Hokkaido Mathematical Journal Vol. 42 (2013) p. 121–130

On the existence of local frames of CR vector bundles

Tomonori Kajisa

(Received May 17, 2011; Revised October 31, 2011)

Abstract. Given a CR manifold D, we shall show that existence of a CR local frame of a certain CR vector bundle over D is equivalent to the local imbeddability of D. This will imply that there exists a CR vector bundle which doesn't have CR local frames. Using this bundle, we shall construct CR line bundles over 3-dimensional non-imbeddable CR manifolds which don't have CR local frames.

Key words: CR manifold, CR imbedding, CR vector bundle.

1. Introduction

In CR geometry, CR vector bundle is a basic notion. In contrast to holomorphic vector bundles over complex manifolds, CR local frames do not always exist, although it was shown by Webster [6] that CR vector bundles always admit CR local frames if the manifold is strongly pseudoconvex (spc) and of dimension ≥ 7. In this paper we shall say that a CR vector bundle is CR framable (framable for short) if it has CR local frames around any point and consider framability problem of a CR vector bundle over a 3-dim CR manifold mainly. First we discuss a relation between local imbeddability of a CR manifold and framability of a CR vector bundle. This relation was studied in [2] and [6]. We refine the result of Webster [6].

Theorem 1 *Let $(D, T^{0,1}D)$ be a $2n-1$ $(n \geq 2)$ dimensional CR manifold. Then $(D, T^{0,1}D)$ has a CR coframe locally if and only if it admits a local imbedding to $\mathbb{C}^n$.*

A CR coframe is a CR frame of a certain CR vector bundle (see Section 2). So it will imply that there exists a non-framable CR vector bundle over any non-imbeddable CR manifold. Particularly we obtain a non-framable CR vector bundle of rank 2 over every non-imbeddable 3-dim CR manifold. (There exist a lot of examples of non-imbeddable spc manifolds. See [3].) Next we ask whether there exist non-framable CR line bundles over a non-imbeddable 3-dim CR manifold. We introduce CR vector bundle structure

2010 Mathematics Subject Classification : 32V99.

space $\mathcal{H}''(F)$ over a $\mathbb{C}$-vector bundle F and discuss this problem differential geometrically. Using the non-framable CR vector bundle of rank 2 we can construct non-framable CR line bundles under some condition.

Theorem 2 *Let D be a 3-dim CR manifold, F' be a $\mathbb{C}$-line bundle of over D, F'' be a $\mathbb{C}$-vector bundle of rank 2 over D and $p \in D$. Assume the existence of a non-framable CR vector bundle structure $\omega_0 \in \mathcal{H}''(F'')$. If there are no CR local frames but there is a nowhere-vanishing CR local section around p for ω_0, then there exist line bundle structures in $\mathcal{H}''(F')$ which are non-framable around p.*

Furthermore it is shown that if there exists a non-framable CR line bundle structure, we can find a lot of non-framable structures.

Theorem 3 *Let D be a 3-dim CR manifold, $E' = \mathbb{C} \times D$, and F' be a $\mathbb{C}$-line bundle over D. If there is a non-framable CR line bundle structure over E', then there exist non-framable structures in $\mathcal{H}''(F')$ arbitrarily close to any framable structure in $\mathcal{H}''(F')$.*

2. Preliminaries

Let D be a $2n-1$ $(n \geq 2)$ dimensional C^∞ manifold and $T^{0,1}D$ be a subbundle of $\mathbb{C}TD := \mathbb{C} \otimes TD$ of rank $n-1$ such that $T^{0,1}D \cap \overline{T^{0,1}D} = \{0\}$ and $[\Gamma(T^{0,1}D), \Gamma(T^{0,1}D)] \subset \Gamma(T^{0,1}D)$, where $\Gamma(T^{0,1}D)$ denotes the set of C^∞ sections of $T^{0,1}D$ on D. The pair $(D, T^{0,1}D)$ is called a CR manifold. We set $T^{1,0}D = \overline{T^{0,1}D}$. It is possible that we define a CR manifold in another way using differential forms. Namely, let G be a subbundle of $\mathbb{C}TD^*$ of rank n and $\mathcal{I}(G)$ be the exterior ideal of complex differential forms on D generated by G. If $G + \overline{G} = \mathbb{C}TD^*$ and $d\mathcal{I}(G) \subset \mathcal{I}(G)$ are satisfied, the pair (D, G) is called a CR manifold. In these two definitions $T^{0,1}D^\perp := \{w \in \mathbb{C}TD^*; w(v) = 0, \textit{for} \text{ any } v \in T^{0,1}D\}$ coincides with G. Let $\mathcal{A}^p (0 \leq p)$ be the sheaf of C^∞ $\mathbb{C}$-valued p-forms, let $\mathcal{A}^p(F)$ be the sheaf of C^∞ F-valued p-forms for a C^∞ $\mathbb{C}$-vector bundle F and let $\Gamma(\mathcal{A}^p)$ be sections of $\mathcal{A}^p$ over D. We may define locally free subsheaves of $\mathcal{A}^0$ modules

