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Generalized wave operators for a system of semilinear wave equations in three space dimensions

(In memory of Professor Rentaro Agemi)

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Abstract. This paper is concerned with the final value problem for a system of semilinear wave equations. The main issue is to solve the problem when the nonlinearity is of a long-range type. By assuming that the solution is spherically symmetric, we shall show global solvability of the final value problem around a suitable final state, and hence, the generalized wave operator and long range-scattering operator can be constructed.

Key words: Final value problem, system of semilinear wave equations, generalized wave operator, long-range scattering operator.

0. Introduction

In this paper we consider the final value problem for the following system of semilinear wave equations:

$$\begin{cases} \partial_t^2 u - \Delta u = |\partial_t v|^p & \text{in } \ \mathbb{R}^3 \times \mathbb{R}, \\ \partial_t^2 v - \Delta v = |\partial_t u|^q & \text{in } \ \mathbb{R}^3 \times \mathbb{R}, \end{cases} \tag{0.1}$$

where $1 < p \le q$, $\Delta = \sum_{j=1}^3 \partial_j^2$, $\partial_j = \partial/\partial x_j$, and $\partial_t = \partial/\partial t$.

First of all, we shall recall known results for single wave equations with the corresponding nonlinearity:

$$\partial_t^2 u - \Delta u = |\partial_t u|^p \quad \text{in } \ \mathbb{R}^3 \times \mathbb{R}, \tag{0.2}$$

where $p > 1$. If $1 < p \le 2$, then the classical solution of the initial value problem for (0.2) generically blows up in finite time no matter how small the initial data are (see [6]). On the other hand, if $p > 2$, then there exists globally in time a mild solution of the problem for small initial data (see for instance, [5], [11]). Moreover, the final value problem for (0.2) has been

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treated in [5]. Namely, for a given final state which is a solution of the homogeneous wave equation:

$$\partial_t^2 u^+ - \Delta u^+ = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \tag{0.3}$$

one can find a unique solution to (0.2) satisfying

$$\|u(t) - u^+(t)\|_E \to 0 \quad \text{as} \quad t \to \infty,$$

provided $\|u^+(t)\|_E$ is small enough. Here $\|w(t)\|_E$ stands for the energy norm of $w(x,t)$, i.e.,

$$\|w(t)\|_E^2 = \frac{1}{2}\int_{\mathbb{R}^3} (|\partial_t w(x,t)|^2 + |\partial_x w(x,t)|^2)dx.$$

As a consequence, the wave operator W_+ is defined by

$$(u^+, \partial_t u^+)(x,0) \longmapsto (u, \partial_t u)(x,0).$$

Solving (0.2) in $(-\infty, 0] \times \mathbb{R}^3$ around a free solution, we would define W_-, similarly. Then the scattering operator $S = (W_+)^{-1}W_-$ would be constructed in a neighborhood of the origin in the energy space.

We shall now return to the semilinear system (0.1). It was shown by Deng [1] that if $q(p-1) \leq 2$, then the classical solution of the initial value problem for (0.1) blows up even for the small initial data in general. Thus we assume in what follows that

$$q(p-1) > 2. \tag{0.4}$$

Then, it was shown by Kubo, Kubota, and Sunagawa [7] that there exists globally a radially symmetric solution of the problem for small initial data, by assuming (0.4). Moreover, the global solution tends to a solution of the following system of homogeneous wave equations:

$$\partial_t^2 w_0 - \Delta w_0 = 0, \quad \partial_t^2 v_0 - \Delta v_0 = 0 \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}, \tag{0.5}$$

if $p > 2$. On the other hand, when $1 < p \leq 2$, the global solution does not approach any solution to (0.5) of finite energy, but tends to a solution of the following type of system with a suitably chosen $F(x,t)$:

$$\partial_t^2 w - \Delta w = F(x,t), \quad \partial_t^2 v - \Delta v = 0 \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}$$

in the sense of the energy (see also Kubo and Takaki [8] where the case of $p = 2$ was handled without assuming the radial symmetry). Hence, we may call the nonlinearity in (0.1) a long-range type when $1 < p \le 2$.

In order to formulate the final value problem for (0.1), we need to specify the final state to which the solution of (0.1) tends as $t \to \pm\infty$. In view of the result about the single wave equation (0.2), a natural choice of the final state may be the solution to (0.5). However, we meet difficulties to solve (0.1) around it for $1 < p \le 2$, because the nonlinearity then becomes a long-range type. Therefore, we need to choose the final state, by making use of the nonlinear structure. Moreover, when $1 < p < 2$, a function $x \mapsto |x|^p$ is of class C^1, so that the loss of derivatives would not be recovered. For this reason, we treat only radially symmetric solutions to the system (0.1) throughout this paper. If we write

$$u(x,t) = u_1(|x|,t), \quad v(x,t) = u_2(|x|,t), \tag{0.6}$$

then (0.1) becomes to

$$\begin{cases} \partial_t^2 u_1 - \left(\partial_r^2 + \frac{2}{r}\partial_r\right) u_1 = |\partial_t u_2|^p & \text{in} \quad r > 0,\ t \in \mathbb{R}, \\ \partial_t^2 u_2 - \left(\partial_r^2 + \frac{2}{r}\partial_r\right) u_2 = |\partial_t u_1|^q & \text{in} \quad r > 0,\ t \in \mathbb{R}. \end{cases} \tag{0.7}$$

Now, the final state is chosen as in the following way. When $p > 2$, we simply take the free solution (w_0, v_0) as the final state (see Theorem 2.2 below). When $1 < p \le 2$, we first assume $(p-1)(q(p-1)-1) > 1$. Then we may choose the first iterate (w_1, v_0) of (w_0, v_0) as the final state, where $w_1(r,t)$ is a solution of

$$\partial_t^2 w_1 - \left(\partial_r^2 + \frac{2}{r}\partial_r\right) w_1 = |\partial_t v_0|^p \quad \text{for} \quad r > 0,\ t \in \mathbb{R} \tag{0.8}$$

(see Theorem 2.4 below). If we drop the additional assumption, we need to iterate (w_0, v_0) several times so as to gain the integrability of the right hand member (see (2.30), (2.31) below). Then one can solve (0.7) around the ℓ-th iterate in a suitable metric space given by (4.36) below, provided

$$(p-1)^2(q-1) > 1 \tag{0.9}$$

(see Theorem 2.5 below). We remark that this is a stronger condition than (0.4) and that it is an open question whether (0.9) can be removed or not. Since the initial value problem for (0.7) can be solved in the same function space used for the final value problem (see Theorem 2.6 below), we are able to construct a long-range scattering operator for (0.7).

We observe that this kind of modification goes back to the seminal work of Ozawa [9] for the nonlinear Schrödinger equation (see also [2], [3], [4], [10], for instance). To our knowledge, this paper provides the first result on the wave equation in this direction.

This paper is organized as follows. In the next section we collect notation. In the section 2 we present our main results. The section 3 is a summary of [7, Section 4]. We refine Theorem 6 and Theorem 7 in [7] so that one can take a parameter γ to be positive. The section 4 is devoted to proving the main theorems.

1. Notation

First we introduce a class of initial data:

$$\begin{aligned} Y_\nu(\varepsilon) = \Big\{ &\vec{f} = (f,g) \in C^1(\mathbb{R}) \times C(\mathbb{R}); \vec{f}(-r) = \vec{f}(r)\ (r \in \mathbb{R}), \\ &r\vec{f}(r) \in C^2(\mathbb{R}) \times C^1(\mathbb{R}) \text{ and } \sup_{r>0} (1+r)^\nu \big|\!\big|\!\big| \vec{f}(r) \big|\!\big|\!\big| \le \varepsilon \Big\}, \end{aligned}$$

where $\nu \in \mathbb{R}$, $\varepsilon > 0$ and

$$\big|\!\big|\!\big| \vec{f}(r) \big|\!\big|\!\big| = |f(r)| + (1+r)(|f'(r)| + |g(r)|) + r(|f''(r)| + |g'(r)|).$$

Next we define several function spaces and norms. Let $s = 1$ or $s = 2$. First of all, we introduce a basic space of our argument:

$$\begin{aligned} X^s = \big\{ &u(r,t) \in C^{s-1}(\mathbb{R} \times [0,\infty)); ru(r,t) \in C^s(\mathbb{R} \times [0,\infty)), \\ &u(-r,t) = u(r,t) \text{ for } (r,t) \in \mathbb{R} \times [0,\infty) \big\}. \end{aligned}$$

For $r > 0$ and $t \ge 0$ we put

$$[u(r,t)]_2 = |u(r,t)| + (1+r) \sum_{|\alpha|=1} |\partial^\alpha u(r,t)| + r \sum_{|\alpha|=2} |\partial^\alpha u(r,t)|$$

if $u \in X^2$, and

$$[u(r,t)]_1 = |u(r,t)| + r \sum_{|\alpha|=1} |\partial^\alpha u(r,t)|$$

if $u \in X^1$, where $\partial = (\partial_r, \partial_t)$ and α is a multi-index. For $\nu \in \mathbb{R}$, we define Banach spaces:

$$X^s(\nu) = \{u(r,t) \in X^s; \|u\|_{X^s(\nu)} < \infty\},$$
$$Z^s(\nu) = \{u(r,t) \in X^s; \|u\|_{Z^s(\nu)} < \infty\},$$

where we have set

$$\|u\|_{X^s(\nu)} = \sup_{r>0,\, t\geq 0} [u(r,t)]_s \, (1+|r-t|)^\nu, \tag{1.1}$$

$$\|u\|_{Z^s(\nu)} = \sup_{r>0,\, t\geq 0} [u(r,t)]_s \, (1+r+t)^{\nu-1}(1+|r-t|). \tag{1.2}$$

Notice that $X^s(\nu) \subset Z^s(\nu)$ if $\nu \leq 1$, while $Z^s(\nu) \subset X^s(\nu)$ if $\nu \geq 1$. In the application, we shall use the wider space to which a component of the solution of (0.7) belongs. For example, when $1 < p < 2$ and $q(p-1) > 2$, we look for a solution (u_1, u_2) of (0.7) in $Z^2(p-1) \times X^2(q(p-1)-1)$.

For notational symplicity, we shall denote $\|w(|\cdot|,t)\|_E$ by $\|w(t)\|_E$ for a function $w(r,t)$.

