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# ON PROBLEMS BY BAER AND KULIKOV USING $V = L$

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ABSTRACT. Let $T$ be a torsion abelian group and $\lambda$ a cardinal. Among all torsion-free abelian groups $H$ of rank less than or equal to $\lambda$ satisfying $\mathrm{Ext}(H, T) = 0$ a group $G$ is called $\lambda$-universal for $T$ if it is universal with respect to group-embedding. We show that in Gödel's constructible universe ($V = L$) there always exists a $\lambda$-universal group for $T$ if $T$ has only finitely many non-trivial bounded $p$-components. This answers a question by Kulikov in the affirmative. Moreover, we prove that in $V = L$ for a large class of torsion-free abelian groups $G$ there exists a completely decomposable group $C$ such that $\mathrm{Ext}(G, T') = 0$ if and only if $\mathrm{Ext}(C, T') = 0$ for any torsion abelian group $T'$. This is related to a question of Baer.

## Introduction

In 1936 R. Baer [B] asked for a characterization of all pairs of torsion-free abelian groups $G$ and torsion abelian groups $T$ satisfying $\mathrm{Ext}(G, T) = 0$. This is a simpler version of the problem of characterizing the pairs $G$ and $T$ such that any mixed abelian group $M$ with torsion subgroup $T$ and torsion-free quotient $M/T \cong G$ has to split, i.e., that $M \cong T \oplus G$ in a canonical way. Baer himself [B] gave such a characterization for countable $G$. The question was first considered again by Wallutis and the author [SW] who, in the framework of cotorsion theories that are singly cogenerated by a torsion-free abelian group $G$ (as introduced by Salce [S]), introduced the class $\mathcal{TC}(G)$ of all torsion abelian groups $T$ satisfying $\mathrm{Ext}(G, T) = 0$. The characterization of the class $\mathcal{TC}(G)$ for torsion-free abelian groups $G$ is closely related to Griffith's solution of the Baer problem [G], which in this terminology can be stated as follows: A torsion-free abelian group $G$ is free if and only if $\mathcal{TC}(G)$ is the class of all torsion groups, i.e., $G$ is free if and only if every mixed abelian group $M$ with $M/t(M) \cong G$ splits. For rational groups $R \subseteq \mathbb{Q}$, and hence for completely decomposable groups $C$, a complete description was obtained for $\mathcal{TC}(R)$ and

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$\mathcal{TC}(C)$, respectively (see [SW]). Moreover, a necessary and sufficient criterion was given for a class of torsion abelian groups $\mathfrak{C}$ to be of the form $\mathcal{TC}(C)$ for some completely decomposable group $C$. A similar result was obtained by the author [St] for rational groups instead of completely decomposable groups, and it was shown that any finite rank torsion-free abelian group $G$ satisfies $\mathcal{TC}(G) = \mathcal{TC}(R)$ for some rational group $R$. In Section 2 of this paper we will prove that the criterion from [SW] is satisfied for a large class of torsion-free abelian groups, assuming Gödel's universe of constructibility. Hence, for a group $G$ in this class, we have $\mathcal{TC}(G) = \mathcal{TC}(C)$ for some completely decomposable group $C$. Since $\mathcal{TC}(C)$ is well understood, this gives a characterization of $\mathcal{TC}(G)$ for torsion-free abelian groups $G$ in this class. We are not able to show that in $V = L$ every torsion-free abelian group is of this kind, but we formulate a conjecture ("$\mathcal{TC}$-Conjecture") stating that this is in fact the case.

In Section 3 we consider Kulikov's problem on the existence of $\lambda$-universal groups. As defined in the abstract, a torsion-free abelian group $G$ is $\lambda$-universal for a torsion abelian group $T$ and a cardinal $\lambda$ if $G$ is of rank less than or equal to $\lambda$, $\mathrm{Ext}(G,T) = 0$, and every torsion-free abelian group $H$ of rank less than or equal to $\lambda$ satisfying $\mathrm{Ext}(H,T) = 0$ embeds into $G$. Clearly, the existence of a $\lambda$-universal group $G$ for $T$ answers the question which torsion-free groups $H$ (of cardinality at most $\lambda$) satisfy $\mathrm{Ext}(H,T) = 0$: namely, these groups are precisely the subgroups of $G$. Kulikov [KN] asked if for uncountable $\lambda$ and arbitrary $T$ there is always a $\lambda$-universal group for $T$. We first deal with the case when $\lambda$ is a positive integer or $\omega$, and we obtain satisfactory classification results in this case. Moreover, we show that in $V = L$ for every $\lambda$ and every torsion abelian group $T$ with only finitely many non-trivial bounded $p$-components there is always a $\lambda$-universal group for $T$.

Our notations are standard; notations that are not explained here can be found in [F] or [EM]. All groups under consideration are abelian. The set of primes is denoted by $\Pi$. All rational groups $R \subseteq \mathbb{Q}$ are assumed to contain the element 1. Moreover, we identify rational groups with their types since this does not cause any confusion. However, if $S$ and $R$ are rational groups, we write $S \subseteq R$ if we mean set inclusion and $S \leq R$ if we mean inequality as types.

## 1. Preliminaries

We first recall a definition from [SW].

Definition 1.1. Let $G$ be a (torsion-free) group. By $\mathcal{TC}(G)$ we denote the class of all torsion groups $T$ such that $\mathrm{Ext}(G,T) = 0$.

It is easy to see that $\mathcal{TC}(G)$ is closed under taking epimorphic images and contains all finite groups. Moreover, a torsion group $T$ is in $\mathcal{TC}(G)$ if and only if its reduced part is in $\mathcal{TC}(G)$.

We shall need the following lemma, which is well known. For the convenience of the reader we provide a proof.

First recall that a basic subgroup $B$ of a torsion group $T$ is the direct sum $B = \bigoplus_{p\in\Pi} B_p$ of basic subgroups $B_p$ of the $p$–components $T_p$; for each prime $p$, $B_p$ is a direct sum of cyclic $p$–groups, $B_p$ is a pure subgroup of $T_p$, and the quotient $T_p/B_p$ is divisible (see [F]).