$$\hat{\mathcal{A}}^{p,q} = \left\{\omega \in \mathcal{A}^{p+q} | v_0 \wedge v_1 \wedge \cdots \wedge v_q \lrcorner \omega = 0, \text{ for any } v_0, \ldots, v_q \in T^{0,1}D\right\}$$

for $p \geq 1$ and $q \geq 0$, and set $\hat{\mathcal{A}}^{0,q} = \mathcal{A}^q$ $(q \geq 0)$ and $\hat{\mathcal{A}}^{p,-1} = 0$, where $\lrcorner$

denotes the interior product. We now define smooth (p,q)-forms $\mathcal{A}^{p,q}$ by

$$\mathcal{A}^{p,q} := \hat{\mathcal{A}}^{p,q}/\hat{\mathcal{A}}^{p+1,q-1} \cong \mathcal{A}^0\big(\wedge^p (T^{0,1}D^\perp) \otimes \wedge^q T^{0,1}D^*\big).$$

From the integrability condition $d\mathcal{I}(T^{0,1}D^\perp) \subset \mathcal{I}(T^{0,1}D^\perp)$, we have $d(\hat{\mathcal{A}}^{p,q}) \subset \hat{\mathcal{A}}^{p,q+1}$. So we can define an operator $\overline{\partial}_b : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$ as the exterior derivative d composed with the projection $\pi : \hat{\mathcal{A}}^{p,q} \to \mathcal{A}^{p,q}$ as follows.

$$\overline{\partial}_b[v] = \pi \cdot dv \quad \text{for } [v] \in \mathcal{A}^{p,q}.$$

For each fixed p, $\{\mathcal{A}^{p,q}\}$ forms a complex. We will often call $\{\mathcal{A}^{0,q}\}$ simply the $\overline{\partial}_b$ complex. We refer the reader to [6] and [4] to follow up the fundamental materials of CR manifolds. Let F be a $\mathbb{C}$-vector bundle of rank r over $(D, T^{0,1}D)$. A CR vector bundle structure over F is defined by a linear differential operator $\overline{\partial}_F : \mathcal{A}^0(F) \to \mathcal{A}^0(T^{0,1}D^* \bigotimes F)$ such that $\overline{\partial}_F(af) = (\overline{\partial}_b a)f + a\overline{\partial}_F f$ for $a \in \mathcal{A}^0$, $f \in \mathcal{A}^0(F)$ and $\overline{\partial}_F \cdot \overline{\partial}_F = 0$ hold, where $\overline{\partial}_F$ is extended to $\overline{\partial}_F : \mathcal{A}^0(T^{0,1}D^* \bigotimes F) \to \mathcal{A}^0(\bigwedge^2 T^{0,1}D^* \bigotimes F)$ so that $\overline{\partial}_F\phi(X,Y) = \frac{1}{2}\{(\overline{\partial}_F\phi(Y))(X) - (\overline{\partial}_F\phi(X))(Y) - \phi([X,Y])\}$ holds for any $\phi \in \mathcal{A}^0(T^{0,1}D^* \bigotimes F)$ and any $X, Y \in \mathcal{A}^0(T^{0,1}D)$ and, $\overline{\partial}_F \cdot \overline{\partial}_F$ means their composition. The pair $(F, \overline{\partial}_F)$ is called a CR vector bundle. Let $e = \langle e_i \rangle$ $(1 \le i \le r)$ be a local frame on an open set $U \subset D$ and let $\overline{\partial}_F e = \omega e$, where ω is a $\mathcal{A}^{0,1}$-valued r $\times$ r matrix function. Then ω satisfies $\overline{\partial}_b\omega - \omega \wedge \omega = 0$ from the integrability condition $\overline{\partial}_F \cdot \overline{\partial}_F = 0$. Let e' be another local frame on U. Then, there is a $\mathrm{GL}(r,\mathbb{C})$ valued function a such that $e' = ae$. Then $\overline{\partial}_F e' = (\overline{\partial}_b a)e + a\overline{\partial}_F e = (\overline{\partial}_b a + a\omega)e$. If there exists a local section u of F such that $\overline{\partial}_F u = 0$, we call it a CR local section and if a set of nowhere-vanishing CR sections forms a local frame of F, we call it a CR local frame. A CR vector bundle has a CR local frame around $p \in D$ if and only if a non-linear PDE $a^{-1}\overline{\partial}_b a = -\omega$ has a local solution such that $\det a \neq 0$ around p. We say that a CR vector bundle is CR framable, or framable for short if there exist CR local frames everywhere. Examples of CR vector bundles are given in [4]. CR vector bundles $\bigwedge^p(T^{0,1}D)^\perp$ $(1 \le p \le n)$ are particularly important. Because they are determined by CR structure of the base space D. A section ϕ of $\bigwedge^p(T^{0,1}D)^\perp$ such that $d\phi \in \bigwedge^{p+1}(T^{0,1}D)^\perp$ is called a CR p-form. A frame of $T^{0,1}D^\perp$ composed of CR 1-forms is called a CR coframe. A CR n-form is also important. If a Levi non-degenerate CR manifold has a nowhere-vanishing CR n-form, it admits a pseudo-Einstein structure. (See [5]).

