $\tilde{\phi}$  such that the following diagram commutes.



PROOF. Let s and t be maximal sections of S. Then  $\phi(s)$  is a dense section of T and hence it can be uniquely extended to the maximal section  $\overline{\phi(s)}$  of T. Let p = [s=t].  $Es \leq [i_T(\phi(s))=(\overline{\phi(s)})_1]$  by Theorem 1 and hence  $p \leq [(\overline{\phi(s)})_1] = (\overline{\phi(t)})_1]$  by Lemma 5. By Lemma 4 and the fact that  $Es \wedge Et$  is dense  $[s_1=t_1]$   $= R(p) = [(\overline{\phi(s)})_1 = (\overline{\phi(t)})_1]$ . By Theorem 1 and Lemma 6, the conclusion holds

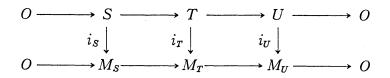
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THEOREM 4. Let S, T and U are simple sheaves. If the upper sequence is exact in the following diagram, then the lower is also exact.



PROOF. It is clear that  $\tilde{\psi} \cdot \tilde{\phi} = 0$ . Let t be a maximal section of T and  $p = \llbracket \psi(t_1) = 0 \rrbracket$ . Then,  $p \wedge Et \leq \llbracket \psi(i_T(t)) = 0 \rrbracket \leq \llbracket i_U(\psi(t)) = 0 \rrbracket \leq \llbracket \psi(t) = 0 \rrbracket$ . By the exactness of the upper sequence,  $p \wedge Et \leq \bigvee_{s \in [S]} Es \wedge \llbracket \phi(s) = t \rrbracket$ . Since Et is dense, there exist a family  $\{s_{\alpha}; \alpha \in I\}$  of sections of S and a pairwise disjoint family  $\{p_{\alpha}; \alpha \in I\}$  of  $\Omega$  such that  $Et \wedge p_{\alpha} \leq Es_{\alpha} \wedge \llbracket \phi(s_{\alpha}) = t \rrbracket$  and  $\bigvee_{\alpha \in I} p_{\alpha}$  is dense under p. Then there exists a maximal section  $s_{\infty}$  of S such that  $Et \wedge p_{\alpha} \leq \llbracket s_{\infty} = s_{\alpha} \rrbracket$  for each  $\alpha \in I$ . Hence  $Et \wedge p \leq \llbracket \phi(s_{\infty}) = t \rrbracket$ . Since  $Et \wedge p \leq \llbracket \phi(s_{\infty}) = i_T(t) \rrbracket \leq \llbracket \phi(i_S(s_{\infty})) = i_T(t) \rrbracket$ ,  $Es_{\infty} \wedge Et \wedge p \leq \llbracket \phi((s_{\infty})_1) = t_1 \rrbracket$  by Lemma 5. Therefore,  $p \leq \llbracket \phi((s_{\infty})_1) = t_1 \rrbracket$  by Lemma 4 and the fact that  $Es_{\infty} \wedge Et$  is dense. Since  $M_s$  is flabby, the above shows that  $\phi: M_s \rightarrow \text{Ker } \phi$  is an epimorphism.

COROLLARY 1. Let S, T and U be simple sheaves. If the upper sequence of the following diagram is exact, then the lower is also exact.



**PROOF.** Since the zero sheaf O is simple and  $M_0 = O$ , it is clear by Theorem 4.

In the rest of this section, we must show that  $\widehat{M_S}$  is the abelian group which consists of the global sections of some abelian group G in the Boolean extension  $V^{(B)}$ , where  $B = R(\Omega)$ . We shall directly define G and do not use the general theory of change of base for  $\Omega$ -sets. Our notations are common with [3], [4] and [5] (Ref. [13]) and consistent with the preceding ones of this paper.

For  $s \in \widehat{M_s}$ ,  $s^*$  is the element of  $V^{(B)}$  such that dom  $s^* = \{\check{t}; t \in \widehat{M_s}\}$  and  $s^*(\check{t}) = R(\bigvee \{p; \rho_p^{Es}(s) = \rho_p^{Et}(t)\})$ . G and + are elements of  $V^{(B)}$  satisfy the following:

(1) dom  $G = \{s^*; s \in \widehat{M_s}\}$ ;

(2) dom += {
$$\langle u^* \langle s^*t^* \rangle^B \rangle^B$$
; *u* is the maximal extension of  $\rho_p^{Es}(s) + \rho_p^E(t)$ ,  
where  $p = Es \wedge Et$  and  $s, t \in M_s$ }:

(3) range  $G = \operatorname{range} + = \{1\}$ .