Lemma 1.2. *Let $T$ be a torsion group and $B \subseteq T$ a basic subgroup of $T$. Then, for any group $G$, $T$ is an element of $\mathcal{TC}(G)$ if and only if $B$ is.*

*Proof.* The short exact sequence $0 \to B \to T \to T/B \to 0$ induces the exact sequence $\mathrm{Ext}(G, B) \to \mathrm{Ext}(G, T) \to \mathrm{Ext}(G, T/B) = 0$, where the last term is zero since $T/B$ is divisible. Thus $\mathrm{Ext}(G, B) = 0$ implies $\mathrm{Ext}(G, T) = 0$, i.e., if $B \in \mathcal{TC}(G)$ then $T \in \mathcal{TC}(G)$.

Conversely, assume $T \in \mathcal{TC}(G)$. By [F, Theorem 36.1] $B$ is an epimorphic image of $T$ and thus also belongs to $\mathcal{TC}(G)$. □

It is well known that the functor $\mathrm{Ext}(G, -)$ is closed under taking epimorphic images, but in general not closed under taking (pure) subgroups. However, if we restrict ourselves to the class $\mathcal{TC}(G)$, then this property holds.

Lemma 1.3. *Let $G$ be any group and $T$ a torsion group. Then $T \in \mathcal{TC}(G)$ if and only if $T' \in \mathcal{TC}(G)$ for all pure subgroups $T'$ of $T$ such that $|T'| \leq |G|$. Moreover, $T' \in \mathcal{TC}(G)$ for all pure subgroups $T'$ of $T$.*

*Proof.* Assume that $T \in \mathcal{TC}(G)$ and let $T'$ be a pure subgroup of $T$. Choose a basic subgroup $B'$ of $T'$. Then $B'$ is pure in $T$, and by [F, Corollary 36.2] $B'$ is an epimorphic image of $T$. Thus $B' \in \mathcal{TC}(G)$, and hence $T' \in \mathcal{TC}(G)$ by Lemma 1.2. Conversely, assume that $T \notin \mathcal{TC}(G)$, but $T' \in \mathcal{TC}(G)$ for all pure subgroups $T'$ of $T$ of cardinality less than or equal to $|G|$. Since $T \notin \mathcal{TC}(G)$, $T$ is infinite. Let

$$0 \longrightarrow T \xrightarrow{\mathrm{id}_T} M \xrightarrow{\varphi} G \longrightarrow 0 \tag{1.1}$$

be a non-splitting short exact sequence. For $g \in G$ choose $m_g \in \varphi^{-1}(\{g\})$ and put $M' = \langle m_g : g \in G\rangle \subseteq M$. Note that $|M'| = |G|$. By [F, Proposition 26.2] there exists a pure subgroup $M^* \subseteq M$ such that $M' \subseteq M^*$ and $|M^*| = |M'| = |G|$. We obtain the short exact sequence

$$0 \longrightarrow T' \xrightarrow{\mathrm{id}_{T'}} M^* \xrightarrow{\varphi\restriction_{M^*}} G \longrightarrow 0, \tag{1.2}$$

where $T' = M^* \cap T$. Since $T'$ is pure in $T$ the sequence splits and we obtain $\psi \in \mathrm{Hom}(G, M^*)$ such that $\varphi \restriction_{M^*} \circ \psi = \mathrm{id}_G$. Thus also $\varphi \circ \psi = \mathrm{id}_G$ and therefore (1.1) splits, which is a contradiction. □

To conclude this section we recall some of the basic results obtained in [St, Theorem 2.5] and [SW, Proposition 2.2 and Corollary 3.7], which we shall need in the next sections. For a torsion-free group $G$ we denote by $\mathrm{OT}(G)$ its *outer type* (see [A, page 84]).

PROPOSITION 1.4 ([St], [SW]). *Let $G$ be a torsion-free group and $T$ a reduced torsion group with $T_p$ its $p$-component (where $p$ is a prime). Then we have:*

(i) *If $G \subseteq \mathbb{Q}$, then $T \in \mathcal{TC}(G)$ if and only if the following conditions are satisfied, where $r_p = \chi_p^G(1)$:*
  (a) *$T_p$ is bounded for all $p$ such that $r_p = \infty$.*
  (b) *$T_p = 0$ for almost all $p$ such that $r_p \neq 0$.*

(ii) *If $G$ is countable, then there exists a completely decomposable group $C$ such that $\mathcal{TC}(G) = \mathcal{TC}(C)$.*

(iii) *If $G$ is of finite rank, then $\mathcal{TC}(G) = \mathcal{TC}(\mathrm{OT}(G))$.*

## 2. Characterization of $\mathcal{TC}(G)$ in $V = L$

In [B] Baer asked to characterize $\mathcal{TC}(G)$ for all torsion-free groups $G$. For countable groups Proposition 1.4 gives a satisfactory description of $\mathcal{TC}(G)$ since the structure of $\mathcal{TC}(C)$ is well understood for completely decomposable groups $C$ (see [SW]). In this section we shall show that assuming $V = L$ a large class of torsion-free groups $G$ satisfy $\mathcal{TC}(G) = \mathcal{TC}(C)$ for some completely decomposable group $C$.

We first recall Theorem 3.6 from [SW], which characterizes the classes of torsion groups that are of the form $\mathcal{TC}(C)$ for some completely decomposable group $C$.

THEOREM 2.1 ([SW]). *Let $\mathfrak{C}$ be a class of torsion groups. Then $\mathfrak{C} = \mathcal{TC}(C)$ for some completely decomposable group $C$ if and only if the following conditions are satisfied:*

(i) *$\mathfrak{C}$ contains all torsion cotorsion groups.*

(ii) *$\mathfrak{C}$ is closed under epimorphic images.*

(iii) *For all primes $p$, $\bigoplus_{n\in\omega} \mathbb{Z}(p^n) \in \mathfrak{C}$ if and only if $\mathfrak{C}$ contains all $p$–groups.*

(iv) *If $P$ is an infinite set of primes, then $\bigoplus_{p\in P} \mathbb{Z}(p) \in \mathfrak{C}$ if and only if $\bigoplus_{p\in P} T_p \in \mathfrak{C}$ for all $p$–groups $T_p \in \mathfrak{C}$.*

(v) *If $P$ is an infinite set of primes such that $\bigoplus_{p\in P} \mathbb{Z}(p) \notin \mathfrak{C}$, then there exists an infinite subset $P'$ of $P$ such that $\bigoplus_{p\in X} \mathbb{Z}(p) \notin \mathfrak{C}$ for all infinite $X \subseteq P'$.*

Note that, by Proposition 1.4, for countable torsion-free groups $G$, $\mathcal{TC}(G)$ satisfies all conditions of Theorem 2.1, Moreover, it was shown in [SW, Corollary 3.9] that under the assumption $V = L$ Theorem 2.1(iii) is always satisfied for $\mathcal{TC}(G)$ when $G$ is a torsion-free group.