The following formula for a $\mathbb{C}$-line bundle over a CR manifold is easily proved.

Proposition 1 *Let $(D, T^{0,1}D)$ be a CR manifold and f, g be nowhere-vanishing functions on an open set $U \subset D$. Then*

$$(fg)^{-1}\overline{\partial}_b(fg) = f^{-1}\overline{\partial}_b f + g^{-1}\overline{\partial}_b g. \tag{1}$$

3. Local imbeddability and framability of a CR vector bundle

Let $(D, T^{0,1}D)$ be a $2n-1$ $(n \geq 2)$ dimensional CR manifold and $p \in D$. The CR imbeddimg problem can be described from the viewpoint related to local 1-parameter group of CR diffeomorphism. We shall quote several lemmas from [2]. For a real vector field X let $\mathcal{L}_X\omega$ denote the Lie derivative acting on forms and vector fields. If $Y = X_1 + iX_2$ is a complex vector field, $\mathcal{L}_Y$ means the operator $\mathcal{L}_{X_1} + i\mathcal{L}_{X_2}$. Note that the identity

$$\mathcal{L}_Y\omega = d(i_Y\omega) + i_Y(d\omega) \tag{2}$$

is valid, where ω is any differential form.

Lemma 1 *The following are equivalent*:

(1) *$(D, T^{0,1}D)$ is locally imbeddable around p.*
(2) *There exists a vector field Y around p with $\mathcal{L}_Y T^{0,1}D \subset T^{0,1}D$ and $Y_p \notin T^{0,1}_p D + T^{1,0}_p D$.*

Proof. See [2]. □

Lemma 2 *For any vector field Y the following are equivalent.*

(1) *$\mathcal{L}_Y T^{0,1}D \subset T^{0,1}D$*
(2) *$\mathcal{L}_Y \bigwedge^n (T^{0,1}D)^\perp \subset \bigwedge^n (T^{0,1}D)^\perp$.*
(3) *For every nowhere-vanishing section Ω of $\bigwedge^n (T^{0,1}D)^\perp$ there is some function λ such that $\mathcal{L}_Y\Omega = \lambda\Omega$.*
(4) *There is some nowhere-vanishing section Ω of $\bigwedge^n (T^{0,1}D)^\perp$ and some function λ such that $\mathcal{L}_Y\Omega = \lambda\Omega$.*

Proof. See [2]. □

Proof of Theorem 1. Let U be an open set in D such that the local triviality (in the sense of C^∞) $TD|_U \cong U \times \mathbb{R}^{2n-1}$ holds. Then, there is a nowhere-vanishing real vector field T on U and we have a decomposition

$$\mathbb{C}TU = T^{0,1}U + T^{1,0}U + \mathbb{C}T. \tag{3}$$

From the canonical isomorphisms $T^{0,1}U^* \cong (T^{1,0}U+\mathbb{C}T)^\perp$, $\mathbb{C}T^* \cong (T^{0,1}U+T^{1,0}U)^\perp$,

$$\mathbb{C}TU^* = T^{0,1}U^* + T^{1,0}U^* + \mathbb{C}T^* \tag{4}$$

also holds.