2. Main Results

2.1. Existence of wave operators

When $p > 2$, the evolution obeying (0.7) is well characterized by the homogeneous wave equation. For this, we first recall known facts about the initial value problem for the homogeneous wave equation (see e.g. [7]):

$$u_{tt} - \left(u_{rr} + \frac{2}{r}u_r\right) = 0 \quad \text{in} \quad (0,\infty)\times(0,\infty), \tag{2.1}$$

$$(u, \partial_t u)(r,0) = \vec{f}(r) \quad \text{for} \quad r > 0. \tag{2.2}$$

The solution of this problem is expressed by

$$K[\vec{f}](r,t) = \frac{1}{2r}\left\{ \int_{r-t}^{r+t} \lambda g(\lambda)d\lambda + \frac{\partial}{\partial t}\int_{r-t}^{r+t} \lambda f(\lambda)d\lambda \right\}. \tag{2.3}$$

Moreover, we have

Proposition 2.1 *Let $\varepsilon > 0$, $\nu > 0$. If $\vec{f} \in Y_\nu(\varepsilon)$, then $K[\vec{f}] \in X^2(\nu)$ and*

$$\left\|K[\vec{f}]\right\|_{X^2(\nu)} \le C\varepsilon \tag{2.4}$$

holds, where C is a constant depending only on ν.

We set

$$\kappa_1 = p-1, \quad \kappa_2 = q-1 \tag{2.5}$$

for $p > 2$. Our main result in this subsection is as follows.

Theorem 2.2 (Existence of a wave operator) *Let $2 < p \le q$. Then there is a positive number ε_0 (depending only on p and q) such that for any $\varepsilon \in (0, \varepsilon_0]$, one can define $W_+ = (W_+^{(1)}, W_+^{(2)})$ from $Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$ to $Y_{\kappa_1}(2\varepsilon) \times Y_{\kappa_2}(2\varepsilon)$ by*

$$W_+^{(j)}[\vec{f_1}, \vec{f_2}](r) = (u_j, \partial_t u_j)(r, 0) \quad (j = 1, 2), \tag{2.6}$$

where $(u_1, u_2) \in X^2(\kappa_1) \times X^2(\kappa_2)$ is a unique solution of (0.7) satisfying

$$\left\|u_1(t) - K[\vec{f_1}](t)\right\|_E + \left\|u_2(t) - K[\vec{f_2}](t)\right\|_E \to 0 \quad as \quad t \to \infty \tag{2.7}$$

for each $(\vec{f_1}, \vec{f_2}) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$. Moreover, we have for $r > 0$

$$\left|\!\left|\!\left|W_+^{(1)}[\vec{f_1}, \vec{f_2}](r) - \vec{f_1}(r)\right|\!\right|\!\right|(1+r)^{\kappa_1} \le C\varepsilon^p, \tag{2.8}$$

$$\left|\!\left|\!\left|W_+^{(2)}[\vec{f_1}, \vec{f_2}](r) - \vec{f_2}(r)\right|\!\right|\!\right|(1+r)^{\kappa_2} \le C\varepsilon^q, \tag{2.9}$$

provided $\vec{f_j} \in Y_{\kappa_j}(\varepsilon)$ $(j = 1, 2)$ and $0 < \varepsilon \le \varepsilon_0$, where C is a constant depending only on p and q.

Our next step is to construct the inverse of W_+, based on the existence result given in Theorem 1 of [7] about the initial value problem for (0.7) with

$$(u_1, \partial_t u_1)(r,0) = \vec{\varphi}_1(r), \ (u_2, \partial_t u_2)(r,0) = \vec{\varphi}_2(r) \quad \text{for} \quad r > 0, \tag{2.10}$$

where $(\vec{\varphi}_1, \vec{\varphi}_2) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$. Let $(u_1, u_2) \in X^2(\kappa_1) \times X^2(\kappa_2)$ be the unique solution of the problem satisfying

$$\|u_1\|_{X^2(\kappa_1)} + \|u_2\|_{X^2(\kappa_2)} \leq 2C_0\varepsilon \tag{2.11}$$

with C_0 the constant in (2.4). Note that (u_1, u_2) satisfies the following system of integral equations:

$$u_1 = K[\vec{\varphi}_1] + L(|\partial_t u_2|^p), \quad u_2 = K[\vec{\varphi}_2] + L(|\partial_t u_1|^q). \tag{2.12}$$

Using the solution (u_1, u_2), we define

$$w = u_1 - R(|\partial_t u_2|^p), \quad v = u_2 - R(|\partial_t u_1|^q). \tag{2.13}$$

Here, L and R are integral operators associated with the inhomogeneous wave equation whose definition will be given in (3.3) and (3.8) below, respectively. If we set

$$\vec{f}_1(r) = (w(r,0), \partial_t w(r,0)), \quad \vec{f}_2(r) = (v(r,0), \partial_t v(r,0)) \tag{2.14}$$

for $r > 0$, then we see that

$$w = K[\vec{f}_1], \quad v = K[\vec{f}_2]. \tag{2.15}$$

Now we state the result for the inverse of W_+.

Theorem 2.3 (Existence of the inverse of a wave operator) *Let $2 < p \leq q$. Then there exists a positive number ε_0 (depending only on p and q) such that for any $\varepsilon \in (0, \varepsilon_0]$, one can define $(W_+)^{-1}$ by $(\vec{\varphi}_1, \vec{\varphi}_2) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon) \longmapsto (\vec{f}_1, \vec{f}_2) \in Y_{\kappa_1}(2\varepsilon) \times Y_{\kappa_2}(2\varepsilon)$ so that (2.7) is valid. Here (u_1, u_2) is the solution of (2.12) satisfying (2.11), and $(\vec{f}_1, \vec{f}_2)$ is defined by (2.14) for $(\vec{\varphi}_1, \vec{\varphi}_2) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$.*

Moreover, we have for $r > 0$

$$\left\| \vec{f}_1(r) - \vec{\varphi}_1(r) \right\| (1+r)^{\kappa_1} \leq C\varepsilon^p, \tag{2.16}$$

$$\left\| \vec{f}_2(r) - \vec{\varphi}_2(r) \right\| (1+r)^{\kappa_2} \leq C\varepsilon^q, \tag{2.17}$$

provided $\vec{\varphi}_j \in Y_{\kappa_j}(\varepsilon)$ $(j = 1, 2)$ *and* $0 < \varepsilon \leq \varepsilon_0$*, where* C *is a constant depending only on* p *and* q*.*

Remark Now we are in a position to conclude the existence of a scattering operator for (0.1). As in Theorem 2.2, there exists a positive number ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$ one can define $W_- = (W_-^{(1)}, W_-^{(2)}) : Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon) \longrightarrow Y_{\kappa_1}(2\varepsilon) \times Y_{\kappa_2}(2\varepsilon)$ by

$$W_-^{(j)}[\vec{f}_1, \vec{f}_2](r) = (u_j, \partial_t u_j)(r, 0) \quad (j = 1, 2),$$

where $(u_1, u_2) \in X^2(\kappa_1) \times X^2(\kappa_2)$ is a unique solution of (0.7) satisfying (2.7) with $t \to \infty$ replaced by $t \to -\infty$. Therefore, if we put $\varepsilon_2 = \min\{\varepsilon_1, \varepsilon_0/2\}$ with ε_0 being from Theorem 2.3, then we are able to define $S = (W_+)^{-1} W_- : Y_{\kappa_1}(\varepsilon_2) \times Y_{\kappa_2}(\varepsilon_2) \longrightarrow Y_{\kappa_1}(2\varepsilon_0) \times Y_{\kappa_2}(2\varepsilon_0)$, which is called a scattering operator, thanks to Theorem 2.3.

2.2. Existence of generalized wave operators

In this subsection we consider the case where $1 < p \leq 2$. We set

$$\kappa_1 = p - 1, \quad \kappa_2 = q(p-1) - 1 \tag{2.18}$$

for $1 < p < 2$. While, when $p = 2$, we take κ_1 and κ_2 in such a way that

$$0 < \kappa_1 < 1 < \kappa_2 < q - 1, \quad q\kappa_1 = \kappa_2 + 1. \tag{2.19}$$

For instance, $\kappa_1 = (q+2)/(2q)$, $\kappa_2 = q/2$ satisfy the above conditions. Note that $0 < \kappa_1 < 1$ and $\kappa_2 > 1$ in both cases, by the assumption (0.4).

First of all, we present a result for a special case of Theorem 2.5 below, because it would make easy to recognize the statement for the general case. Namely, we assume that $1 < p < 2$ and the following stronger condition on p, q than (0.9):

$$\kappa_1 \kappa_2 = (p-1)(q(p-1) - 1) > 1. \tag{2.20}$$

In order to have an analogue to Theorem 2.2, we define

$$K_1[\vec{f}_1, \vec{f}_2] = K[\vec{f}_1] + L\big(|\partial_t K[\vec{f}_2]|^p\big)$$

(for the definition of L, see (3.3) below), and replace the final state

$(K[\vec{f_1}], K[\vec{f_2}])$ by $(w_1, v_0) = (K_1[\vec{f_1}, \vec{f_2}], K[\vec{f_2}]) \in Z^2(\kappa_1) \times X^2(\kappa_2)$ which is a solution of the initial value problem for

$$\begin{cases} \partial_t^2 w_1 - \big(\partial_r^2 + \frac{2}{r}\partial_r\big) w_1 = |\partial_t v_0|^p & \text{for} \quad r > 0,\ t \in \mathbb{R}, \\ \partial_t^2 v_0 - \big(\partial_r^2 + \frac{2}{r}\partial_r\big) v_0 = 0 & \text{for} \quad r > 0,\ t \in \mathbb{R} \end{cases} \tag{2.21}$$

with

$$(w_1, \partial_t w_1)(r, 0) = \vec{f_1}(r), \quad (v_0, \partial_t v_0)(r, 0) = \vec{f_2}(r) \quad \text{for} \quad r > 0, \tag{2.22}$$

where $(\vec{f_1}, \vec{f_2}) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$. Then, we have the following.