Lemma 2.2 ($V = L$, [SW]). *Let $G$ be a torsion–free group and let $p$ be any prime number. Then the following are equivalent:*

(i) *$\mathcal{TC}(G)$ contains all $p$–groups.*
(ii) *$\mathcal{TC}(G)$ contains $\bigoplus_{n\in\omega}\mathbb{Z}(p^n)$.*
(iii) *$\mathbb{Z}_{(p)} \otimes G$ is a free $\mathbb{Z}_{(p)}$–module.*

*Hence, Theorem 2.1(iii) holds for $\mathcal{TC}(G)$.*

The following example of a Shelah group shows that Lemma 2.2 does not hold in ZFC (see [SW, Lemma 3.10]).

Example 2.3 (MA + ¬ CH, [SW]). For any prime number $p$ there exists a non–free $\mathbb{Z}_{(p)}$–module $G$ of cardinality $\aleph_1$ such that $\operatorname{Ext}(G, \bigoplus_{n\in\omega}\mathbb{Z}(p^n)) = 0$.

Next we prove that Theorem 2.1(iv) holds under the assumption $V = L$. We need the following basic lemma on the vanishing of Ext (see [ET, Lemma 1]).

Lemma 2.4 ([ET]). *Let $T$ be a torsion group. Suppose that the torsion-free group $G$ is the union of a continuous ascending chain of subgroups $G_\alpha$ ($\alpha < \lambda$) such that $T \in \mathcal{TC}(G_0)$ and $T \in \mathcal{TC}(G_{\alpha+1}/G_\alpha)$ for all $\alpha < \lambda$. Then $T \in \mathcal{TC}(G)$.*

Using Lemma 1.3 and Theorem XII.1.15 from [EM] it is now easy to prove that a torsion-free group $G$ satisfies $\operatorname{Ext}(G, T) = 0$ (where $T$ is a torsion group) if and only if $G$ is the union of a continuous well-ordered ascending chain $\{G_\alpha : \alpha < \lambda\}$ of subgroups ($G_0 = 0$) such that $|G_\alpha| < |G|$ and $\operatorname{Ext}(G_{\alpha+1}/G_\alpha, T) = 0$ for all $\alpha < \lambda$. But we can do even better using results from [BFS, Theorem 3.1].

Proposition 2.5 ($V = L$). *Let $G$ be a torsion-free group of infinite rank and $T$ a torsion group. Then $\operatorname{Ext}(G, T) = 0$ if and only if $G$ is the union of a continuous well-ordered ascending chain $\{G_\alpha : \alpha < \lambda\}$ of subgroups ($G_0 = 0$) such that $G_{\alpha+1}/G_\alpha$ is countable, $|G_\alpha| < |G|$, and $\operatorname{Ext}(G_{\alpha+1}/G_\alpha, T) = 0$ for all $\alpha < \lambda$.*

*Proof.* The proof of [BFS, Theorem 3.1] carries over verbatim to the present situation. All one has to do is to replace the property of being Whitehead by the condition $\operatorname{Ext}(G, T) = 0$. □

PROPOSITION 2.6 ($V = L$). *Let $G$ be a torsion-free group and let $T$ be a torsion group. If $G$ is of singular cardinality $\lambda$, then* $\operatorname{Ext}(G,T) = 0$ *if and only if* $\operatorname{Ext}(H,T) = 0$ *for all $H \subset G$ of smaller cardinality than $\lambda$.*

*Proof.* See [E, Theorem 5.5] or the proof of [BFS, Theorem 3.1] which uses Shelah's Singular Compactness Theorem [Sh]. The notion of freeness is defined as follows: $G$ is "free" if and only if there exists a chain $\{G_\alpha\}$ of the type described in Proposition 2.5. It is readily checked that this definition satisfies the assumptions of Shelah's Singular Compactness Theorem. $\square$

We remark that Proposition 2.6 does not hold in ZFC; see [SS1].

THEOREM 2.7 ($V = L$). *Let $G$ be a torsion-free abelian group. If $P$ is an infinite set of primes, then $\bigoplus_{p\in P} \mathbb{Z}(p) \in \mathcal{TC}(G)$ if and only if $\bigoplus_{p\in P} T_p \in \mathcal{TC}(G)$ for all $p$–groups $T_p \in \mathcal{TC}(G)$. In particular, Theorem* 2.1(iv) *holds for $\mathcal{TC}(G)$.*

*Proof.* We first note that we only have to prove that $\bigoplus_{p\in P} \mathbb{Z}(p) \in \mathcal{TC}(G)$ implies $\bigoplus_{p\in P} T_p \in \mathcal{TC}(G)$ for all $p$–groups $T_p \in \mathcal{TC}(G)$; the converse implication is trivial.