Let $\Gamma(T^{1,0}U) = \langle v_i \rangle_{1\le i\le n-1}$. Then $\Gamma(\mathbb{C}TU) = \langle v_i, \overline{v_i}, T\rangle_{1\le i\le n-1}$. Taking dual basis, $\Gamma(\mathbb{C}TU^*) = \langle u_i, \overline{u_i}, \eta\rangle_{1\le i\le n-1}$, where $u_i(v_i) = 1$ and $\eta(T) = 1$.

Assume the existence of a CR coframe $\langle \theta_i \rangle_{1\le i\le n}$ on U. Set $\Omega = \theta_1 \wedge \cdots \wedge \theta_n$. We want to find $Y \notin \Gamma(T^{0,1}U + T^{1,0}U)$ such that $\mathcal{L}_Y\Omega = \lambda\Omega$ for some function λ. $\mathcal{L}_Y\Omega = d(i_Y\Omega) + i_Y(d\Omega) = d(\sum_{i=1}^{i=n}(-1)^{i+1}\theta_i(Y)\theta_1 \wedge \cdots \hat{\theta}_i \wedge \cdots \wedge \theta_n)$. Set $Y = \sum_{i=1}^{i=n-1} f_i v_i + f_n T$ for some functions f_i $(1 \le i \le n)$. We determine f_i so that Y satisfies the condition above. $\langle\theta_i\rangle$ can be written as follows.

$$\big(\theta_1, \dots, \theta_n\big) = \big(u_1, \dots, u_{n-1}, \eta\big)A \tag{5}$$

for some A such that $\det A \neq 0$. From (5),

$$\big(\theta_1(Y), \dots, \theta_n(Y)\big) = \big(f_1, \dots, f_n\big)A. \tag{6}$$

For $i = 1, \dots, n$, set

$$\big(f_1, \dots, f_n\big) = \big(0, \dots, 0, \overset{i}{1}, 0, \dots, 0\big)A^{-1}. \tag{7}$$

Then, from $\mathrm{rank}_{\mathbb{C}} A = n$ we can obtain $(f_1, \dots, f_n)$ such that $f_n \neq 0$ for some i $(1 \le i \le n)$. Then $\mathcal{L}_Y\Omega = \lambda\Omega$ holds for $Y = \sum_{i=1}^{i=n-1} f_i v_i + f_n T$ and CR imbeddability is shown. The converse is trivial. Let ι be a CR imbedding map from U to $\mathbb{C}^n$ and $(z_1, \cdots, z_n)$ be a coordinate in $\mathbb{C}^n$. Then $\langle \iota^* dz_i\rangle$ $(1 \le i \le n)$ is a CR coframe on U. $\square$

Remark 1 As examples of non-imbeddable CR manifolds besides 3-dim spc manifolds, a class of CR twister manifolds is also famous. It was given by LeBrun [4].