Theorem 2.4 (Existence of a generalized wave operator; a special case) *Let $1 < p \le q$. Suppose that $1 < p < 2$ and (2.20) holds. Then there exists a positive number ε_0 (depending only on p and q) such that for any $\varepsilon \in (0, \varepsilon_0]$, one can define $\widetilde{W}_+ = (\widetilde{W}_+^{(1)}, \widetilde{W}_+^{(2)})$ from $Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$ to $Y_{\kappa_1}(2\varepsilon) \times Y_{\kappa_2}(2\varepsilon)$ by*

$$\widetilde{W}_+^{(j)}[\vec{f_1}, \vec{f_2}](r) = (u_j, \partial_t u_j)(r, 0) \quad (j = 1, 2), \tag{2.23}$$

where $(u_1, u_2) \in Z^2(\kappa_1) \times X^2(\kappa_2)$ is a unique solution of (0.7) satisfying

$$\big\| u_1(t) - K_1[\vec{f_1}, \vec{f_2}](t) \big\|_E + \big\| u_2(t) - K[\vec{f_2}](t) \big\|_E \to 0 \quad as \quad t \to \infty \tag{2.24}$$

for each $(\vec{f_1}, \vec{f_2}) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$. Moreover, we have for $r > 0$

$$\big|\!\big|\!\big| \widetilde{W}_+^{(1)}[\vec{f_1}, \vec{f_2}](r) - \vec{f_1}(r) \big|\!\big|\!\big| (1 + r)^{\kappa_1} \le C\varepsilon^{1+(p-1)q} (1 + r)^{-\kappa_1(\kappa_2 - 1)}, \tag{2.25}$$

and

$$\big|\!\big|\!\big| \widetilde{W}_+^{(2)}[\vec{f_1}, \vec{f_2}](r) - \vec{f_2}(r) \big|\!\big|\!\big| (1 + r)^{\kappa_2} \le C\varepsilon^q, \tag{2.26}$$

provided $\vec{f_j} \in Y_{\kappa_j}(\varepsilon)$ $(j = 1, 2)$ and $0 < \varepsilon \le \varepsilon_0$, where C is a constant depending only on p and q.

Remark When $p = 2$, we have only to assume $q > 2$, instead of (2.20). In fact, if we replace the right hand side of (2.25) by $C\varepsilon^{1+q}(1+r)^{-(\kappa_2 - \kappa_1)}$, then the conclusions of Theorem 2.4 remain valid (for the needed modification of the proof, see the remark after the proof of Theorem 2.5).

Next we relax the condition (2.20) to

$$\kappa_1\kappa_2 > 1 + \kappa_1^2 - \kappa_1, \tag{2.27}$$

which is equivalent to (0.9), while we shall keep $1 < p < 2$. In the previous case, it suffices to iterate just once for getting $w_1 = K_1[\vec{f_1}, \vec{f_2}]$ as a final state for u_1. However, in order to treat the general case, we need to iterate several times to obtain a suitable final state for u_1.

First we define a sequence $\{a_j\}_{j=0}^{\infty}$ by $a_0 = 1$ and

$$a_{j+1} = \kappa_1(a_j - 1) + \kappa_2 \quad \text{for } j \geq 0, \tag{2.28}$$

explicitly we have

$$a_j = \frac{\kappa_2 - \kappa_1}{1 - \kappa_1} - \frac{(\kappa_2 - 1)(\kappa_1)^j}{1 - \kappa_1} \quad \text{for } j \geq 0.$$

Observe that $\{a_j\}_{j=0}^{\infty}$ is strictly increasing, $a_j < (\kappa_2 - \kappa_1)/(1 - \kappa_1)$ for $j \geq 1$, and $\lim_{j\to\infty} a_j = (\kappa_2 - \kappa_1)/(1 - \kappa_1)$. Since $p < 2$ and (2.27) yield

$$a_0 = 1 < \frac{1}{\kappa_1} < \frac{\kappa_2 - \kappa_1}{1 - \kappa_1} = \lim_{j\to\infty} a_j,$$

there exists a nonnegative integer ℓ such that

$$a_{\ell+1} > \frac{1}{\kappa_1}, \quad a_\ell \leq \frac{1}{\kappa_1}. \tag{2.29}$$

Next we introduce a sequence $\{(w_j, v_j)\}_{j=0}^{\ell+1}$ as follows: For $(\vec{f_1}, \vec{f_2}) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$ we set

$$w_1 = w_0 + L(|\partial_t v_0|^p), \quad w_0 = K[\vec{f_1}],$$
$$v_1 = v_0 + R(|\partial_t w_1|^q), \quad v_0 = K[\vec{f_2}].$$

Moreover, we define

$$w_{j+1} = w_j + L(|\partial_t v_j|^p - |\partial_t v_{j-1}|^p)), \tag{2.30}$$
$$v_{j+1} = v_j + R(|\partial_t w_{j+1}|^q - |\partial_t w_j|^q) \tag{2.31}$$

for $1 \le j \le \ell$. Here L and R are integral operators given by (3.3) and (3.8), respectively. Since w_j is determined by $\vec{f_1}$ and $\vec{f_2}$, we shall write $w_j = K_j[\vec{f_1}, \vec{f_2}]$. Notice that for $1 \le j \le \ell+1$, we have

$$\begin{cases} \partial_t^2 w_j - \left(\partial_r^2 + \frac{2}{r}\partial_r\right) w_j = |\partial_t v_{j-1}|^p & \text{for} \quad r>0,\ t \in \mathbb{R}, \\ \partial_t^2 v_j - \left(\partial_r^2 + \frac{2}{r}\partial_r\right) v_j = |\partial_t w_j|^q & \text{for} \quad r>0,\ t \in \mathbb{R} \end{cases} \tag{2.32}$$

and $(w_j, \partial_t w_j)(r,0) = \vec{f_1}(r)$ for $r>0$.

The following theorem shows that $K_{\ell+1}[\vec{f_1}, \vec{f_2}]$ is a final state for u_1.

Theorem 2.5 (Existence of a generalized wave operator) *Let $1 < p \le q$. Suppose that $1 < p < 2$ and (2.27). Assume $\kappa_1 a_\ell < 1$ in addition to (2.29). Then there exists a positive number ε_0 (depending only on p and q) such that for $\varepsilon \in (0, \varepsilon_0]$, one can define $\widetilde{W}_+ = (\widetilde{W}_+^{(1)}, \widetilde{W}_+^{(2)})$ from $Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$ to $Y_{\kappa_1}(2\varepsilon) \times Y_{\kappa_2}(2\varepsilon)$ by (2.23), where $(u_1, u_2) \in Z^2(\kappa_1) \times X^2(\kappa_2)$ is a unique solution of (0.7) satisfying*

$$\left\|u_1(t) - K_{\ell+1}[\vec{f_1}, \vec{f_2}](t)\right\|_E + \left\|u_2(t) - K[\vec{f_2}](t)\right\|_E \to 0 \quad \text{as} \quad t \to \infty \tag{2.33}$$

for each $(\vec{f_1}, \vec{f_2}) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$. Moreover, for $r > 0$, we have (2.26) and

$$\left\|\left|\widetilde{W}_+^{(1)}[\vec{f_1}, \vec{f_2}](r) - \vec{f_1}(r)\right\|\right|(1+r)^{\kappa_1} \le C\varepsilon^{B_\ell}(1+r)^{-\kappa_1(a_{\ell+1}-1)}, \tag{2.34}$$

provided $\vec{f_j} \in Y_{\kappa_j}(\varepsilon)$ $(j = 1, 2)$ and $0 < \varepsilon \le \varepsilon_0$, where C is a constant depending only on p and q. Here we put

$$B_\ell = 1 + (p-1)(q + \ell(p+q-2)).$$

Remark If $1 < p < 2$ and (2.20) holds, then (2.27) is valid and $\kappa_1 a_\ell < 1$ is satisfied for $\ell = 0$. Therefore, Theorem 2.4 follows from Theorem 2.5.

On the other hand, suppose $\kappa_1 a_\ell = 1$ (notice that we have $\ell \ge 1$ in this case). Then we need to modify the statement of Theorem 2.5 a little. Letting δ be a number satisfying

$$0 < \delta < a_\ell - a_{\ell-1}, \quad \kappa_1^2 \delta < \kappa_1 a_{\ell+1} - 1, \tag{2.35}$$

we define

$$a'_\ell = a_\ell - \delta, \quad \text{and} \quad a'_{\ell+1} = \kappa_1(a'_\ell - 1) + \kappa_2 \,(= a_{\ell+1} - \kappa_1\delta). \tag{2.36}$$

Observing that

$$a_{\ell-1} < a'_\ell < a'_{\ell+1}, \quad \kappa_1 a'_\ell < 1, \quad \text{and} \quad \kappa_1 a'_{\ell+1} > 1, \tag{2.37}$$

we can show the statement of the theorem with $a_{\ell+1}$ in (2.34) replaced by $a'_{\ell+1}$.

Our next step is to construct the inverse of $\widetilde{W}_+$, based on the existence result in Theorem 1 of [7] for the initial value problem (0.7) and (2.10). Let $(u_1, u_2) \in Z^2(\kappa_1) \times X^2(\kappa_2)$ be the unique solution of (2.12) satisfying

$$\|u_1\|_{Z^2(\kappa_1)} + \|u_2\|_{X^2(\kappa_2)} \le 2C_0\varepsilon \tag{2.38}$$

with C_0 the constant in (2.4). Using the solution, we set $w_0^* = K[\vec{\varphi}_1]$, $v_0^* = u_2 - R(|\partial_t u_1|^q)$. Moreover, when $\ell \ge 1$, we define for $1 \le j \le \ell$

$$w_j^* = w_0^* + L\big(|\partial_t v_{j-1}^*|^p\big), \tag{2.39}$$

$$v_j^* = v_0^* + R\big(|\partial_t w_j^*|^q\big). \tag{2.40}$$

We further define

$$w^* = u_1 - R(|\partial_t u_2|^p - |\partial_t v_\ell^*|^p), \tag{2.41}$$

which we wish to regard as a final state for u_1. If we set

$$\vec{f}_1(r) = (w^*(r,0), \partial_t w^*(r,0)) \quad \text{for} \quad r > 0, \tag{2.42}$$

$$\vec{f}_2(r) = (v_0^*(r,0), \partial_t v_0^*(r,0)) \quad \text{for} \quad r > 0, \tag{2.43}$$

then we see that v_0^* and w^* are represented as

$$w^* = K[\vec{f}_1] + L(|\partial_t v_\ell^*|^p), \quad v_0^* = K[\vec{f}_2]. \tag{2.44}$$

Now we state the result for the inverse of $\widetilde{W}_+$.

