ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 3, 1999

ON GEOMETRIC STABILITY AND POISSON MIXTURES

NADJIB BOUZAR

ABSTRACT. The purpose of this paper is to introduce a generalized notion of geometric stability for distributions with support in $\mathbf{Z}_+$ and $\mathbf{R}_+$. Several characterizations are obtained. A related concept of geometric attraction is also studied. Importantly, Poisson mixtures are used to deduce results for the $\mathbf{R}_+$-case from their $\mathbf{Z}_+$- counterparts.

1. Introduction

The notion of infinite divisibility and the related concepts of self-decomposability (or class L distributions) and stability derive their importance in probability theory from the fact that they are solutions to general central limit problems (for example, see Feller (1971) and Loève (1977).) These concepts are also intimately connected with stochastic processes with stationary independent increments (cf. Bertoin (1996)). Other applications can be found in the area of $\mathbf{R}_+$-valued autoregressive processes; see Pillai and Jayakumar (1994) and references therein.

Steutel and van Harn (1979) introduced an operation $\alpha \odot X$ for a $\mathbf{Z}_+$-valued rv X (here $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$) and $\alpha \in (0, 1)$ in such a way that $\alpha \odot X$ is also $\mathbf{Z}_+$-valued. The authors viewed this operation as an analogue of the ordinary multiplication and used it to define notions of self-decomposability and stability for distributions on $\mathbf{Z}_+$. Subsequently, van Harn et al (1982) and van Harn and Steutel (1993) introduced the generalized multiplications $\odot_F$ and $\odot_C$ (the definitions are recalled below). They further extended the concepts of self-decomposability and stability for distributions with support in $\mathbf{Z}_+$ and $\mathbf{R}_+$ and obtained some central limit-type theorems.

In a related context Klebanov et al (1984) introduced the class of geometrically infinite divisible distributions. This led the authors to naturally consider the subclass of geometrically strictly stable (gss) distributions. A rv X is said to have a gss distribution if for any $p \in (0, 1)$, there exists $\alpha(p) \in (0, 1)$ such that

$$X \stackrel{d}{=} \alpha(p) \sum_{i=1}^{N_p} X_i, \tag{1.1}$$

where $\{X_i\}$ is a sequence of iid rv's, $X_i \stackrel{d}{=} X$, N_p has the geometric distribution with parameter p, and $\{X_i\}$ and N_p are independent. Klebanov et al (1984) obtained several representation theorems for gss distributions, including analogues of the canonical

Received June 11, 1998.
1991 Mathematics Subject Classification. Primary 60E07.

representations of classical stability. A theory of attraction has also been developed in connection with geometric stability (for example, see Gnedenko and Korolev (1996).)

The purpose of this paper is to present a generalized notion of geometric stability for distributions on $\mathbf{Z}_+$ and $\mathbf{R}_+$. In order to arrive at the right definitions, we use the generalized multiplications $\odot_F$ and $\odot_C$ in lieu of the standard multiplication in (1.1). We establish several characterizations of the $\odot_F$ and $\odot_C$-geometric stability, including representation theorems analogous to the ones obtained by van Harn and Steutel (1993) for F-stable and C-stable laws. A related concept of geometric attraction is also introduced and characterized. Poisson mixtures are used to deduce important results for the $\mathbf{R}_+$-case from those for the $\mathbf{Z}_+$-case. Several examples are mentioned.

In the rest of this section we briefly recall some definitions and results that are needed in the sequel. For proofs and further details we refer to van Harn et al (1982) and van Harn and Steutel (1993).