We use induction on the cardinality of $G$. If $G$ is countable, then the claim is true by Proposition 1.4. Hence assume that $\lambda = |G|$ is greater than or equal to $\aleph_1$. If $\lambda$ is singular, then $\bigoplus_{p\in P} \mathbb{Z}(p) \in \mathcal{TC}(G)$ implies that $\bigoplus_{p\in P} \mathbb{Z}(p) \in \mathcal{TC}(H)$ for all subgroups $H$ of $G$ of smaller cardinality. Moreover, $\mathcal{TC}(G) \subseteq \mathcal{TC}(H)$. Hence the induction hypothesis implies that $\bigoplus_{p\in P} T_p \in \mathcal{TC}(H)$ for all $p$–groups $T_p \in \mathcal{TC}(G) \subseteq \mathcal{TC}(H)$. Thus Proposition 2.6 shows that $\bigoplus_{p\in P} T_p \in \mathcal{TC}(G)$ for all $p$–groups $T_p \in \mathcal{TC}(G)$. Finally, assume that $\lambda$ is regular. Fix $T_p \in \mathcal{TC}(G)$ for $p \in P$. By Lemma 1.3, we may assume without loss of generality that $T_p$ is of cardinality less than or equal to $\lambda$. Let $G = \bigcup_{\beta<\lambda} G_\beta$ be an appropriate $\lambda$-filtration of $G$ of the type described in Proposition 2.5, i.e., such that $\bigoplus_{p\in P} \mathbb{Z}(p) \in \mathcal{TC}(G_{\alpha+1}/G_\alpha)$ for all $\alpha < \lambda$. Similarly, for each $p \in P$ choose $\lambda$-filtrations $G = \bigcup_{\beta<\lambda} G_{\beta,p}$ of $G$ such that $T_p \in \mathcal{TC}(G_{\alpha+1,p}/G_{\alpha,p})$ for all $\alpha < \lambda$. It is well known that for each $p \in P$ there is a cub $D_p$ of $\lambda$ such that $G_\beta = G_{\beta,p}$ for all $\beta \in D_p$. Since $\lambda = cf(\lambda) > \aleph_0$, the intersection $D = \bigcap_{p\in P} D_p$ is still a cub in $\lambda$. Thus, for $\alpha < \gamma \in D$ we have $\bigoplus_{p\in P} \mathbb{Z}(p) \in \mathcal{TC}(G_\gamma/G_\alpha)$ and $T_p \in \mathcal{TC}(G_\gamma/G_\alpha)$ for all $p \in P$. Therefore, by the induction hypothesis, for $\alpha < \gamma \in D$ we obtain $\bigoplus_{p\in P} T_p \in \mathcal{TC}(G_\gamma/G_\alpha)$. Hence, by [EM, Proposition XII.1.14] or Lemma 2.4, we have $\bigoplus_{p\in P} T_p \in \mathcal{TC}(G)$. $\square$

Note that Theorem 2.7 already implies that for a large class of torsion-free groups $G$ there exists a completely decomposable group $C$ such that $\mathcal{TC}(G) = \mathcal{TC}(C)$, if we assume $V = L$. In fact, this holds for all torsion-free groups $G$ satisfying Theorem 2.1(v).

However, we consider condition (v) in $V = L$. We will prove that the smallest torsion-free group $G$ violating Theorem 2.1(v) (if it exists) must be of size $\aleph_1$ in $V = L$.

THEOREM 2.8 ($V = L$). *Let $G$ be a torsion-free group and assume that for all groups $H$ of size less than or equal to $\aleph_1$ Theorem* 2.1(v) *is satisfied. If $P$ is an infinite set of primes such that $\bigoplus_{p\in P}\mathbb{Z}(p) \notin \mathcal{TC}(G)$, then there exists an infinite subset $P'$ of $P$ such that $\bigoplus_{p\in X}\mathbb{Z}(p) \notin \mathcal{TC}(G)$ for all infinite $X \subseteq P'$. Hence Theorem* 2.1(v) *is satisfied for $G$.*

*Proof.* Assume that the claim is not true and let $G$ be a counterexample of minimal cardinality. By assumption $|G| = \lambda \geq \aleph_2$, and for all groups $H$ of smaller cardinality than $\lambda$ Theorem 2.1(v) holds. If $\lambda$ is singular, then, by Proposition 2.6, $\bigoplus_{p\in P}\mathbb{Z}(p) \notin \mathcal{TC}(G)$ implies that $\bigoplus_{p\in P}\mathbb{Z}(p) \notin \mathcal{TC}(H)$ for some subgroup $H$ of $G$ of smaller cardinality. Thus, by assumption, our claim holds for $H$ and therefore also for $G$. Finally, assume that $\lambda$ is regular. For any infinite subset $X$ of $P$ put $T_X = \bigoplus_{p\in X}\mathbb{Z}(p)$. Hence $T_P \notin \mathcal{TC}(G)$. Choose a $\lambda$-filtration $G = \bigcup_{\alpha\in\lambda} G_\alpha$ such that $T_P \notin \mathcal{TC}(G_{\alpha+1}/G_\alpha)$ if, for some $\beta > \alpha$, $T_P \notin \mathcal{TC}(G_\beta/G_\alpha)$. Let $W = \{P_\alpha : \alpha \in 2^{\aleph_0}\}$ be an enumeration of all infinite subsets of $P$. Note that $|W| = 2^{\aleph_0} = \aleph_1$ since we work in $V = L$. Since $G$ is a counterexample to Theorem 2.1(v) for every $P_\alpha \in W$, there exists a set $X_\alpha \in W$ such that $X_\alpha \subseteq P_\alpha$ and $T_{X_\alpha} \in \mathcal{TC}(G)$. Choose $\lambda$-filtrations $G = \bigcup_{\beta<\lambda} G_{\beta,\rho}$ of $G$ for each $P_\rho \in W$ ($\rho < \aleph_1$) as in Proposition 2.5, such that $T_{X_\rho} \in \mathcal{TC}(G_{\alpha+1,\rho}/G_{\alpha,\rho})$ for all $\alpha < \lambda$. For all $\mu < \nu < \lambda$ we therefore have $T_{X_\rho} \in \mathcal{TC}(G_{\nu,\rho}/G_{\mu,\rho})$. It is well known that for each $\rho < \aleph_1$ there is a cub $D_\rho$ of $\lambda$ such that $G_{\beta,\rho} = G_\beta$ for all $\beta \in D_\rho$. Since $\lambda = cf(\lambda) > \aleph_1$, the intersection $C = \bigcap_{\rho<\aleph_1} D_\rho$ is still a cub in $\lambda$. Assume that there exists $\beta \in C$ such that $T_P \notin \mathcal{TC}(G_{\beta+1}/G_\beta)$. Since $G_{\beta+1}/G_\beta$ is of smaller cardinality than $G$, there exists an infinite subset $X \subseteq P$ such that $T_Y \notin \mathcal{TC}(G_{\beta+1}/G_\beta)$ for all infinite subsets $Y \subseteq X$. Choose $\beta + 1 \leq \gamma \in C$. Then also $T_Y \notin \mathcal{TC}(G_\gamma/G_\beta)$ for all infinite subsets $Y \subseteq X$. But this implies $X = P_\rho$ for some $P_\rho \in W$, and hence $T_{X_\rho} \in \mathcal{TC}(G_\gamma/G_\beta) = \mathcal{TC}(G_{\gamma,\rho}/G_{\beta,\rho})$, which is a contradiction. Thus, for all $\beta \in C$ we have $T_P \in \mathcal{TC}(G_{\beta+1}/G_\beta)$. Hence the relative $\Gamma$-invariant $\Gamma_{T_P}(G)$ is equal to 0, and therefore, by [EM, Proposition XII.1.14], $T_P \in \mathcal{TC}(G)$, which is a contradiction. □