Theorem 2.6 (Existence of the inverse of a generalized wave operator) *Let the assumptions of Theorem* 2.5 *be fulfilled. Then there exists a positive* *number* ε_0 *(depending only on* p *and* q*) such that, for any* $\varepsilon \in (0, \varepsilon_0]$*, one can define* $(\widetilde{W}_+)^{-1}$ *by* $(\vec{\varphi}_1, \vec{\varphi}_2) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon) \longmapsto (\vec{f}_1, \vec{f}_2) \in Y_{\kappa_1}(2\varepsilon) \times Y_{\kappa_2}(2\varepsilon)$ *so that*

$$\left\| u_1(t) - (K[\vec{f}_1] + L(|\partial_t v_\ell^*|^p))(t) \right\|_E \to 0 \quad as \ \ t \to \infty, \tag{2.45}$$

$$\left\| u_2(t) - K[\vec{f}_2](t) \right\|_E \to 0 \quad as \ \ t \to \infty \tag{2.46}$$

hold. Here (u_1, u_2) *is the solution of* (2.12) *satisfying* (2.38)*,* $(\vec{f}_1, \vec{f}_2)$ *is defined by* (2.42)*,* (2.43)*, and* v_ℓ^* *is given by* (2.40) *for* $(\vec{\varphi}_1, \vec{\varphi}_2) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$*.*

Moreover, for $r > 0$*, we have* (2.17) *and*

$$\left|\left|\left| \vec{f}_1(r) - \vec{\varphi}_1(r) \right|\right|\right| (1+r)^{\kappa_1} \le C\varepsilon^{B_\ell}(1+r)^{-\kappa_1(a_{\ell+1}-1)}, \tag{2.47}$$

provided $\vec{\varphi}_j \in Y_{\kappa_j}(\varepsilon)$ $(j = 1, 2)$ *and* $0 < \varepsilon \le \varepsilon_0$*, where* C *is a constant depending only on* p *and* q*.*

Remark In view of Theorems 2.5 and 2.6, we find that a long-range scattering operator is defined by $\widetilde{S} = (\widetilde{W}_+)^{-1}\widetilde{W}_-$, similarly to the remark given in the below of Theorem 2.3.

3. Inhomogeneous wave equations

In this section we give a refinement of Theorems 6 and 7 from [7], by assuming $a = c = 1$.

First we consider the following initial value problem:

$$u_{tt} - \left(u_{rr} + \frac{2}{r} u_r \right) = F(r, t) \quad \text{in} \ \ (0, \infty) \times (0, \infty), \tag{3.1}$$

$$u(r, 0) = (\partial_t u)(r, 0) = 0 \qquad \text{for} \ \ r > 0. \tag{3.2}$$

The solution of this problem is given by

$$L(F)(r, t) = \frac{1}{2r} \int_0^t ds \int_{r-(t-s)}^{r+(t-s)} \lambda F(\lambda, s) d\lambda. \tag{3.3}$$

In order to study the qualitative property of $L(F)$, we set

$$M_0(F) = \sup_{r>0,\, t\ge 0} |F(r,t)| r^{\alpha}(1+r)^{\beta}(1+r+t)^{\gamma}(1+|r-t|)^{\delta}, \tag{3.4}$$

$$M_1(F) = M_0(F) + \sup_{r>0,\, t\ge 0} |\partial_r F(r,t)| r^{\alpha+1}(1+r)^{\beta-1}(1+r+t)^{\gamma}(1+|r-t|)^{\delta}. \tag{3.5}$$

for α, β, γ, and $\delta \in \mathbb{R}$. Then we have

Proposition 3.1 *Let $F \in X^1$. Then we have $L(F) \in X^2$. Moreover, if $M_{s-1}(F)$ with $s = 1$ or $s = 2$ is finite for $\alpha < 3 - s$, $\beta \in \mathbf{R}$, $\gamma \ge 0$, and $\delta > 1$, then there exists a constant C depending only on α, β, γ, and δ such that*

$$\|L(F)\|_{X^s(\nu)} \le CM_{s-1}(F) \quad \text{if } \alpha+\beta+\gamma > 2, \tag{3.6}$$

$$\|L(F)\|_{Z^s(\alpha+\beta+\gamma-1)} \le CM_{s-1}(F) \quad \text{if } 1 < \alpha+\beta+\gamma < 2, \tag{3.7}$$

where $\nu = \min(\alpha+\beta+\gamma-1, \delta)$.

Proof. Note that the statement follows from the case $\gamma = 0$, since $(1+\lambda+s)^{-\gamma} \le (1+\lambda)^{-\gamma}$ when $\gamma > 0$. Therefore, applying Theorem 6 in [7] where the case $\gamma = 0$ was shown, we conclude the proof. □

Next we study an integral operator

$$R(F)(r,t) = \frac{1}{2r}\int_t^{\infty} ds \int_{(s-t)-r}^{(s-t)+r} \lambda F(\lambda, s)\, d\lambda, \tag{3.8}$$

related to the final value problem. Indeed, if $F \in X^1$ and

$$\sup_{r\ge 1,\, t\ge 0} |F(r,t)|(1+r)^{\beta}(1+r+t)^{\gamma}(1+|r-t|)^{\delta} < \infty \tag{3.9}$$

for $\beta+\gamma > 2$, $\delta \in \mathbf{R}$, then we have $R(F) \in X^1$ and it satisfies

$$(\partial_t^2 - \Delta)R(F)(|x|,t) = F(|x|,t) \tag{3.10}$$

in the distributional sense on $\mathbf{R}^3 \times (0,\infty)$. The following result will play an essential role in this paper.

Proposition 3.2 *If $F \in X^1$ and $M_{s-1}(F)$ with $s = 1$ or $s = 2$ is finite* *for $\alpha < 3-s$, β, $\gamma \in \mathbf{R}$, and $\delta > 1$ satisfying $\alpha+\beta+\gamma > 2$, then $R(F) \in X^s$ and there exists a constant C depending only on α, β, γ, and δ such that*

$$\|R(F)\|_{Z^s(\mu+\gamma)} \leq CM_{s-1}(F), \tag{3.11}$$

where $\mu = \min(\alpha+\beta-1, \delta)$.

Proof. Since the statement for $\gamma \leq 0$ was shown in Theorem 7 in [7], it suffices to prove it for $\gamma > 0$. The argument in [7] imply that $R(F) \in X^s$ is valid also for $\gamma > 0$. Hence it remains to show (3.11).

It follows from (3.8) that

$$R(F)(r,t) = \frac{1}{2r}\int_t^\infty ds \int_{|(s-t)-r|}^{(s-t)+r} \lambda F(\lambda, s) d\lambda,$$

since $\lambda F(\lambda, s)$ is odd in λ. Observe that if $\lambda \geq |(s-t)-r|$ and $s \geq t$, then we have $\lambda + s \geq r + t$, so that $(1+\lambda+s)^{-\gamma} \leq (1+r+t)^{-\gamma}$ when $\gamma > 0$. Therefore, if $\alpha+\beta > 2$, one can reduce the proof to the case $\gamma = 0$ which was already shown in [7].

Suppose, on the contrary, that $\alpha + \beta \leq 2$. We take a positive number $\rho > 0$ satisfying

$$2 - (\alpha+\beta) < \rho \leq \delta + 1 - (\alpha+\beta), \quad \rho < \gamma \tag{3.12}$$

and set $\beta' = \beta + \rho$, $\gamma' = \gamma - \rho$. Then we have $\alpha + \beta' > 2$, $\gamma' > 0$ and $\delta > 1$. Applying the result in the preceding case with β and γ replaced by β' and γ' respectively, we obtain the needed conclusion, because $\min(\alpha+\beta'-1, \delta) = \min(\alpha+\beta-1+\rho, \delta) = \alpha+\beta-1+\rho$ and $\mu = \min(\alpha+\beta-1, \delta) = \alpha+\beta-1$. This completes the proof. □

4. Proof of Main Results

4.1. Proof of Theorems 2.2 and 2.3

First we prove Theorem 2.2. Suppose $p > 2$. Let $(\vec{f_1}, \vec{f_2}) \in Y_{\kappa_1}(\varepsilon) \times Y_{\kappa_2}(\varepsilon)$ with $0 < \varepsilon \leq 1$, and set $w_0 = K[\vec{f_1}]$, $v_0 = K[\vec{f_2}]$. Then it follows from (2.4) that

$$\|w_0\|_{X^2(\kappa_1)} + \|v_0\|_{X^2(\kappa_2)} \leq C\varepsilon. \tag{4.1}$$

Recall that $\kappa_1 = p-1 > 1$ and $\kappa_2 = q-1 > 1$.

We shall solve the following system of integral equations:

$$u_1 = w_0 + R(|\partial_t u_2|^p), \quad u_2 = v_0 + R(|\partial_t u_1|^q), \tag{4.2}$$

where R is defined by (3.8). To this end, we define $T(u_1, u_2) = (T^{(1)}(u_2), T^{(2)}(u_1))$ by

$$T^{(1)}(u_2) = w_0 + R(|\partial_t u_2|^p), \quad T^{(2)}(u_1) = v_0 + R(|\partial_t u_1|^q). \tag{4.3}$$

For $\varepsilon > 0$ we introduce a metric space

$$D_\varepsilon = \{(u_1, u_2) \in X^2 \times X^2; d((u_1, u_2), (w_0, v_0)) \le \varepsilon\}, \tag{4.4}$$

where we have set

$$d((u_1, u_2), (u_1^*, u_2^*)) = \|u_1 - u_1^*\|_{Z^2(\kappa_1)} + \|u_2 - u_2^*\|_{Z^2(\kappa_2)}.$$

First we prepare the following.