4. Non-framable CR vector bundle structures

Let D be a 3-dim CR manifold, $E = D \times \mathbb{C}^r$ and $p \in D$. E has a trivial CR vector bundle structure $\overline{\partial}_b$, so we can write any CR vector bundle structure over E as $\overline{\partial}_E = \overline{\partial}_b + \omega$, where $\omega \in \Gamma(\mathcal{A}^{0,1}(\mathrm{End}\, E))$. We regard $\mathcal{A}^{0,1}(\mathrm{End}\, E)$ as the $\mathcal{A}^{0,1}$-valued $\mathrm{r} \times \mathrm{r}$ matrix space $\mathcal{M}(\mathrm{r}, \mathcal{A}^{0,1})$. As the integrability condition of $\overline{\partial}_E$, ω satisfies $\overline{\partial}_b \omega + \omega \wedge \omega = 0$. Since $\mathcal{A}^{0,2} = 0$, we have $\mathcal{H}''(E) = \{\overline{\partial}_b + \omega; \omega \in \Gamma(\mathcal{M}(\mathrm{r}, \mathcal{A}^{0,1})), \overline{\partial}_b \omega + \omega \wedge \omega = 0\} = \{\overline{\partial}_b + \omega; \omega \in \Gamma(\mathcal{M}(\mathrm{r}, \mathcal{A}^{0,1}))\}$, where $\mathcal{H}''(E)$ denotes the set of all CR vector bundle structures over E. We choose the natural frame $e = \langle e_i \rangle_{1 \leq i \leq r}$ of the trivial bundle E. Then for $\overline{\partial}_E = \overline{\partial}_b + \omega$, $\overline{\partial}_E e = \omega e$ holds. In this section, we shall ask whether there exist non-framable CR line bundles over a 3-dim non-imbeddable CR manifold D. In [1], Hörmander gives a necessary condition for a linear PDE $Pu = f$ to have a local solution for every $\mathbb{C}$-valued function $f \in C^\infty$. (See Theorem 6.1.1, Theorem 6.1.2 in [1].) Since the PDE for framability of a CR line bundle is $a^{-1}\overline{\partial}_b a = -\omega$ and it is reduced to a $\overline{\partial}_b$-equation $\overline{\partial}_b(\log a) = -\omega$. However it is hard to check whether Hörmander's condition holds or not. So we will try another approach using a non-framable CR vector bundle structure $\overline{\partial}_{T^{0,1}D^\perp}$ obtained in the previous section. As a result we give an answer partially. In the end we shall ask how many non-framable CR line bundle structures exist in a CR line bundle structure space and how they exist there. Through this section, note the following two facts. For any $\mathbb{C}$-vector bundle F over a 3-dim CR manifold D, there exist CR vector bundle structures (i.e. $\mathcal{H}''(F) \neq \emptyset$). This is verified in the same way as construction of connections in vector bundles. (Take a covering $\{U_\alpha\}$ of D and the associate partition of unity $\{\rho_\alpha\}$. Then set $\omega_\alpha = \sum_\beta \rho_\beta \cdot (\overline{\partial}_b A_{\alpha\beta}^{-1} \cdot A_{\alpha\beta})$, where $\{A_{\alpha\beta}\}$ is a family of transition functions.) CR vector bundle can be defined through a connection satisfying a certain integrability condition (see [6]), so checking that on a 3-dim CR manifold D any connection satisfies this integrability condition is another way. The key is that rank $T^{0,1}D^* = 1$ ($\mathcal{A}^{0,2} = 0$). The second fact is that E is framable around p for any CR vector bundle structure over E if and only if a $\mathbb{C}$-vector bundle F over D of rank r is framable around p for any CR vector bundle structure over F. This is also easily verified from $\mathcal{H}''(F) \neq \emptyset$ and $\mathcal{A}^{0,2} = 0$. By these facts we may prove Theorem 2 and Theorem 3 in the following simpler forms.

Theorem 2′ *Let D be a 3-dim CR manifold, $E' = D \times \mathbb{C}$, and $E'' = D \times \mathbb{C}^2$. Assume the existence of a non-framable CR vector bundle structure $\omega_0 \in \mathcal{H}''(E'')$. If there are no CR local frames but there is a nowhere-vanishing CR local section around p for ω_0, then there exist line bundle structures in $\mathcal{H}''(E')$ which are non-framable around p.*

Theorem 3′ *Let D be a 3-dim CR manifold and $E' = D \times \mathbb{C}$. If there is a non-framable CR line bundle structure over E', then there exist non-framable structures arbitrarily close to any framable structure in $\mathcal{H}''(E')$.*

Proposition 2 *Let $E' = D \times \mathbb{C}$ be a $\mathbb{C}$-line bundle over a $2n-1$ $(n \geq 2)$ dimensional CR manifold $(D, T^{0,1}D)$. Then all CR line bundle structures over E' are framable if and only if $\varinjlim_{p \in U} H^{0,1}(U) = 0$ for every $p \in D$, where U runs through the neighborhoods of p.*

