Illinois Journal of Mathematics
Volume 46, Number 4, Winter 2002, Pages 1077–1078
S 0019-2082

COARSE SHEAF COHOMOLOGY FOR FOLIATIONS: CORRIGENDUM

JAMES L. HEITSCH

ABSTRACT. We point out an error in the computation of the coarse cohomology of the Sullivan foliation in our paper "Coarse sheaf cohomology for foliations."

In [H], we claimed to have computed the coarse cohomology of the Sullivan foliation. However, Proposition 6 of [H] contains an error, which makes the computation incomplete.

Recall that the Sullivan foliation F is a one dimensional foliation of the compact manifold $M = T^1S^2 \times S^2$ with all leaves compact, but with the volumes of the leaves unbounded; see [S]. T^1S^2 is the unit tangent bundle of S^2. For $-1 \leq t \leq 1$ set $T_t = T^1S^2 \times \{(x,y,z) \,|\, z = t\}$, where (x,y,z) are global coordinates on $S^2 \subset \boldsymbol{R}^3$. Then the foliation on $T_{\pm 1}$ is given by the fibers of $\pi : T^1S^2 \to S^2$. For $-1 < t < 1$, T_t is saturated by the leaves of F and the length of each leaf in T_t is $f(t)$, where $f : (-1,1) \to \boldsymbol{R}$ is a smooth positive function with $\lim_{t \to \pm 1} f(t) = \infty$. Set $T = T_1 \cup T_{-1}$. The computations $\mathcal{H}^0(\mathcal{L}^*) = \mathcal{R}_{M-T}$ and $\mathcal{H}^q(\mathcal{L}^*)_x = 0$ for $x \in M - T$ and $q > 0$ given in [H] are correct. Proposition 6 of [H] is now

CONJECTURE. *For all $x \in T$, and $q > 0$, $\mathcal{H}^q(\mathcal{L}^*)_x = 0$.*

Given this, the isomorphism $HX^*(F) \simeq H^*(M, T; \boldsymbol{R})$ follows immediately since T is closed in M.

In the proof of Proposition 6 of [H], we claimed that we could assume that for $x \in T$, any element of $\mathcal{H}^*(\mathcal{L}^*)_x$ is represented by a cochain which is completely independent of u. This does not follow from our argument. What the construction in [H] does is to eliminate any dependence on the T coordinates, but it does not eliminate dependence on those M coordinates transverse to T. This has the effect of reducing the computation of $\mathcal{H}^*(\mathcal{L}^*)_x$

Received May 22, 2003.
2000 *Mathematics Subject Classification.* Primary 55N30. Secondary 57R30.

to the computation of $\mathcal{H}^*(\mathcal{L}^*)_{(0,0)}$ for the metric family

$$\mathcal{F}_x = \{(U \times \boldsymbol{R})/\boldsymbol{Z}, d^U, \pi, U\}.$$

Here $U \simeq D^2$ with coordinates u_1, u_2, and if $(u_1, u_2) \neq (0,0)$, then a generator of the $\boldsymbol{Z}$ action takes (u_1, u_2, r) to $(u_1, u_2, r + (u_1^2 + u_2^2)^{-1})$. There is no identification on the fiber $(0,0) \times \boldsymbol{R}$. $\pi : (U \times \boldsymbol{R})/\boldsymbol{Z} \to U$ is the natural projection and for each $u \in U$, d_u^U is the metric on $\{u\} \times \boldsymbol{R}/\boldsymbol{Z}$ induced from the usual metric on $\boldsymbol{R}$.

In support of the conjecture, we have the following result.

Proposition. *For all $x \in T$, $\mathcal{H}^1(\mathcal{L}^*)_x = 0$.*

Proof. As stated above, this is equivalent to showing that $\mathcal{H}^1(\mathcal{L}^*)_{(0,0)} = 0$ for the metric family $\mathcal{F}_x$. Let $\alpha = [\omega_{1,0} + \omega_{0,1}] \in HX^1(\mathcal{F}_x)$. Then $\omega_{1,0}$ is a 1 form on $G_1 = (U \times \boldsymbol{R})/\boldsymbol{Z}$ which satisfies the support condition for coarse cochains, so it has compact support. By restricting to a subset of U if necessary, we may assume that its support is contained in the subset of G_1 determined by the subset of $U \times \boldsymbol{R}$ given by

$$\{(u_1, u_2, t) \mid |t| \leq 1/2(u_1^2 + u_2^2)\}.$$

Apply the Controlled Poincaré Lemma of [H] to $\omega_{1,0}$ to obtain a new representative of α which has $\omega_{1,0}$ completely independent of u. Then $\delta\omega_{1,0}$ is also completely independent of u, and as $\delta\omega_{1,0} = d\omega_{0,1}$ (since $D(\omega_{1,0} + \omega_{0,1}) = 0$), $\omega_{0,1}$ must also be independent of u. Thus $\omega_{0,1}$ is a function on $G_2 = (U \times \boldsymbol{R} \times \boldsymbol{R})/(\boldsymbol{Z} \times \boldsymbol{Z})$, which is independent of u, and which satisfies the support condition for coarse cochains. Therefore we may view $\omega_{0,1}$ as a function on $\boldsymbol{R}^2$ which satisfies the support condition for coarse cochains (since it satisfies it on G_2 at $(0,0)$) and which is periodic in each variable of period $(u_1^2 + u_2^2)^{-1}$ for all $(u_1, u_2) \in U$, with $(u_1, u_2) \neq (0,0)$ (since it defines a function on G_2). The only such function is the zero function, so we have $\omega_{0,1} = 0$. Now write $\omega_{1,0} = \psi(t)dt$. Then $\delta\omega_{1,0}(t_0, t_1) = \psi(t_1)dt_1 - \psi(t_0)dt_0$. As $\delta\omega_{1,0} = d\omega_{0,1}$, we have $\delta\omega_{1,0} = 0$, which immediately gives $\omega_{1,0} = 0$. $\square$

References

[H] J. L. Heitsch, *Coarse sheaf cohomology for foliations*, Illinois J. Math. **44** (2000), 860–867.

[S] D. Sullivan, *A counterexample to the periodic orbit conjecture*, Publ. Math. Inst. Hautes Etudes Sci. **46** (1976), 5–14.

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607-7045, USA

E-mail address: heitsch@math.uic.edu