By Lemmas 3, 4 and 5, [+] is the operation on  $G]^{(B)}=1$  is assured. Hence  $[\langle G, + \rangle ]$  is an abelian group  $]^{(B)}=1$ . By the simplicity of  $S \ \widehat{G} \simeq \widehat{M_S}$  as abelian groups.

# $\S 2$ . An abelian group consisting of all the global sections of a sheaf.

First we investigate  $\hat{Z}_T$ , where  $Z_T$  is the constant Z-sheaf over a topological space T.  $\hat{Z}_T$  is the abelian group which consists of all the continuous functions from T to Z.

PROPOSITION 1. (G.M. Bergmann [1] or [9]) If T is compact, then  $\hat{Z}_T$  is free.

**PROPOSITION 2.** If T is discrete, then Hom  $(\widehat{Z_T}, Z)$  is free.

**PROOF.** Since  $\widehat{Z_T} \simeq Z^T$ , it holds by Corollary 1 of [3].

PROPOSITION 3. Let T be a topological space such that any countable intersection of open subsets is still open. Then,  $\operatorname{Hom}(\widehat{Z_T}, Z)$  is free.<sup>(\*)</sup>

PROOF. Let CO(T) be the Boolean algebra consisting of all clopen subsets of T. Then, CO(T) is countably complete and  $\bigvee_{n \in N} {}^{CO(T)} b_n = \bigcup_{n \in N} b_n$  for  $b_n \in CO(T)$ by the condition for T. Hence,  $\widehat{Z}_T$  is isomorphic to the Boolean power  $Z^{(CO(T))}$ . Hom  $(\widehat{Z}_T, Z)$  is free, by Corollary 1 of [3] and the remark at the end of Theorem 1 of [3].

It should be noted that without any hypothesis for T,  $\bigvee_{n \in N} {}^{CO(T)}b_n$ ,  $\bigvee_{n \in N} {}^{R(O(T))}b_n$ and  $\bigcup_{n \in N} b_n (= \bigvee_{n \in N} {}^{O(T)}b_n)$  are not equal in general.

Neither  $\widehat{Z}_{T}$  nor Hom  $(Z_{T}, Z)$  has such a simple structure in general

DEFINITION 5. ([12] or [2]) An abelian group is a Z-kernel group, if it can be obtained from Z by iterating direct products and direct sums, i.e.,

(1) Z is a Z-kernel group;

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<sup>(\*)</sup> We have now the following. Let  $G^{0*}=G$  and  $G^{(n+1)*}=\text{Hom}(G^{n*}, \mathbb{Z})$ . Suppose that X is a 0-dimensional Hausdorff space.  $(C(X, \mathbb{Z}))^{2n*}$  is free iff X is pseudo-compact.  $(C(X, \mathbb{Z}))^{(2n+1)*}$  is free iff any compact subset of the N-compactification of X is finite.

- (2)  $\prod_{\alpha \in I} G_{\alpha}$  and  $\bigoplus_{\alpha \in I} G_{\alpha}$  are Z-kernel groups in the case that  $G_{\alpha}$  is a Z-kernel group for each  $\alpha \in I$ ;
- (3) No other group than defined in the above manner is a Z-kernel group.

PROPOSITION 4. For any Z-kernel group G, there exists a topological space T such that  $\widehat{Z_T} \simeq G$ .

PROOF. If G is a Z-kernel group and not isomorphic to  $\bigoplus_{F} Z$  for any finite F, G is isomorphic to  $Z \oplus G$ .

Now let T be a space with one point, then  $\widehat{Z_T} \simeq Z$ .

Suppose that  $\widehat{Z}_{T_{\alpha}} \simeq G_{\alpha}$  for each  $\alpha \in I$ . Let  $T (= \sum_{\alpha \in I} T_{\alpha})$  be the topological sum of the  $T_{\alpha}$ , then  $\widehat{Z}_{T} \simeq \prod_{\alpha \in I} G_{\alpha}$ . Next  $T (= \sum_{\alpha \in I} T_{\alpha} \cup \{\infty\})$  be the extension space of the topological sum  $\sum_{\alpha \in I} T_{\alpha}$  such that the neighborhoods of  $\infty$  are  $\sum_{\alpha \in I-F} T_{\alpha}$ for finite subsets F of I. Then,  $\widehat{Z}_{T} \simeq Z \oplus \bigoplus_{\alpha \in I} G_{\alpha}$ . We may assume that I is infinite. In the case that  $Z \oplus G_{\alpha} \simeq G_{\alpha}$  for some  $\alpha \in I$ ,  $Z_{T} \simeq \bigoplus_{\alpha \in I} G_{\alpha}$ . Otherwise, every  $G_{\alpha}$  is isomorphic to  $\bigoplus_{F} Z$  for some finite F. Hence  $\widehat{Z}_{T} \simeq \bigoplus_{\alpha \in I} G_{\alpha}$ .