Proof. In the $\mathbb{C}$-line bundle case, the set of all CR line bundle structures over E' is $\{\overline{\partial}_b + \omega; \omega \in \Gamma(\mathcal{A}^{0,1}), \overline{\partial}_b \omega = 0\}$. And the PDE for framability can be written $a^{-1}\overline{\partial}_b a = \overline{\partial}_b(\log a) = -\omega$. Suppose a PDE $\overline{\partial}_b f = \omega$ can't be solved for some $\omega \in \Gamma(\mathcal{A}^{0,1})$ such that $\overline{\partial}_b \omega = 0$, where f is an unknown function. Then this ω gives a non-framable CR line bundle structure. If a PDE $\overline{\partial}_b f = -\omega$ can be solved locally around every point in D for any $\omega \in \Gamma(\mathcal{A}^{0,1})$ such that $\overline{\partial}_b \omega = 0$, $\overline{\partial}_b(\log a) = -\omega$ has a nowhere-vanishing local solution $a = e^f$. Therefore ω is framable. □

Proposition 3 *Let D be a 3-dim CR manifold, $E' = D \times \mathbb{C}^r$ and $\omega \in \mathcal{H}''(E')$. Put $\omega' = \overline{\partial}_b S^{-1} S + S^{-1}\omega S$ for a $GL(r, \mathbb{C})$ valued function $S \in \Gamma(D \times GL(r, \mathbb{C}))$. Then ω is framable if and only if ω' is framable.*

Proof. The PDE for framability of ω is

$$a^{-1}\overline{\partial}_b a = -\omega. \tag{8}$$

We consider the PDE

$$a'^{-1}\overline{\partial}_b a' = -\omega' = -\big(\overline{\partial}_b S^{-1} S + S^{-1}\omega S\big). \tag{9}$$

Noting that

$$\overline{\partial}_b(S^{-1}S) = \overline{\partial}_b S^{-1}S + S^{-1}\overline{\partial}_b S = 0, \tag{10}$$

(9) can be written as

$$a'^{-1}\overline{\partial}_b a' - S^{-1}\overline{\partial}_b S = -S^{-1}\omega S. \tag{11}$$

Here, we consider a PDE

$$-\omega = (a'S^{-1})^{-1}\overline{\partial}_b(a'S^{-1}) = S\big(a'^{-1}\overline{\partial}_b a' - S^{-1}\overline{\partial}_b S\big)S^{-1}. \tag{12}$$

(12) is solvable if and only if (11) is solvable. This shows that ω is framable if and only if ω' is framable. $\square$

Proof of Theorem 2′. Let $\omega_0 = \begin{pmatrix} \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{pmatrix}$ be some non-framable CR vector bundle structure in $\mathcal{H}''(E'')$ and $S' = \begin{pmatrix} s_1' & 0 \\ 0 & s_2' \end{pmatrix}$ be a $\mathrm{GL}(2,\mathbb{C})$ valued matrix function. Let

$$\omega_0' = \overline{\partial}_b S'^{-1}S' + S'^{-1}\omega_0 S' = \begin{pmatrix} -s_1'^{-1}\overline{\partial}_b s_1' + \omega_1^1 & s_1'^{-1}s_2'\omega_1^2 \\ s_2'^{-1}s_1'\omega_2^1 & -s_2'^{-1}\overline{\partial}_b s_2' + \omega_2^2 \end{pmatrix}. \tag{13}$$

Then the PDE $-s_1'^{-1}\overline{\partial}_b s_1' + \omega_1^1 = 0$ or $-s_2'^{-1}\overline{\partial}_b s_2' + \omega_2^2 = 0$ can be solved if all CR line bundle structures are framable, where s_1' and s_2' are unknown functions. By picking up these solutions, we may assume

$$\omega_0' = \begin{pmatrix} 0 & s_1'^{-1}s_2'\omega_1^2 \\ s_2'^{-1}s_1'\omega_2^1 & 0 \end{pmatrix} \tag{14}$$

around $p \in D$.

If we assume that the second component of the local frame is a nowhere vanishing CR local section, we can set $\omega_0 = \begin{pmatrix} \omega_1^1 & \omega_1^2 \\ 0 & 0 \end{pmatrix}$. In addition, using the above argument we can reset $\omega_0' = \begin{pmatrix} 0 & \omega_1' \\ 0 & 0 \end{pmatrix}$. If we set $S = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ $(s \neq 0)$, then $\overline{\partial}_b S^{-1}S + S^{-1}\omega_0' S$

$$= \begin{pmatrix} 0 & -\overline{\partial}_b s + \omega_1' \\ 0 & 0 \end{pmatrix}. \tag{15}$$

If all CR line bundle structures are framable, we can pick up $s(\neq 0)$ such that $-\overline{\partial}_b s + \omega_1' = 0$, and it is contradictory because $\omega = 0$ is framable around p. □

Theorem 2 implies that, on a 3-dim non-imbeddable CR manifold D, if there is a nowhere-vanishing CR 1-form locally around every point in D, there exist non-framable CR line bundle structures over any $\mathbb{C}$-line bundle F'.