Lemma 4.1 *Let $(u_1, u_2) \in D_\varepsilon$. Then we have*

$$\left\|T^{(1)}(u_2) - w_0\right\|_{Z^2(\kappa_1)} \le C\varepsilon^p, \tag{4.5}$$

$$\left\|T^{(2)}(u_1) - v_0\right\|_{Z^2(\kappa_2)} \le C\varepsilon^q. \tag{4.6}$$

Moreover we have

$$\left\|T^{(1)}(u_2) - T^{(1)}(u_2^*)\right\|_{Z^2(\kappa_1)} \le C\varepsilon^{p-1}\|u_2 - u_2^*\|_{Z^2(\kappa_2)}, \tag{4.7}$$

$$\left\|T^{(2)}(u_1) - T^{(2)}(u_1^*)\right\|_{Z^2(\kappa_2)} \le C\varepsilon^{q-1}\|u_1 - u_1^*\|_{Z^2(\kappa_1)} \tag{4.8}$$

for (u_1, u_2), $(u_1^*, u_2^*) \in D_\varepsilon$.

Proof. First we observe that if $(u_1, u_2) \in D_\varepsilon$, then we have

$$\|u_1\|_{X^2(\kappa_1)} + \|u_2\|_{X^2(\kappa_2)} \le C\varepsilon, \tag{4.9}$$

due to κ_1, $\kappa_2 > 1$ and (4.1).

We start with the proof of (4.5). In view of (4.3), it suffices to show

$$\|R(|\partial_t u_2|^p)\|_{Z^2(\kappa_1)} \le C\varepsilon^p. \tag{4.10}$$

We see from (4.9) that $M_1(|\partial_t u_2|^p) \le C\varepsilon^p$ holds for $\alpha = \gamma = 0$, $\beta = p$ and $\delta = p\kappa_2$, where $M_1(F)$ is defined by (3.4) and (3.5). Since $\alpha + \beta + \gamma - 1 = \kappa_1 > 1$, by (3.11) with $s = 2$ we get (4.10), which implies (4.5). Analogously we obtain (4.6), because $\kappa_2 > 1$.

Next we show (4.7). It follows from (4.3) that

$$T^{(1)}(u_2) - T^{(1)}(u_2^*) = R(|\partial_t u_2|^p - |\partial_t u_2^*|^p). \tag{4.11}$$

Since $p > 2$, we see from (4.9) that

$$M_1(|\partial_t u_2|^p - |\partial_t u_2^*|^p) \le C\varepsilon^{p-1}\|u_2 - u_2^*\|_{Z^2(\kappa_2)}$$

for $\alpha = 0$, $\beta = p$, $\gamma = \kappa_2 - 1$, and $\delta = 1 + (p-1)\kappa_2$. Since $\alpha + \beta + \gamma - 1 = \kappa_1 + \kappa_2 - 1 > 1$, by (3.11) with $s = 2$ we obtain

$$\left\|R(|\partial_t u_2|^p - |\partial_t u_2^*|^p)\right\|_{Z^2(\kappa_1+\kappa_2-1)} \le C\varepsilon^{p-1}\|u_2 - u_2^*\|_{Z^2(\kappa_2)}.$$

In view of (4.11), we get (4.7), since $\kappa_2 > 1$. Analogously we have (4.8), because $q \ge p > 2$. This completes the proof. $\square$

End of the proof of Theorem 2.2. We see from Lemma 4.1 that there exists a positive number ε_0 depending only on p and q such that if $0 < \varepsilon \le \varepsilon_0$, then we have $T(u_1, u_2) \in D_\varepsilon$ and

$$d(T(u_1, u_2), T(u_1^*, u_2^*)) \le 2^{-1} d((u_1, u_2), (u_1^*, u_2^*))$$

for (u_1, u_2), $(u_1^*, u_2^*) \in D_\varepsilon$, namely, T is a contraction on D_ε. Hence we find a unique solution $(u_1, u_2) \in D_\varepsilon$ of (4.2). Here and in what follows, we suppose that $0 < \varepsilon \le \varepsilon_0$ and (u_1, u_2) is the solution.

Since $T^{(1)}(u_2) = u_1$ and $w_0 = K[\vec{f_1}]$, it follows from (4.5) that

$$\left[(u_1 - K[\vec{f_1}])(r,t)\right]_2 \le C\varepsilon^p (1 + r + t)^{-(\kappa_1 - 1)}(1 + |r - t|)^{-1}. \tag{4.12}$$

Therefore we have

$$\left\|(u_1 - K[\vec{f_1}])(t)\right\|_E \le C\varepsilon^p(1+t)^{-(\kappa_1-1)}\left(\int_0^\infty (1+|r-t|)^{-2}dr\right)^{1/2}$$
$$\le C\varepsilon^p(1+t)^{-(\kappa_1-1)} \tag{4.13}$$

for $t \ge 0$. Analogously by (4.6) we get

$$\left\|(u_2 - K[\vec{f_2}])(t)\right\|_E \le C\varepsilon^q(1+t)^{-(\kappa_2-1)} \tag{4.14}$$

for $t \ge 0$. Hence we obtain (2.7).

Moreover, we easily get (2.8) by taking $t = 0$ in (4.12). Analogously, (2.9) follows from (4.6). Thus we prove Theorem 2.2. □

Next we show Theorem 2.3. Let $(u_1, u_2) \in X^2(\kappa_1) \times X^2(\kappa_2)$ be the unique solution of (2.12) satisfying (2.11). Then we see from (2.13) and (2.15) that (2.7), (2.16) and (2.17) follows from (4.10) and

$$\|R(|\partial_t u_1|^q)\|_{Z^2(\kappa_2)} \le C\varepsilon^q. \tag{4.15}$$

By virtue of (2.11), (4.10) can be shown as before. In the same way we obtain (4.15). Thus we prove Theorem 2.3. □

4.2. Proof of Theorems 2.5 and 2.6

Throughout this subsection, we assume the assumptions of Theorem 2.5 are valid. In particular, κ_1 and κ_2 are then defined by (2.18). We start by showing the following basic estimates.

Lemma 4.2 *Let $v \in X^2(\kappa_2)$. Then $L(|\partial_t v|^p) \in Z^2(\kappa_1)$ and we have*

$$\|L(|\partial_t v|^p)\|_{Z^2(\kappa_1)} \le C\|v\|^p_{X^2(\kappa_2)}. \tag{4.16}$$

While, let $w \in Z^2(\kappa_1)$. Then $R(|\partial_t w|^q) \in Z^2(\kappa_2)$ and we have

$$\|R(|\partial_t w|^q)\|_{Z^2(\kappa_2)} \le C\|w\|^q_{Z^2(\kappa_1)}. \tag{4.17}$$

Proof. For $v \in X^2(\kappa_2)$, we have

$$[v(r,t)]_2 \le \|v\|_{X^2(\kappa_2)}(1+|r-t|)^{-\kappa_2},$$

so that $M_1(|\partial_t v|^p) \le p\|v\|^p_{X^2(\kappa_2)}$ holds for $\alpha = 0$, $\beta = p$, $\gamma = 0$, and $\delta = p\kappa_2$.

By Proposition 3.1 with $s = 2$ we get $L(|\partial_t v|^p) \in Z^2(\kappa_1)$ and (4.16), since $\alpha + \beta + \gamma - 1 = \kappa_1 < 1$ and $\delta = p\kappa_2 > 1$.

On the other hand, for $w \in Z^2(\kappa_1)$, we have

$$[w(r,t)]_2 \le \|w\|_{Z^2(\kappa_1)}(1+r+t)^{1-\kappa_1}(1+|r-t|)^{-1},$$

so that $M_1(|\partial_t w|^q) \le q\|w\|^q_{Z^2(\kappa_1)}$ holds for $\alpha = 0$, $\beta = \delta = q$, and $\gamma = q\kappa_1 - q = \kappa_2 + 1 - q$. By Proposition 3.2 with $s = 2$ we get $R(|\partial_t w|^q) \in Z^2(\kappa_2)$ and (4.17), since $\alpha + \beta + \gamma - 1 = \kappa_2 > 1$. This completes the proof. $\square$

Next we examine the qualitative property of $\{w_j\}_{j=0}^{\ell+1}$ and $\{v_j\}_{j=0}^{\ell+1}$ defined by (2.30) and (2.31). As a corollary of Lemma 4.2, we derive the following estimates.

Corollary 4.3 *Let $0 \le j \le \ell + 1$, $0 < \varepsilon \le 1$ and $\vec{f_i} \in Y_{\kappa_i}(\varepsilon)$ with $i = 1, 2$. Then $w_j \in Z^2(\kappa_1)$, $v_j \in X^2(\kappa_2)$, and we have*

$$\|w_j\|_{Z^2(\kappa_1)} \le C\varepsilon, \tag{4.18}$$

$$\|v_j\|_{X^2(\kappa_2)} \le C\varepsilon. \tag{4.19}$$

Besides, we have

$$\|w_1 - w_0\|_{Z^2(\kappa_1)} \le C\varepsilon^p, \tag{4.20}$$

$$\|v_1 - v_0\|_{Z^2(\kappa_2)} \le C\varepsilon^q. \tag{4.21}$$

Proof. Since $\vec{f_2} \in Y_{\kappa_2}(\varepsilon)$, by Proposition 2.1 we get $v_0 \in X^2(\kappa_2)$ and (4.19) for $j = 0$. Analogously we have $w_0 \in X^2(\kappa_1)$ and $\|w_0\|_{X^2(\kappa_1)} \le C\varepsilon$. Since $0 < \kappa_1 < 1$, we find $w_0 \in Z^2(\kappa_1)$ and (4.18) for $j = 0$.

Next suppose that (4.18) and (4.19) hold for some j with $0 \le j \le \ell$. Since $w_{j+1} - w_0 = L(|\partial_t v_j|^p)$ by (2.30), we have $w_{j+1} - w_0 \in Z^2(\kappa_1)$ and $\|w_{j+1} - w_0\|_{Z^2(\kappa_1)} \le C\varepsilon^p$, using (4.16) and (4.19). Hence we get $\|w_{j+1}\|_{Z^2(\kappa_1)} \le C\varepsilon$ for $0 < \varepsilon \le 1$ and (4.20) by taking $j = 0$ in the above. Analogously we obtain $\|v_{j+1} - v_0\|_{Z^2(\kappa_2)} \le C\varepsilon^q$ by (2.31), (4.17) and (4.18). Therefore, $\|v_{j+1}\|_{X^2(\kappa_2)} \le C\varepsilon$ for $0 < \varepsilon \le 1$ and (4.21) holds, because $\kappa_2 > 1$. The proof is complete. $\square$

The following estimates are crucial in the proof of Theorem 2.5 for $\ell \ge 1$.