Concerning Hom  $(G, \mathbb{Z})$  for  $\mathbb{Z}$ -kernel group G, we refer the reader to [2], [4], [6], [7] and [9].

Next we investigate  $\widehat{M_s}$  for a simple sheaf S. For a slender group and a Fuchs-44-group, we refer the reader to [9] and [10] respectively.

A topological space T satisfies  $\kappa$ -c.c. if there exists no pairwise disjoint family of non-empty open subsets of T with the cardinality  $\kappa$ . Therefore, Tsatisfies  $\kappa$ -c.c. iff the *cBa* R(O(T)) satisfies  $\kappa$ -c.c.. If T is a Hausdorff space without an isolated point, then R(O(T)) is atomless, i.e., for any nonzero element b there is a nonzero element which is strictly less than b.

As in [3] and [4],  $M_c$  is the least measurable cardinal. (Ref. [11] and [9])

THEOREM 5. Let T be a Hausdorff space which satisfies  $M_c$ -c.c. and has no isolated point. Let S be a simple sheaf over T. If G is a slender group, then Hom  $(\widehat{M_s}, G)=0$ . In addition,  $\widehat{M_s}$  is a Fuchs-44-group.

PROOF. By the comment preceding the theorem and Lemma 2 of [3], R(O(T)) has no c.c. max-filter. (Ref. [3] and [4]) Hence the conclusions follow Theorem 1 of [4], Corollary 3 of [5] and the fact that  $\widehat{M}_S$  is isomorphic to  $\widehat{G}$  where G is an abelian group in  $V^{(R(O(T)))}$ .

In the following we say that an abelian group is a summand of A, if it is

isomorphic to a summand of A.

COROLLARY 2. Let  $G_{\alpha}$  be a slender group for each  $\alpha \in I$ . Under the same conditions of Theorem 5, if  $\prod_{\alpha \in I} G_{\alpha}$  is a summand of  $M_s/S$ , then it is a summand of  $H^1(T, S)$ .

PROOF. Let  $0 \to \widehat{S} \to \widehat{M_S} \xrightarrow{\hat{\pi}} \widehat{M_S/S}$  be the derived exact sequence. Let  $\sigma_{G_{\alpha}} : \widehat{M_S/S} \to G_{\alpha}$  be the projection for each  $\alpha \in I$ , then  $\sigma_{G_{\alpha}} \cdot \hat{\pi} = 0$  by Theorem 5. Hence,  $\prod_{\alpha \in I} G_{\alpha}$  is a summand of  $H^1(T, S)$ .

COROLLARY 3. Under the same condition of Theorem 1, a free abelian group is a summand of  $M_s/S$  iff it is a summand of  $H^1(T, S)$ .

PROOF. If a free abelian group is a summand of a quotient group of an abelian group G, then it is a summand of G. The conclusion follows from this and Corollary 2, since a free abelian group is slender.

COROLLARY 4. Under the same condition of Theorem 5, let  $\bigoplus_{i \in I} G_i$  be a summand of  $M_S/S$  with the following:

- (1)  $G_i$  is reduced for each  $i \in I$ ;
- (2) For each m there exists a finite subset F of I such that every non-zero element of  $G_i$  has the order greater than m for each  $i \in F$ .

Then, there exists a finite subset  $F^*$  of I such that  $\bigoplus_{i\in I-F^*} G_i$  is a summand of  $H^1(T, S)$ .

PROOF. By Corollary 3 of [5],  $\widehat{M_S}$  is a Fuchs-44-group. Hence,  $\widehat{\pi}(\widehat{M_S})$  is also a Fuchs-44-group. Let  $\sigma: \widehat{M_S/S} \to \bigoplus_{i \in I} G_i$  be the projection. Then, there exist m and a finite subset F' of I such that  $m\sigma \cdot \widehat{\pi}(\widehat{M_S}) \subseteq \bigoplus_{i \in F'} G_i$ . By the condition of the theorem, there exists a finite subset  $F^*$  of I such that  $\sigma \cdot \widehat{\pi}(\widehat{M_S}) \subseteq \bigoplus_{i \in F^*} G_i$ . Hence,  $\bigoplus_{i \in I - F^*} G_i$  is a summand of  $H^1(T, S)$ .

For the next corollary we must know the structure of the quotient sheaf  $M_S/S$ . We use higher-order  $\Omega$ -sets [8]. For the intuitionistic argument, we define "torsion free" and "pure" explicitly. A group G is torsion free if nx=0 implies n=0 or x=0 for any  $n \in N$  and  $x \in G$ . A subgroup H of G is pure if  $nx \in H$  implies  $nx \in nH$ .