Corollary 1 *Let D be a 3-dim non-imbeddable CR manifold and F' be a $\mathbb{C}$-line bundle. If there is a local CR function f such that $df_p \neq 0$ at every $p \in D$, there exist non-framable CR line bundle structures in $\mathcal{H}''(F')$.*

Proof. df is a nowhere-vanishing CR 1-form. □

Remark 2 Corollary 1 also follows from Theorem 2 in [2].

Hereafter, framability of CR vector bundle structures around a framable structure $\omega_1 \in \mathcal{H}''(E)$ is discussed. We consider framability of a CR vector bundle structure $\omega_1 + \omega_\delta$, which is a perturbation of ω_1. Since ω_1 is framable, there exists a $\mathrm{GL}(r, \mathbb{C})$ valued a_1 such that

$$a_1^{-1}\overline{\partial}_b a_1 = -\omega_1. \tag{16}$$

The PDE for framability of $\omega_1 + \omega_\delta$ is

$$a^{-1}\overline{\partial}_b a = -(\omega_1 + \omega_\delta) = -\big(\overline{\partial}_b a_1^{-1} a_1 + \omega_\delta\big). \tag{17}$$

Set $\omega_\delta = a_1^{-1}\omega_\delta' a_1$. Then from Proposition 3, $\omega_1 + \omega_\delta$ is framable if and only if ω_δ' is framable. This implies that if we can construct arbitrarily small perturbations which are non-framable, we can find non-framable structures around every framable structure in $\mathcal{H}''(E)$. From here we consider the case of CR line bundles. We set $\omega_\delta' = \delta\omega_0$, $0 < \delta \leq 1$. Let $c = \inf\{\delta; \omega_\delta' \text{ is non-framable}\}$. If $c = 0$, we can obtain the small perturbations as above. We consider the case $c > 0$. In this case, ω_δ' $(0 < \delta < c)$ are framable. For δ_1 $(0 < \delta_1 < c)$, there exists L_{δ_1} such that

$$L_{\delta_1}\overline{\partial}_b L_{\delta_1}^{-1} = -\delta_1\omega_0 \tag{18}$$

Let $\delta_2 \geq c$ and $\omega_{\delta_2} = \delta_2\omega_0$ be non-framable. Then,

$$\overline{\partial}_b L_{\delta_1}^{-1} L_{\delta_1} + L_{\delta_1}^{-1}(\delta_2\omega_0)L_{\delta_1} = (\delta_2 - \delta_1)\omega_0. \tag{19}$$

Therefore, from Proposition 3 $(\delta_2 - \delta_1)\omega_0$ are non-framable and $\delta_2 - \delta_1 > 0$ can be arbitrarily small. It's contradictory to $c > 0$. The argument above proves Theorem 3′.

Acknowledgment The author is grateful to Professor T. Ohsawa for his patient direction and constant support for about three years, and to the referee for his valuable comments.

References

[1] Hörmander L., Linear partial differential operators, Springer-Verlag, Berlin Heidelberg New York, 1963.

[2] Jacobowitz H., *The canonical bundle and realizable CR hypersurfaces*. Pacific J. of Math. **127**(1) (1987).

[3] Jacobowitz H. and Treves F., *Non-realizable CR structures*. Invent. Math. **66** (1982), 231–249.

[4] LeBrun C., *Twistor CR manifolds and three-dimentional conformal geometry*. Trans. Amer. Math. Soc. **284** (1984), 601–616.

[5] Lee J. M., *Pseudo-Einstein structures on CR manifolds*. American J. Math. **110** (1988), 157–178.

[6] Webster S., *The integrability problem for CR vector bundles*. Proc Symp in Pure Math. **52** (1991), Part 3.

Tomonori Kajisa
ACRO YASHIRODAI 103
Yashirodai 3-59, Meitou-Ku, Nagoya, 465-0092, Japan
E-mail: t-kajisa@citv.jp