COROLLARY 2.9 ($V = L$). *Assume that for all torsion-free groups $H$ of size less than or equal to $\aleph_1$ there exists a completely decomposable group $C_H$ such that $\mathcal{TC}(H) = \mathcal{TC}(C_H)$. Then all torsion-free groups $G$ satisfy $\mathcal{TC}(G) = \mathcal{TC}(C)$ for some completely decomposable group $C$.*

*Proof.* Let $G$ be any torsion-free group. By Lemma 2.2 and Theorem 2.7, conditions (i)–(iv) of Theorem 2.1 are satisfied for $\mathcal{TC}(G)$. Moreover, Theorem

2.8 shows that Theorem 2.1(v) is also satisfied. Hence, by Theorem 2.1, $\mathcal{TC}(G) = \mathcal{TC}(C)$ for some completely decomposable group $C$. □

As a corollary we obtain a special case of [EFS, Theorem C].

Corollary 2.10 ($V = L$). *Let $G$ be a torsion-free abelian group. Then $G$ is free if and only if $\bigoplus_{p\in\Pi}\bigoplus_{n\in\omega}\mathbb{Z}(p^n) \in \mathcal{TC}(G)$. Hence there exists a countable test-group for freeness.*

*Proof.* If $G$ is free, then trivially $\bigoplus_{p\in\Pi}\bigoplus_{n\in\omega}\mathbb{Z}(p^n) \in \mathcal{TC}(G)$. Hence assume that $\bigoplus_{p\in\Pi}\bigoplus_{n\in\omega}\mathbb{Z}(p^n) \in \mathcal{TC}(G)$. Since we assume $V = L$, we know that conditions (i)–(iv) of Theorem 2.1 are satisfied for $\mathcal{TC}(G)$. Therefore, by Theorem 2.1(iii), $\bigoplus_{n\in\omega}\mathbb{Z}(p^n) \in \mathcal{TC}(G)$ implies that $\mathcal{TC}(G)$ contains all $p$-groups for all primes $p \in \Pi$. Thus $\bigoplus_{p\in\Pi}\mathbb{Z}(p) \in \mathcal{TC}(G)$ and Theorem 2.1(iv) imply that $\mathcal{TC}(G)$ contains all direct sums of arbitrary $p$-groups (for $p \in \Pi$), and hence contains all torsion groups. Thus $\mathcal{TC}(G)$ is the class of all torsion groups, and by Griffith's solution of the Baer problem [G] it follows that $G$ is free. □

Remark 2.11. The Example 2.3 of a Shelah group under Martin's axiom shows that the above corollary does not hold in ZFC.

We were not able to show that in $V = L$ every torsion-free group $G$ satisfies $\mathcal{TC}(G) = \mathcal{TC}(C)$ for some completely decomposable group $C$. Hence we conclude this section with a conjecture.

Conjecture 2.12 ($\mathcal{TC}$-Conjecture). *In Gödel's constructible universe $V = L$, for every torsion-free group $G$ there exists a completely decomposable group $C$ such that $\mathcal{TC}(G) = \mathcal{TC}(C)$.*

## 3. Kulikov's problem

Let $T$ be any torsion group and $\lambda$ any cardinal. A torsion-free group $G$ of rank $\lambda$ is called *$\lambda$-universal for $T$* if it satisfies $\operatorname{Ext}(G, T) = 0$ and every torsion-free group $H$ of rank less than or equal to $\lambda$ satisfying $\operatorname{Ext}(H, T) = 0$ can be embedded into $G$. Kulikov [KN, Question 1.66] asked if, for arbitrary $T$ and uncountable $\lambda$, there is always a $\lambda$-universal group. For a large class of torsion groups we will show that this is the case in $V = L$. Since, to the author's knowledge, there are no published results on Kulikov's question for countable or finite $\lambda$, we begin with the case when $\lambda$ is an integer. We first note that it is easy to see that $G$ is $\lambda$-universal for $T$ if and only if $G$ is $\lambda$-universal for the reduced part of $T$. Hence we may always assume that $T$ is reduced. Recall that a group $T$ is called *cotorsion* if $\operatorname{Ext}(\mathbb{Q}, T) = 0$.

Lemma 3.1. *If $T$ is a torsion cotorsion group, then for any cardinal $\lambda$ there exists a $\lambda$-universal group $G$ for $T$.*

*Proof.* Clearly the direct sum of $\lambda$ copies of the rationals $\mathbb{Q}$ forms a $\lambda$-universal group for $T$ since every torsion-free group can be embedded into its divisible hull. □

LEMMA 3.2. *Let $T$ be a torsion group and let $G$ be $n$-universal for $T$ for some $n > 0$. Then $G$ is homogeneous completely decomposable.*

*Proof.* Since $G$ is of finite rank $n$, it follows from Proposition 1.4(iii) that the outer type $R = OT(G)$ satisfies $\mathcal{TC}(R) = \mathcal{TC}(G)$, and hence $\mathrm{Ext}(R,T) = 0$. Thus, by universality, $\bigoplus_{i \leq n} R$ can be embedded into $G$. Therefore there exists a maximal linearly independent set $\{x_1, \cdots, x_n\}$ of elements of $G$ having type greater than or equal to $R$. Thus the inner type $IT(G)$ is greater than or equal to $R$ (for the definition of inner type see [A, page 84]). Hence $R = OT(G) = IT(G)$ and it follows from [A, Proposition 3.1.13] that $G$ must be homogeneous completely decomposable of type $R$. □

THEOREM 3.3. *Let $T$ be a torsion group and $n \in \mathbb{N}$. Then there exists an $n$-universal group $G$ for $T$ if and only if $T$ has only finitely many non-trivial bounded $p$-components. In this case $G$ is completely decomposable.*

*Proof.* Without loss of generality we may assume that $T$ is reduced. Assume that $G$ is $n$-universal for $T$ for some positive integer $n$. Then, by Lemma 3.2, $G$ must be homogeneous completely decomposable, and hence $\mathcal{TC}(G) = \mathcal{TC}(S)$ for some rational group $S \subseteq \mathbb{Q}$. Assume that $T$ has infinitely many bounded $p$-components, say $T_p$ is bounded and non-trivial for $p \in P$, where $P$ is an infinite set of primes. Then $\mathrm{Ext}(\mathbb{Q}^{(p)}, T) = 0$ for all $p \in P$. Hence $\mathbb{Q}^{(p)}$ embeds into $G$ and thus, by Proposition 1.4(i), $\chi_p^S(1) = \infty$ for all $p \in P$. But then, again by Proposition 1.4(i), we have $\mathrm{Ext}(S,T) \neq 0$, which is a contradiction. Thus $T$ can have only finitely many bounded non-trivial $p$-components.