Lemma 4.4 *Let $1 \le j \le \ell$, $0 < \varepsilon \le 1$ and $\vec{f_i} \in Y_{\kappa_i}(\varepsilon)$ with $i = 1, 2$. Then $w_{j+1} - w_j \in Z^2(\kappa_1 a_j)$, $v_{j+1} - v_j \in Z^2(a_{j+1})$, and we have*

$$\|w_{j+1} - w_j\|_{Z^1(\kappa_1 a_j)} \le C\varepsilon^{b_{j-1}+p-1}, \tag{4.22}$$

$$\|v_{j+1} - v_j\|_{Z^1(a_{j+1})} \le C\varepsilon^{b_j}, \tag{4.23}$$

where we put $b_k = q + k(p + q - 2)$ for a nonnegative integer k, and

$$\|w_{j+1} - w_j\|_{Z^2(\kappa_1 a_j)} \le C\varepsilon^{B_{j-1}}, \tag{4.24}$$

$$\|v_{j+1} - v_j\|_{Z^2(a_{j+1})} \le C\varepsilon^{B_{j-1}+q-1}, \tag{4.25}$$

where we put $B_k = 1 + (p - 1)b_k$ for a nonnegative integer k.

Proof. Observe that (4.21) implies (4.23) for $j = 0$, since $b_0 = q$ and $a_1 = \kappa_2$.

First we show that if (4.23) holds for some j with $0 \le j \le \ell - 1$, then (4.22) with j replaced by $j + 1$ holds. It follows from (2.30) that

$$w_{j+2} - w_{j+1} = L(G(v_{j+1}, v_j)), \quad G(v, v^*) = |\partial_t v|^p - |\partial_t v^*|^p \tag{4.26}$$

for $0 \le j \le \ell - 1$. Note that if v, $v^* \in X^2(\kappa_2)$ and $v - v^* \in Z^1(a_{j+1})$, then we have

$$\begin{aligned}
&|G(v, v^*)(r, t)| \\
&\quad \le p|\partial_t(v - v^*)|(|\partial_t v| + |\partial_t v^*|)^{p-1} \\
&\quad \le p\|v - v^*\|_{Z^1(a_{j+1})}(\|v\|_{X^2(\kappa_2)} + \|v^*\|_{X^2(\kappa_2)})^{p-1} \\
&\qquad \times r^{-1}(1 + r)^{-(p-1)}(1 + r + t)^{-(a_{j+1}-1)}(1 + |r - t|)^{-1-\kappa_1\kappa_2}.
\end{aligned} \tag{4.27}$$

In addition, we have

$$\begin{aligned}
&(1 + r + t)^{-(a_{j+1}-1)}(1 + |r - t|)^{-1-\kappa_1\kappa_2} \\
&\qquad \le (1 + r + t)^{-\kappa_1(a_{j+1}-1)}(1 + |r - t|)^{-\kappa_1-\kappa_2},
\end{aligned} \tag{4.28}$$

since $0 < \kappa_1 < 1$ and $a_{j+1} \ge \kappa_2$.

Applying (4.27) to $G(v_{j+1}, v_j)$ and using (4.19), (4.23) and (4.28), we obtain

$$M_0(G(v_{j+1}, v_j)) \leq C\varepsilon^{b_j+p-1}$$

for $\alpha = 1$, $\beta = \kappa_1$, $\gamma = \kappa_1 a_{j+1} - \kappa_1\,(> 0)$, and $\delta = \kappa_1 + \kappa_2$. Since $\alpha + \beta + \gamma - 1 = \kappa_1 a_{j+1} \leq \kappa_1 a_\ell$ and $\kappa_1 a_\ell < 1$ from the assumption in Theorem 2.5, if we apply (3.7) with $s = 1$ to the right hand side on (4.26), then the desired estimate holds. In particular, we have (4.22) with $j = 1$, since (4.23) is valid for $j = 0$.

Next we show that if (4.22) holds for some j with $1 \leq j \leq \ell$, then (4.23) is valid for the same j. It follows from (2.31) that

$$v_{j+1} - v_j = R(H(w_{j+1}, w_j)), \quad H(w, w^*) = |\partial_t w|^q - |\partial_t w^*|^q \tag{4.29}$$

for $1 \leq j \leq \ell$. Note that if w, $w^* \in Z^2(\kappa_1)$ and $w - w^* \in Z^1(\kappa_1 a_j)$, then we have

$$\begin{aligned} &|H(w, w^*)(r, t)| \\ &\quad \leq q|\partial_t(w - w^*)|(|\partial_t w| + |\partial_t w^*|)^{q-1} \\ &\quad \leq q\|w - w^*\|_{Z^1(\kappa_1 a_j)}(\|w\|_{Z^2(\kappa_1)} + \|w^*\|_{Z^2(\kappa_1)})^{q-1} \\ &\qquad \times r^{-1}(1 + r)^{-(q-1)}(1 + r + t)^{-(q-1)(\kappa_1-1)-(\kappa_1 a_j-1)}(1 + |r - t|)^{-q}. \end{aligned} \tag{4.30}$$

Applying (4.30) to $H(w_{j+1}, w_j)$ and using (4.18), (4.22), we obtain

$$M_0(H(w_{j+1}, w_j)) \leq C\varepsilon^{b_j}$$

for $\alpha = 1$, $\beta = q - 1$, $\gamma = a_{j+1} + 1 - q$, and $\delta = q$. Since $\alpha + \beta + \gamma - 1 = a_{j+1} > 1$, if we apply (3.11) with $s = 1$ to the right hand side on (4.29), then the desired estimate holds. In conclusion, we have proven (4.22) and (4.23) for $1 \leq j \leq \ell$.

Next we show (4.24) and (4.25). Observe that if we put $B_{-1} = 1$, then (4.25) with $j = 0$ follows from (4.21).

First we show that if (4.25) holds for some j with $0 \leq j \leq \ell - 1$, then it, in combination with (4.23), implies (4.24) with j replaced by $j + 1$. Note that if v, $v^* \in X^2(\kappa_2)$ and $v - v^* \in Z^2(a_{j+1})$, then we have

$$\begin{aligned}
&\{(1+r)|G(v,v^*)(r,t)| + r|\partial_r G(v,v^*)(r,t)|\}\\
&\qquad \times r^{p-1}(1+r+t)^{\kappa_1(a_{j+1}-1)}(1+|r-t|)^{\kappa_1+\kappa_2}\\
&\quad \le 2p\big\{\|v-v^*\|^{p-1}_{Z^1(a_{j+1})}\|v\|_{X^2(\kappa_2)}\\
&\qquad\qquad + \|v-v^*\|_{Z^2(a_{j+1})}(\|v\|_{X^2(\kappa_2)} + \|v^*\|_{X^2(\kappa_2)})^{p-1}\big\}. \qquad (4.31)
\end{aligned}$$

In fact, similarly to (4.27), we have

$$\begin{aligned}
|G(v,v^*)(r,t)| &\le p\|v-v^*\|_{Z^2(a_{j+1})}\big(\|v\|_{X^2(\kappa_2)} + \|v^*\|_{X^2(\kappa_2)}\big)^{p-1}\\
&\qquad \times (1+r)^{-p}(1+r+t)^{-(a_{j+1}-1)}(1+|r-t|)^{-1-\kappa_1\kappa_2}.
\end{aligned}$$

Since $1<p<2$, we obtain

$$\begin{aligned}
&|\partial_r G(v,v^*)(r,t)|\\
&\quad \le 2p|\partial_t(v-v^*)|^{p-1}|\partial_r\partial_t v| + p|\partial_r\partial_t(v-v^*)||\partial_t v^*|^{p-1}\\
&\quad \le 2p\|v-v^*\|^{p-1}_{Z^1(a_{j+1})}\|v\|_{X^2(\kappa_2)}\\
&\qquad\quad \times r^{-p}(1+r+t)^{-\kappa_1(a_{j+1}-1)}(1+|r-t|)^{-\kappa_1-\kappa_2}\\
&\qquad + p\|v-v^*\|_{Z^2(a_{j+1})}\|v^*\|^{p-1}_{X^2(\kappa_2)}\\
&\qquad\quad \times r^{-1}(1+r)^{-(p-1)}(1+r+t)^{-(a_{j+1}-1)}(1+|r-t|)^{-1-\kappa_1\kappa_2}.
\end{aligned}$$

By (4.28) we get (4.31).

Applying (4.31) to $G(v_{j+1},v_j)$ and using (4.19), (4.23) and (4.25), we obtain

$$M_1(G(v_{j+1},v_j)) \le C\varepsilon^{(p-1)b_j+1} + C\varepsilon^{B_{j-1}+p+q-2}$$

for $\alpha = p-1\,(<1)$, $\beta = 1$, $\gamma = \kappa_1 a_{j+1} - \kappa_1$, and $\delta = \kappa_1+\kappa_2$. It is not difficult to see that

$$B_j \le B_{j-1} + p + q - 2 \qquad (4.32)$$

for $0 \le j \le \ell$ (recall $B_{-1}=1$). Therefore we have $M_1(G(v_{j+1},v_j)) \le C\varepsilon^{B_j}$. Since $\alpha+\beta+\gamma-1 = \kappa_1 a_{j+1} \le \kappa_1 a_\ell < 1$, if we apply (3.7) with $s=2$ to the right hand side on (4.26), then the desired estimate holds. In particular, we have (4.24) with $j=1$, since (4.25) is valid for $j=0$.