LEMMA 11. For an abelian sheaf A over a cHa  $\Omega$ , A(p) is torsion free for each  $p \in \Omega$ , iff [A is torsion free] = 1.

PROOF. Suppose that [A] is torsion free  $]\neq 1$ , then there exist x and  $n\neq 0$ such that  $[nx=0] \land [x \in A] \leq [x=0]$  does not hold. Let  $p=[nx=0] \land [x \in A]$ , then A(p) is not torsion free. The other implication is obvious.

LEMMA 12. Let S be a constant sheaf  $A_T$ . Then,  $[S is a pure subgroup of <math>M_S]=1$ .

PROOF. By Theorem 1,  $[S ext{ is a subgroup of } M_S]=1$ . Let  $h=[n(s)_1=i_S(t)]$ for  $n \in N$ , a maximal section s of S and an h-section t of S. Then,  $Es \wedge h \leq [ni_S(s)=i_S(t)]=[ns=t]$ . Since S is a constant sheaf, there exists an h-section s' of S such that  $h\leq [ns'=t]$ . Hence, the conclusion holds.

COROLLARY 5. In addition to the condition of Theorem 5, let S be a constant sheaf  $A_T$ , where A is a torsion-free abelian group. If  $D \bigoplus \bigoplus_{i \in I} G_i$  is a direct decomposition of  $\widehat{M_S/S}$ , where D is divisible and  $\bigoplus_{i \in I} G_i$  is reduced, then there exists a finite subset F of I such that  $\bigoplus_{i \in I-F} G_i$  is a summand of  $H^1(T, S)$ .

PROOF. By virtue of Lemma 11,  $\llbracket M_S$  is torsion free  $\rrbracket = 1$ . Since the proof of the fact that the quotient group of a torsion-free abelian group by a pure subgroup is torsion free can be done intuitionistically,  $\llbracket M_S/S$  is torsion free  $\rrbracket = 1$ by Lemma 12. Hence,  $\widehat{M_S/S}$  is torsion-free by Lemma 11. By Corollary 3 of  $\llbracket 5 \rrbracket$ ,  $\widehat{M_S}$  is a Fuchs-44-group and so  $\widehat{\pi}(\widehat{M_S})$  is also a Fuchs-44-group. Hence, there exists an integer m > 0 and a finite subset F of I such that  $m\widehat{\pi}(\widehat{M_S}) \subseteq D \bigoplus \bigoplus_{i \in F} G_i$ . Since  $\widehat{M_S/S}$  is torsion free,  $\widehat{\pi}(\widehat{M_S}) \subseteq D \bigoplus \bigoplus_{i \in F} G_i$  and hence  $\bigoplus_{i \in I-F} G_i$  is a summand of  $H^1(T, S)$ .

REMARK. Here we contrast the minimal flabby extension  $M_s$  with an injective extension  $I_s$  and the canonical flabby extension  $F_s$  of a simple sheaf S. Let  $D \oplus R$  be the direct decomposition of  $\widehat{I_s/S}$  such that D is the maximal divisible subgroup and R is reduced. Since  $\widehat{I_s}$  is divisible, R becomes a summand of  $H^1(T, S)$ . Hence Theorem 5, Corollaries 2, 3, 4 and 5 hold for an injective extension  $I_s$ , though the minimal flabby extension is seldom injective.

Suppose that T is a non-trivial connected Hausdorff space which is acyclic for a constant co-efficient sheaf and of cardinality less than  $M_c$ , e.g., the unit interval. Let  $0 \rightarrow S \rightarrow F_S \xrightarrow{\pi'} F_S / S \rightarrow 0$ .

(1) Let S be the constant sheaf  $Z_T$ . Then,  $\widehat{F_s} \simeq Z^T$  and  $\widehat{S}$  corresponds to the subgroup of  $Z^T$  consisting of constant functions. Hence,  $\widehat{\pi}'(\widehat{F_s}) \simeq Z^T$ . Since

 $H^{1}(T, \mathbb{Z}_{T}) \simeq 0$ ,  $\widetilde{F_{s}/S}$  must be isomorphic to  $\mathbb{Z}^{T}$ . Compare this fact with Corollaries 2 and 3.

(2) Let S be the constant sheaf  $A_T$  where  $A \simeq \bigoplus_{n \in N} R_n$  for some non-trivial torsion free reduced groups  $R_n$   $(n \in N)$ . Then,  $F_S/S$  is isomorphic to  $A^T$   $(\simeq A^T \bigoplus_{n \in N} R_n)$  as above.

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