Conversely, assume that $T$ has only finitely many non-trivial bounded $p$-components. Let $P = \{p \in \Pi : T_p \text{ is unbounded }\}$ and put $R = \langle 1/p^\infty : p \notin P\rangle \subseteq \mathbb{Q}$. Then, by Proposition 1.4(i), $C = \bigoplus_{i \leq n} R$ clearly satisfies $\mathrm{Ext}(C,T) = 0$. Let $G$ be any torsion-free group of rank less than or equal to $n$ satisfying $\mathrm{Ext}(G,T) = 0$. We will show that $G$ can be embedded into $C$. Since $\mathrm{Ext}(G,T) = 0$ it follows that for any type $S$ in the typeset of $G$ we have $\mathrm{Ext}(S,T) = 0$. Hence $S \leq R$ since $R$ is idempotent. Thus the tensor product $G \otimes R$ is homogeneous of type $R$. Moreover, the short exact sequence

$$0 \to \mathbb{Z} \to R \to D \to 0 \tag{3.1}$$

with $D$ torsion divisible induces the short exact sequence

$$0 \to G \to G \otimes R \to D \otimes R \to 0, \tag{3.2}$$

and thus $G$ is embeddable into $G \otimes R$. Note that $D$ has non-trivial $p$-components only for $p \notin P$. Let $T = T_1 \oplus T_2$ with $T_1$ finite and $T_2 = \bigoplus_{p \in P} T_p$.

Then $\mathrm{Ext}(G \otimes R, T) = 0$ if and only if $\mathrm{Ext}(G \otimes R, T_2) = 0$. Applying the Hom-functor to the sequence (3.2) we obtain

$$\mathrm{Ext}(D \otimes R, T_2) \to \mathrm{Ext}(G \otimes R, T_2) \to \mathrm{Ext}(G, T_2) \to 0, \tag{3.3}$$

and since $D$ is $q$-divisible for all $q \in P$ it follows that $\mathrm{Ext}(D \otimes R, T_2) = 0$, and hence $\mathrm{Ext}(G \otimes R, T) = 0$. By Proposition 1.4(iii) there exists a rational group $S \subseteq \mathbb{Q}$ such that $\mathcal{TC}(G \otimes R) = \mathcal{TC}(S)$. It follows that $S \geq R$, and since $R$ is idempotent and $\mathrm{Ext}(G \otimes R, T) = 0$ we obtain $S = R$. Thus [St, Corollary 2.11] implies that $G \otimes R$ is completely decomposable, and hence embeddable into $C$. □

In the case of $\omega$-universal groups the situation is more delicate.

Lemma 3.4. *If $T$ is a torsion group with only finitely many non-trivial bounded $p$-components, then there is an $\omega$-universal group $C$ for $T$ which is completely decomposable.*

*Proof.* Without loss of generality assume that $T$ is reduced torsion and has only finitely many non-trivial bounded $p$-components. As in the proof of Theorem 3.3 we define $P = \{p \in \Pi : T_p \text{ is unbounded}\}$ and put $R = \langle 1/p^\infty : p \notin P\rangle \subseteq \mathbb{Q}$. We will show that $C = \bigoplus_{n\in\omega} R$ is $\omega$-universal for $T$. Let $G$ be countable torsion-free such that $\mathrm{Ext}(G, T) = 0$. By the same arguments as in the proof of Theorem 3.3 we see that $G \otimes R$ is homogeneous of type $R$ and that $\mathrm{Ext}(G \otimes R, T) = 0$. By Proposition 1.4(ii) there exists a completely decomposable group $H$ such that $\mathcal{TC}(G \otimes R) = \mathcal{TC}(H)$. It follows that $S \leq R$ for all types $S$ in the typeset of $H$ since $R$ is idempotent and $\mathrm{Ext}(G \otimes R, T) = 0$. Therefore $\mathcal{TC}(R) \subseteq \mathcal{TC}(H) = \mathcal{TC}(G \otimes R)$. Moreover, by homogeneity we have $\mathcal{TC}(G \otimes R) \subseteq \mathcal{TC}(R)$, and hence $\mathcal{TC}(G \otimes R) = \mathcal{TC}(R)$. Griffith's solution of the Baer problem [G] then implies that $G \otimes R$ is completely decomposable of type $R$ and therefore embeds into $C$. □

If $T$ has infinitely many non-trivial bounded $p$-components, then we can find at least a completely decomposable $(< \omega)$-universal group for $T$, i.e., a completely decomposable countable torsion-free group $G$ satisfying $\mathrm{Ext}(G, T) = 0$, and every finite rank torsion-free group $H$ such that $\mathrm{Ext}(H, T) = 0$ is embeddable into $G$.

Lemma 3.5. *Let $T$ be a torsion group. Then there exists a $(< \omega)$-universal group for $T$ which is completely decomposable.*

*Proof.* Let $T$ be torsion and define $P_1$ to be the set of all primes $p$ such that the $p$-component $T_p$ of $T$ is bounded but non-trivial. Moreover, let $P_2$ contain all primes such that $T_p = 0$, and let $P_3 = \Pi\backslash(P_1 \cup P_2)$. For a finite subset $Q \subseteq P_1$ we put

$$R_Q = \langle 1/p^\infty : p \in (Q \cup P_3)\rangle$$

and define $C_Q = \bigoplus_{n\in\omega} R_Q$. Finally, let $C = \bigoplus_{Q \text{ (finite) } \subseteq P_1} C_Q$. Then $\operatorname{Ext}(C,T) = 0$ by Proposition 1.4(i).