Finally we show that if (4.24) holds for some j with $1 \le j \le \ell$, then (4.25) is valid for the same j. Note that if $w,\ w^* \in Z^2(\kappa_1)$ and $w - w^* \in Z^2(\kappa_1 a_j)$, then we have

$$\begin{aligned}\{(1+r)&|H(w,w^*)(r,t)| + r|\partial_r H(w,w^*)(r,t)|\} \\ &\times (1+r)^{q-1}(1+r+t)^{(q-1)(\kappa_1-1)+\kappa_1 a_j - 1}(1+|r-t|)^q \\ &\le q^2 \|w-w^*\|_{Z^2(\kappa_1 a_j)}(\|w\|_{Z^2(\kappa_1)} + \|w^*\|_{Z^2(\kappa_1)})^{q-1}, \end{aligned} \tag{4.33}$$

since $q>2$. Applying (4.33) to $H(w_{j+1}, w_j)$ and using (4.18), (4.24), we obtain

$$M_1(H(w_{j+1}, w_j)) \le C\varepsilon^{B_{j-1}+q-1}$$

for $\alpha=0$, $\beta=\delta=q$ and $\gamma = a_{j+1}+1-q$. Since $\alpha+\beta+\gamma-1 = a_{j+1} > 1$, if we apply (3.11) with $s=2$ to the right hand side on (4.29), then the desired estimate holds. In conclusion, all the asserion of the lemma is proven. □

Our next step is to solve the following system:

$$u_1 = w_{\ell+1} + R(G(u_2, v_\ell)), \quad u_2 = v_{\ell+1} + R(H(u_1, w_{\ell+1})), \tag{4.34}$$

where $G(v,v^*)$ and $H(w,w^*)$ are the notations from (4.26) and (4.29), respectively. We define $T(u_1,u_2) = (T^{(1)}(u_2), T^{(2)}(u_1))$ by

$$\begin{aligned} T^{(1)}(u_2) &= w_{\ell+1} + R(G(u_2, v_\ell)), \\ T^{(2)}(u_1) &= v_{\ell+1} + R(H(u_1, w_{\ell+1})). \end{aligned} \tag{4.35}$$

For $\varepsilon > 0$ we introduce a metric space

$$D_\varepsilon = \left\{(u_1,u_2) \in X^2 \times X^2\,;\, d((u_1,u_2),(w_{\ell+1}, v_{\ell+1})) \le \varepsilon^{(p-1)b_\ell}\right\}, \tag{4.36}$$

where $b_\ell = q + \ell(p+q-2)$ and we have set $d((u_1,u_2),(u_1^*,u_2^*)) = d_1(u_1,u_1^*) + d_2(u_2,u_2^*)$ with

$$\begin{aligned} d_1(u_1,u_1^*) &= \|u_1 - u_1^*\|_{Z^2(\kappa_1 a_{\ell+1})} + \|u_1-u_1^*\|^{p-1}_{Z^1(\kappa_1 a_{\ell+1})}, \\ d_2(u_2,u_2^*) &= \|u_2 - u_2^*\|_{Z^2(a_{\ell+1})} + \|u_2-u_2^*\|^{p-1}_{Z^1(a_{\ell+1})}. \end{aligned}$$

We shall show that T is a contraction on D_ε, provided ε is small enough.

First of all, we prepare the following.

Lemma 4.5 *Let $(u_1, u_2) \in D_\varepsilon$ with $0 < \varepsilon \le 1$. Then we have*

$$\|u_1\|_{Z^2(\kappa_1)} + \|u_2\|_{X^2(\kappa_2)} \le C\varepsilon \tag{4.37}$$

and

$$d_1(u_1, w_{\ell+1}) \le \varepsilon^{(p-1)b_\ell}, \quad d_2(u_2, v_\ell) \le C\varepsilon^{(p-1)b_\ell}. \tag{4.38}$$

Proof. First we prove (4.37). Notice that $a_{\ell+1} \ge \kappa_2 > 1$ and $(p-1)b_\ell \ge (p-1)q = \kappa_2 + 1$. Then (4.37) follows from (4.18) and (4.19) with $j = \ell + 1$.

Next we prove (4.38). The first inequality is apparent. On the other hand, in order to get the second one, it suffices to show $d_2(v_{\ell+1}, v_\ell) \le C\varepsilon^{(p-1)b_\ell}$. When $\ell = 0$, it follows from (4.21) that

$$d_2(v_1, v_0) \le C(\varepsilon^q + \varepsilon^{(p-1)q}) \le C\varepsilon^{(p-1)q}$$

for $0 < \varepsilon \le 1$. While, when $\ell \ge 1$, it follows from (4.23) and (4.25) with $j = \ell$ that

$$d_2(v_{\ell+1}, v_\ell) \le C(\varepsilon^{B_{\ell-1}+q-1} + \varepsilon^{(p-1)b_\ell}).$$

Since

$$B_{\ell-1} + q - 1 = (p-1)b_\ell + (2-p)(p+q-1) > (p-1)b_\ell$$

for $\ell \ge 1$, we obtain the needed estimate. This completes the proof. $\square$

The following estimate will play a basic role in proving that T is a contraction on D_ε.

Lemma 4.6 *Let u_2, $u_2^* \in X^2(\kappa_2)$ satisfy $u_2 - u_2^* \in Z^2(a_{\ell+1})$ and*

$$\|u_2\|_{X^2(\kappa_2)} + \|u_2^*\|_{X^2(\kappa_2)} \le C\varepsilon. \tag{4.39}$$

Then we have

$$\|R(G(u_2, u_2^*))\|_{Z^1(\kappa_1 a_{\ell+1})} \le C\varepsilon^{p-1}\|u_2 - u_2^*\|_{Z^1(a_{\ell+1})} \tag{4.40}$$

$$\|R(G(u_2, u_2^*))\|_{Z^2(\kappa_1 a_{\ell+1})} \le C\varepsilon^{p-1} d_2(u_2, u_2^*). \tag{4.41}$$

While, let u_1, $u_1^* \in Z^2(\kappa_1)$ *satisfy* $u_1 - u_1^* \in Z^2(\kappa_1 a_{\ell+1})$ *and*

$$\|u_1\|_{Z^2(\kappa_1)} + \|u_1^*\|_{Z^2(\kappa_1)} \leq C\varepsilon. \tag{4.42}$$

Then we have

$$\|R(H(u_1, u_1^*))\|_{Z^1(a_{\ell+2})} \leq C\varepsilon^{q-1}\|u_1 - u_1^*\|_{Z^1(\kappa_1 a_{\ell+1})}, \tag{4.43}$$

$$\|R(H(u_1, u_1^*))\|_{Z^2(a_{\ell+2})} \leq C\varepsilon^{q-1}\|u_1 - u_1^*\|_{Z^2(\kappa_1 a_{\ell+1})}. \tag{4.44}$$

Proof. First we prove (4.40). It follows from (4.27) with $j = \ell$, (4.28) and (4.39) that

$$M_0(G(u_2, u_2^*)) \leq C\varepsilon^{p-1}\|u_2 - u_2^*\|_{Z^1(a_{\ell+1})}$$

for $\alpha = 1$, $\beta = \kappa_1$, $\gamma = \kappa_1 a_{\ell+1} - \kappa_1$, and $\delta = \kappa_1 + \kappa_2$. Applying (3.11) with $s = 1$ to $G(u_2, u_2^*)$, we get (4.40), because

$$\alpha + \beta + \gamma - 1 = \kappa_1 a_{\ell+1} > 1, \tag{4.45}$$

by virtue of (2.29).

Next we prove (4.41). It follows from (4.31) with $j = \ell$, (4.39) that

$$M_1(G(u_2, u_2^*)) \leq C\varepsilon^{p-1} d_2(u_2, u_2^*)$$

for $\alpha = p - 1$, $\beta = 1$, $\gamma = \kappa_1 a_{\ell+1} - \kappa_1$, and $\delta = \kappa_1 + \kappa_2$. Since (4.45) holds for these α, β and γ, we obtain (4.41) by (3.11) with $s = 2$.

Next we prove (4.43). It follows from (4.30) with $j = \ell + 1$ and (4.42) that

$$M_0(H(u_1, u_1^*)) \leq C\varepsilon^{q-1}\|u_1 - u_1^*\|_{Z^1(\kappa_1 a_{\ell+1})}$$

for $\alpha = 1$, $\beta = q - 1$, $\gamma = a_{\ell+2} + 1 - q$, and $\delta = q$. Since $\alpha + \beta + \gamma - 1 = a_{\ell+2} > 1$, by (3.11) with $s = 1$, we get (4.43).

Finally we prove (4.44). It follows from (4.33) with $j = \ell + 1$ and (4.42) that

$$M_1(H(u_1, u_1^*)) \leq C\varepsilon^{q-1}\|u_1 - u_1^*\|_{Z^2(\kappa_1 a_{\ell+1})}$$

for $\alpha = 0$, $\beta = \delta = q$ and $\gamma = a_{\ell+2} + 1 - q$. Since $\alpha + \beta + \gamma - 1 = a_{\ell+2} > 1$, (3.11) with $s = 2$ yields (4.44). The proof is complete. $\square$

Corollary 4.7 *Let* $(u_1, u_2) \in D_\varepsilon$ *with* $0 < \varepsilon \le 1$. *Then we have*

$$d_1(T^{(1)}(u_2), w_{\ell+1}) \le C\varepsilon^{(p-1)b_\ell + (p-1)^2}, \tag{4.46}$$

$$d_2(T^{(2)}(u_1), v_{\ell+1}) \le C\varepsilon^{(p-1)b_\ell + (p-1)(q-1)}. \tag{4.47}$$

Moreover, we have

$$d_1(T^{(1)}(u_2), T^{(1)}(u_2^*)) \le C\varepsilon^{(p-1)^2} d_2(u_2, u_2^*), \tag{4.48}$$

$$d_2(T^{(2)}(u_1), T^{(2)}(u_1^*)) \le C\varepsilon^{(p-1)(q-1)} d_1(u_1, u_1^*) \tag{4.49}$$

for (u_1, u_2), $(u_1^*, u_2^*) \in D_\varepsilon$ *with* $0 < \varepsilon \le 1$.

Proof. It follows from Lemma 4.5 that if $(u_1, u_2) \in D_\varepsilon$ and $0 < \varepsilon \le 1$, then we have (4.37) and

$$\|u_1 - w_{\ell+1}\|_{Z^2(\kappa_1 a_{\ell+1})} + \|u_2 - v_\ell\|_{Z^2(a_{\ell+1})} \le C\varepsilon^{(p-1)b_\ell}, \tag{4.50}$$

$$\|u_1 - w_{\ell+1}\|_{Z^1(\kappa_1 a_{\ell+1})} + \|u_2 - v_\ell\|_{Z^1(a_{\ell+1})} \le C\varepsilon^{b_\ell}. \tag{4.51}$$

We start with the proof of (4.46). By (4.35) we have $T^{(1)}(u_2) - w_{\ell+1} = R(G(u_2, v_\ell))$. Therefore, applying the preceding lemma, we get (4.46). Similarly, since $T^{(2)}(u_1) - v_{\ell+1} = R(H(u_1, w_{\ell+1}))$, we obtain (4.47).