We will show that every finite rank torsion-free group $H$ such that $\operatorname{Ext}(H,T) = 0$ is embeddable into $C$. Let $H$ be such a group. Then, by Proposition 1.4(iii), $\mathcal{TC}(H) = \mathcal{TC}(K)$ for some rational group $K \subseteq \mathbb{Q}$. Note that $K$ is the outer type of $G$. Thus, if $S$ is in the typeset of $H$ then, by Proposition 1.4(i), clearly $S \le R_{Q_S}$ for some finite subset $Q_S$ of $P_1$. Let $Q = \bigcup_{S\in \operatorname{Tst}(H)} Q_S \subseteq P_1$. Then $Q$ must be finite, for otherwise $\chi_p^K(1) = \infty$ for infinitely many primes $p \in P_1$, which is a contradiction since $\operatorname{Ext}(K,T) = 0$. We let $R = \sup\{R_Q, K\}$ and conclude that $H \otimes R$ must be homogeneous of type $R$. Notice that $R$ is idempotent and that $\operatorname{Ext}(R,T) = 0$. We consider the short exact sequence

$$0 \to \mathbb{Z} \to R \to D \to 0, \tag{3.4}$$

where $D$ is torsion divisible with non-trivial $p$-components for $\chi_p^R(1) = \infty$, say $U = \{p \in \Pi : D_p \neq 0\}$. Note that $U \cap P_1$ is finite by the choice of $R$. By applying first the $\otimes$-functor and then the Hom-functor to (3.4) we obtain the short exact sequence

$$\operatorname{Ext}(D \otimes H, T) \to \operatorname{Ext}(H \otimes R, T) \to \operatorname{Ext}(H,T) = 0. \tag{3.5}$$

From the elementary properties of Ext it follows that

$$\operatorname{Ext}(D \otimes H, T) \cong \operatorname{Ext}\left(\bigoplus_{p\in U\cap P_1} D_p \otimes H, \bigoplus_{p\in U\cap P_1} T_p\right).$$

By the choice of $P_1$ and the finiteness of $U \cap P_1$ we obtain that $\bigoplus_{p\in U\cap P_1} T_p$, and hence also $\operatorname{Ext}(D \otimes H, T)$, is bounded. But $\operatorname{Ext}(H \otimes R, T)$ is divisible and an epimorphic image of $\operatorname{Ext}(D \otimes R, T)$, and hence trivial. Thus $\operatorname{Ext}(H \otimes R, T) = 0$. By Proposition 1.4(iii) there exists a rational group $S \subseteq \mathbb{Q}$ such that $\mathcal{TC}(H\otimes R) = \mathcal{TC}(S)$, and hence $R \le S$. Note that $S$ must be idempotent. Assume now that $S > R$. Then there exists $p \in P_1$ such that $\chi_p^S(1) = \infty$ and $\chi_p^R(1) = 0$. Since $K \le R$, it follows that there exists an unbounded $p$-group $T_1 \in \mathcal{TC}(H)$. We now prove by induction on the rank of $H$ that $T_1 \in \mathcal{TC}(H \otimes R) = \mathcal{TC}(S)$, which yields a contradiction. If $H$ is of rank one, then clearly $\chi_p^{H\otimes R}(1) < \infty$, and hence $\operatorname{Ext}(H \otimes R, T_1) = 0$. Assume that $H$ is of rank $n$ and choose a pure subgroup $M$ of $H$ of rank $n-1$. Let $N = H/M \subseteq \mathbb{Q}$. From the exact sequence

$$0 \to M \to H \to N \to 0 \tag{3.6}$$

we obtain the short exact sequence

$$\operatorname{Ext}(N \otimes R, T_1) \to \operatorname{Ext}(H \otimes R, T_1) \to \operatorname{Ext}(M \otimes R, T_1) \to 0. \tag{3.7}$$

By the induction hypothesis, $\operatorname{Ext}(M \otimes R, T_1) = 0$. Moreover, we have $N \le K$ since $K = OT(G)$, and therefore $\operatorname{Ext}(G, T_1) = 0$ implies $\operatorname{Ext}(K, T_1) = 0 = \operatorname{Ext}(N, T_1)$. Thus, $\chi_p^N(1) < \infty$, and hence also $\chi_p^{N\otimes R}(1) < \infty$. We conclude that $\operatorname{Ext}(N \otimes R, T_1) = 0$, and hence $\operatorname{Ext}(H \otimes R, T_1) = 0$, which is a contradiction. Thus $S = R$ and $\mathcal{TC}(H \otimes R) = \mathcal{TC}(R)$. By Griffith's solution of the Baer problem or [St, Corollary 2.11] it follows that $H \otimes R$ is completely decomposable of type $R$. Hence $H \subseteq H \otimes R$ embeds into C. $\square$

Note that Lemma 3.2 cannot hold for $\omega$-universal groups, for if $G$ is $\omega$-universal for $T$ and $H$ is countable torsion-free satisfying $\operatorname{Ext}(H, T) = 0$, then $G \oplus H$ is also $\omega$-universal for $T$. We can even show that no $\omega$-universal group for $T$ (if it exists) can be completely decomposable if $T$ has infinitely many non-trivial bounded $p$-components. Recall that a torsion-free group $B$ is called a *$B_1$-group* if $\operatorname{Bext}(G, T) = 0$ for all torsion groups $T$.

Lemma 3.6 ([SW]). *Let $B$ be a $B_1$-group. Then we have $\mathcal{TC}(B) = \mathcal{TC}(\bigoplus_{R \in \operatorname{Tst}(B)} R)$, where $\operatorname{Tst}(B)$ denotes the typeset of $B$.*

Proposition 3.7. *Let $T$ be a torsion group with infinitely many non-trivial bounded $p$-components. Then there exists a countable torsion-free group $G$ with $\operatorname{Ext}(G, T) = 0$ such that $G$ is not embeddable into any completely decomposable group $C$ satisfying $\operatorname{Ext}(C, T) = 0$.*

*Proof.* The proof is very similar to that given in [A, Example 3.4.2]. Hence we will only give a brief outline. Let $P$ be the set of primes such that $T_p$ is non-trivial and bounded. Let $S$ be the set of words on the alphabet $\{0, 1\}$ and denote by $\emptyset$ the trivial word. We divide $P$ into two disjoint infinite subsets $P_0$ and $P_1$ and enumerate these by $\omega$, e.g., $P_i = \{p_{i,j} : j < \omega\}$ $(i = 0, 1)$. For $s \in S$ let $X_s = \left\langle 1/p_{i,j}^{\infty} : s(j) = i \right\rangle \subseteq \mathbb{Q}$. Then the family $X_s$ $(s \in S)$ forms a tree, i.e., satisfies $X_s = X_{s0} \cap X_{s1}$ for all $s \in S$, and clearly $\operatorname{Ext}(X_s, T) = 0$. Moreover, if $R$ is a type and for each $n \in \mathbb{N}$ there is $s \in S$ with $l(s) = n$ such that $R \geq X_s$, then $\chi_p^R(1) = \infty$ for infinitely many primes $p \in P$, and hence $\operatorname{Ext}(R, T) \neq 0$.

Now define, for $n \in \mathbb{N}$, $G_n = \bigoplus\{X_s : s \in S, l(s) = n\}$, so that $\operatorname{Ext}(G_n, T) = 0$, and let $G$ be the direct limit of $\{G_n, f_n : n \geq 1\}$, where $f_n : G_n \to G_{n+1}$, $x \mapsto (x, x)$. Then $G$ is a $B_1$-group and the typeset $\operatorname{Tst}(G)$ of the group $G$ is contained in the (even finite) meet closure of the sets $X_s$. Since $\operatorname{Ext}(X_s, T) = 0$ for all $s \in S$, it follows by Lemma 3.6 that $T \in \mathcal{TC}(\bigoplus_{s \in S} X_s) = \mathcal{TC}(G)$, and thus $\operatorname{Ext}(G, T) = 0$. Finally, assume that $G$ is a subgroup of a completely decomposable group $C = \bigoplus_{i \in \omega} C_i$ satisfying $\operatorname{Ext}(C, T) = 0$. Since $G$ is reduced, we may assume that $C$ is reduced. There exists $m \geq 1$ such that $X_\emptyset = \mathbb{Z} \subseteq C_1 \oplus \cdots \oplus C_m$. Let $0 \neq x \in X_\emptyset$ and $n \geq 1$. Then $x = \bigoplus\{x_s : l(s) = n\} \in G_n$ with each $x_s \neq 0$. Moreover, $x_s = c(1, s) \oplus \cdots \oplus c(k, s)$ for some $k \geq m$ and $c(i, s) \in C_i$. Hence, $X_s = \operatorname{type}(x_s) \leq \cap\{\operatorname{type}(c(i, s)) : 1 \leq i \leq k\} \leq \cap\{\operatorname{type}(c(i, s)) : 1 \leq i \leq m\}$, and since $x \neq 0$ there exists $j$ with $1 \leq j \leq m$ such that $c(j, s) \neq 0$. Thus, for every $n \geq 1$ there exists $s \in S$ with $l(s) = n$

and $j$ with $1 \leq j \leq m$ such that $X_s \leq C_j$. By the choice of the sets $X_s$ it follows that $\operatorname{Ext}(C_j, T) \neq 0$, and we have reached a contradiction. $\square$

Corollary 3.8. *If $T$ has infinitely many non-trivial bounded $p$-components and $\lambda$ is an infinite cardinal, then no $\lambda$-universal group $G$ for $T$ can be completely decomposable.*

To conclude this section we show that in $V = L$ every torsion group with only finitely many non-trivial $p$-components has $\lambda$-universal groups for every $\lambda$. This contrasts the consistency result from [SS2], which shows that there is a model of ZFC in which for every torsion group $T$ that is not cotorsion there is a class of cardinals $\lambda$ such that there exist no $\lambda$-universal groups for $T$.

Theorem 3.9 ($V = L$). *If $T$ is a torsion group with only finitely many non-trivial bounded $p$-components and $\lambda$ is a cardinal, then there is a $\lambda$-universal group $C$ for $T$ which is completely decomposable.*

*Proof.* Assume that $T$ is reduced torsion and has only finitely many non-trivial $p$-bounded components. We define $P = \{p \in \Pi : T_p \text{ is unbounded}\}$ and put $R = \langle 1/p^\infty : p \notin P\rangle \subseteq \mathbb{Q}$. Then, by Proposition 1.4(i), $C = \bigoplus_\lambda R$ clearly satisfies $\operatorname{Ext}(C, T) = 0$. We will show that $C$ is $\lambda$-universal for $T$. Let $G$ be torsion-free of cardinality less than or equal to $\lambda$ and satisfying $\operatorname{Ext}(G, T) = 0$. Since $\operatorname{Ext}(G, T) = 0$, it follows that for any type $S$ in the typeset of $G$ we have $\operatorname{Ext}(S, T) = 0$. Hence $S \leq R$, since $R$ is idempotent. Thus the tensor product $G \otimes R$ is homogeneous of type $R$. As in the proof of Theorem 3.3 it follows that $\operatorname{Ext}(G \otimes R, T) = 0$. Since we do not know whether the $\mathcal{TC}$-Conjecture holds, we show directly that Theorem 2.1(v) holds for $\mathcal{TC}(G \otimes R)$. Assume that $Q$ is an infinite set of primes such that $\bigoplus_{p\in Q} \mathbb{Z}(p) \notin \mathcal{TC}(G \otimes R)$. Then $Q \backslash P$ must be infinite since $T' = \bigoplus_{p \in Q \cap P} \mathbb{Z}(p)$ is an epimorphic image of $T$, and hence $T' \in \mathcal{TC}(G \otimes R)$. It follows that condition (v) of Theorem 2.1 holds for $\mathcal{TC}(G \otimes R)$ since it holds for $\mathcal{TC}(R)$. Thus $V = L$ and the results from Section 2 imply that there exists a completely decomposable group $H$ such that $\mathcal{TC}(G) = \mathcal{TC}(H)$. As in the proof of Theorem 3.4 we conclude that $\mathcal{TC}(G \otimes R) = \mathcal{TC}(R)$. Thus Griffith's solution of the Baer problem [G] implies that $G \otimes R$ is completely decomposable of type $R$ and therefore embeds into $C$. $\square$

To the author's knowledge, it is not known whether under $V = L$ for every torsion group $T$ and every infinite cardinal $\lambda$ there exists a $\lambda$-universal group for $T$. The author strongly conjectures that the answer is yes, but the techniques developed in this article are not sufficient to provide a complete solution. Therefore we pose the following open question.

QUESTION 3.10 ($V = L$). *Let $T$ be a torsion group (with infinitely many non-zero p-components) and $\lambda$ an infinite cardinal. Does there exist a $\lambda$-universal group for $T$?*

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