Next we prove (4.48). By (4.35) we have $T^{(1)}(u_2) - T^{(1)}(u_2^*) = R(G(u_2, u_2^*))$. Since $d_2(u_2, u_2^*) \le 2\varepsilon^{(p-1)b_\ell}$ for (u_1, u_2), $(u_1^*, u_2^*) \in D_\varepsilon$, the preceding lemma shows (4.48). Similarly, we obtain (4.49). This completes the proof. $\square$

End of the proof of Theorem 2.5. We see from Corollary 4.7 that there exists a positive number ε_0 depending only on p and q such that if $0 < \varepsilon \le \varepsilon_0$, then we have $T(u_1, u_2) \in D_\varepsilon$ and

$$d(T(u_1, u_2), T(u_1^*, u_2^*)) \le 2^{-1} d((u_1, u_2), (u_1^*, u_2^*))$$

for (u_1, u_2), $(u_1^*, u_2^*) \in D_\varepsilon$, namely, T is a contraction on D_ε. Hence we find a unique solution $(u_1, u_2) \in D_\varepsilon$ of (4.34). Here and in what follows, we suppose that $0 < \varepsilon \le \varepsilon_0$ and (u_1, u_2) is the solution.

Next we prove (2.33). Since (4.19), (4.37) and (4.51) yield

$$\|u_2\|_{X^2(\kappa_2)} + \|v_\ell\|_{X^2(\kappa_2)} \le C\varepsilon, \quad \|u_2 - v_\ell\|_{Z^1(a_{\ell+1})} \le C\varepsilon^{b_\ell}, \tag{4.52}$$

applying (4.40), we obtain

$$[R(G(u_2, v_\ell))(r,t)]_1 \le C\varepsilon^{p-1+b_\ell}(1+r+t)^{-(\kappa_1 a_{\ell+1}-1)}(1+|r-t|)^{-1}. \tag{4.53}$$

In view of (4.34), we have

$$\|(u_1 - w_{\ell+1})(t)\|_E \le C\varepsilon^{p-1+b_\ell}(1+t)^{-(\kappa_1 a_{\ell+1}-1)} \tag{4.54}$$

for $t \ge 0$. Similarly, by (4.18), (4.37), (4.51), and (4.43), we get

$$\|(u_2 - v_{\ell+1})(t)\|_E \le C\varepsilon^{q-1+b_\ell}(1+t)^{-(a_{\ell+2}-1)} \tag{4.55}$$

for $t \ge 0$. Moreover, it follows from (4.21) and (4.23) that

$$[(v_{\ell+1} - v_0)(r,t)]_1 \le C\varepsilon^q(1+r+t)^{-(\kappa_2-1)}(1+|r-t|)^{-1}, \tag{4.56}$$

so that

$$\|(v_{\ell+1} - v_0)(t)\|_E \le C\varepsilon^q(1+t)^{-(\kappa_2-1)}. \tag{4.57}$$

Thus we obtain (2.33) from (4.54), (4.55) and (4.57).

Next we prove (2.34). Since $(w_{\ell+1}, \partial_t w_{\ell+1})(r,0) = \vec{f_1}(r)$, it suffices to prove

$$[R(G(u_2, v_\ell))(r,t)]_2 \le C\varepsilon^{B_\ell}(1+r+t)^{-(\kappa_1 a_{\ell+1}-1)}(1+|r-t|)^{-1}. \tag{4.58}$$

Using (4.31) and (4.52), we have

$$M_1(G(u_2, v_\ell)) \le C(\varepsilon^{B_\ell} + \varepsilon^{p-1}\|u_2 - v_\ell\|_{Z^2(a_{\ell+1})})$$

for $\alpha = \kappa_1$, $\beta = 1$, $\gamma = \kappa_1 a_{\ell+1} - \kappa_1$, and $\delta = \kappa_1 + \kappa_2$. It follows from (4.25) and (4.32) that

$$\varepsilon^{p-1}\|v_{\ell+1} - v_\ell\|_{Z^2(a_{\ell+1})} \le C\varepsilon^{B_\ell}$$

for $0 < \varepsilon \le 1$. We see from (4.34), (4.35) and (4.47) that

$$\varepsilon^{p-1}\|u_2 - v_{\ell+1}\|_{Z^2(a_{\ell+1})} \le C\varepsilon^{(p-1)b_\ell+(p-1)q}.$$

Since $(p-1)q = \kappa_2 + 1 > 2$, we therefore obtain

$$M_1(G(u_2, v_\ell)) \le C\varepsilon^{B_\ell}$$

for $0 < \varepsilon \le 1$. Since (4.45) is satisfied for those α, β, γ, and δ, applying (3.11) with $s = 2$, we get (4.58).

Finally we show (2.26), which follows from

$$\|u_2 - v_0\|_{Z^2(\kappa_2)} \le C\varepsilon^q. \tag{4.59}$$

Let $0 < \varepsilon \le 1$. By (4.33), (4.18) and (4.50) we have

$$M_1(H(u_1, w_{\ell+1})) \le C\varepsilon^{q-1+(p-1)b_\ell} \le C\varepsilon^q$$

for $\alpha = 0$, $\beta = \delta = q$ and $\gamma = a_{\ell+2} + 1 - q$, since $(p-1)b_\ell \ge (p-1)q > 2$. Noting $\alpha + \beta + \gamma - 1 = a_{\ell+2} > 1$ and applying (3.11) with $s = 2$, we get

$$\|u_2 - v_{\ell+1}\|_{Z^2(a_{\ell+2})} \le C\varepsilon^q,$$

in view of (4.34). While, we see from (4.21) and (4.25) that

$$\|v_{\ell+1} - v_0\|_{Z^2(\kappa_2)} \le C\varepsilon^q,$$

since $a_{j+1} \ge \kappa_2$ and $B_{j-1} \ge 1$ for $j \ge 1$. Therefore we obtain (4.59). Thus we complete the proof of Theorem 2.5. □

Remark When $p = 2$, Lemma 4.2 and Corollary 4.3 with $\ell = 0$ remain valid, in view of (2.19). We replace (4.36) by

$$D_\varepsilon = \{(u_1, u_2) \in X^2 \times X^2\,;\ \|u_1 - w_1\|_{Z^2(\kappa_2)} + \|u_2 - v_1\|_{Z^2(\kappa_2)} \le \varepsilon^q\}.$$

Notice that we have

$$\|R(G(u_2, u_2^*))\|_{Z^2(\kappa_2)} \le C\varepsilon\|u_2 - u_2^*\|_{Z^2(\kappa_2)},$$

instead of (4.41), because $M_1(G(u_2, u_2^*)) \leq C\varepsilon \|u_2 - u_2^*\|_{Z^2(\kappa_2)}$ for $\alpha = 0$, $\beta = 2$, $\gamma = \kappa_2 - 1$, and $\delta = 1 + \kappa_2$ (remark that $\alpha + \beta + \gamma - 1 = \kappa_2 > 1$). Proceeding as in the proof of Theorem 2.5, we find the desired conclusion stated after Theorem 2.4.

Next we prove Theorem 2.6. Similarly to the proofs of Corollary 4.3 and Lemma 4.4, one can establish the following lemma.

Lemma 4.8 *Let $0 < \varepsilon \leq 1$ and $\vec{\varphi}_i \in Y_{\kappa_i}(\varepsilon)$ with $i = 1, 2$. Then $w_j^* \in Z^2(\kappa_1)$, $v_j^* \in X^2(\kappa_2)$, and we have*

$$\|w_j^*\|_{Z^2(\kappa_1)} + \|v_j^*\|_{X^2(\kappa_2)} \leq C\varepsilon \tag{4.60}$$

for $0 \leq j \leq \ell$. Moreover, $u_1 - w_j^* \in Z^2(\kappa_1 a_j)$, $u_2 - v_j^* \in Z^2(a_{j+1})$, and we have

$$\|u_1 - w_j^*\|_{Z^1(\kappa_1 a_j)} \leq C\varepsilon^{b_{j-1}+p-1}, \quad \|u_2 - v_j^*\|_{Z^1(a_{j+1})} \leq C\varepsilon^{b_j}, \tag{4.61}$$

$$\|u_1 - w_j^*\|_{Z^2(\kappa_1 a_j)} \leq C\varepsilon^{B_{j-1}}, \qquad \|u_2 - v_j^*\|_{Z^2(a_{j+1})} \leq C\varepsilon^{B_{j-1}+q-1} \tag{4.62}$$

for $1 \leq j \leq \ell$, together with

$$\|u_2 - v_0^*\|_{Z^2(\kappa_2)} \leq C\varepsilon^q. \tag{4.63}$$

Here b_k and B_k are defined in Lemma 4.4.

End of the proof of Theorem 2.6. Let $(u_1, u_2) \in Z^2(\kappa_1) \times X^2(\kappa_2)$ be the unique solution of (2.12) satisfying (2.38).

In order to prove (2.45), it suffices to show

$$[R(G(u_2, v_\ell^*))(r, t)]_1 \leq C\varepsilon^{p-1+b_\ell}(1 + r + t)^{-(\kappa_1 a_{\ell+1} - 1)}(1 + |r - t|)^{-1},$$

in view of (2.41) and (2.44). The needed estimate can be deduced from (2.38), (4.60), (4.61), and (4.63), similarly to (4.53).

Next we show (2.47). By (2.41) and (2.42), it is enough to prove

$$[R(G(u_2, v_\ell^*))(r, t)]_2 \leq C\varepsilon^{B_\ell}(1 + r + t)^{-(\kappa_1 a_{\ell+1} - 1)}(1 + |r - t|)^{-1}.$$

Similarly to (4.58), we obtain the desired estimate from (2.38), (4.60), (4.62), and (4.63).

Finally we show (2.46) and (2.17). Since it follows from (2.44) that $u_2 - K[\vec{f_2}] = u_2 - v_0^*$, we see that (2.46) and (2.17) are consequences of (4.63). This completes the proof of Theorem 2.6. □

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