The following notation will be used. P_X is the probability generating function (pgf) of the $\mathbf{Z}_+$-valued rv X, ϕ_Y is the Laplace-Stieltjes transform (LST) of the $\mathbf{R}_+$-valued rv Y. $F = (F_t;\ t \geq 0)$ is a continuous composition semigroup of pgf's F_t such that $F_t \not\equiv 1$ and $\delta_F = -\log F_1'(1) > 0$. $C = (C_t;\ t \geq 0)$ is a continuous composition semigroup of cumulant generating functions (cfg's), $C_t = -\log L_t$ where L_t is the LST of an infinitely divisible rv such that $\delta_C = -\log(-L_1'(0)) > 0$. We denote by U_F and U_C the infinitesimal generators of the semigroups F and C, respectively. The related A-functions defined by

$$A_F(z) = \exp\left\{-\int_0^z (U_F(x))^{-1}\,dx\right\}, \quad A_C(\tau) = \exp\left\{\int_\tau^1 (U_C(x))^{-1}\,dx\right\}, \tag{1.2}$$

$z \in [0, 1)$, $\tau \geq 0$ satisfy

$$A_F(F_t(z)) = e^{-t}A_F(z), \quad A_C(C_t(\tau)) = e^{-t}A_C(\tau), \quad t \geq 0. \tag{1.3}$$

Let X be a $\mathbf{Z}_+$-valued rv and $\nu \in (0, 1)$. The generalized multiplication $\nu \odot_F X$ is defined in distribution by its pgf as follows:

$$P_{\nu\odot_F X}(z) = P_X(F_t(z)), \qquad t = -\ln \nu. \tag{1.4}$$

Let X be an $\mathbf{R}_+$-valued rv and $\nu \in (0, 1)$. The generalized multiplication $\nu \odot_C X$ is defined in distribution by its LST as follows:

$$\phi_{\nu\odot_C X}(\tau) = \phi_X(C_t(\tau)), \qquad t = -\log \nu. \tag{1.5}$$

2. Discrete geometric stability

Definition 2.1. A $\mathbf{Z}_+$-valued rv X is said to have an F-gss distribution if for any $p \in (0, 1)$, there exists $\alpha(p) \in (0, 1)$ such that

$$X \stackrel{d}{=} \alpha(p) \odot_F \sum_{i=1}^{N_p} X_i, \tag{2.1}$$

where $\{X_i\}$ is a sequence of iid rv's, $X_i \stackrel{d}{=} X$, N_p has the geometric distribution with parameter p, and $\{X_i\}$ and N_p are independent.

We start out with a useful characterization of the F-gss property.

LEMMA 2.2. *A $\mathbf{Z}_+$-valued rv X with pgf P has an F-gss distribution if and only if for any $p \in (0, 1)$ there exists α $(= \alpha(p)) \in (0, 1)$ such that*

$$P(z) = \frac{pP(F_t(z))}{1 - qP(F_t(z))}, \qquad t = -\ln \alpha. \tag{2.2}$$

Proof. If X has an F-gss distribution, then by (1.4) and (2.1),

$$P(z) = P_{\sum_{i=1}^{N_p} X_i}(z) = P_{N_p}(P(F_t(z))) = \frac{pP(F_t(z))}{1 - qP(F_t(z))}, \tag{2.3}$$

where p, X_i, N_p, and α are as in Definition 2.1., and $t = -\ln \alpha$. The converse follows by (2.3) and a simple representaion argument. □

If X has an F-gss distribution, then by (2.3), for any $p \in (0, 1)$, $X \stackrel{d}{=} \sum_{i=1}^{N_p} Y_i^{(p)}$ where N_p is geometric, $Y_i^{(p)}$ are iid with common pgf $P[F_t(z)]$, and N_p and $\{Y_i^{(p)}\}$ are independent. Hence, any F-gss distribution is geometrically infinitely divisible and necessarily infinitely divisible (cf. Klebanov et al (1984).)

The following proposition constitutes the main result of the section.

PROPOSITION 2.3. *Let X be a $\mathbf{Z}_+$-valued rv with pgf P, $0 < P(0) < 1$. The following assertions are equivalent.*
(i) X has an F-gss distribution;
(ii) $H(z) = \exp\{1 - \frac{1}{P(z)}\}$ is the pgf of an F-stable distribution;
(iii) There exist $0 < \gamma \leq \delta_F$, and $d > 0$ such that

$$P(z) = (1 + dA_F(z)^\gamma)^{-1}. \tag{2.4}$$

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If X is F-gss, then X is g.i.d. and hence, by adapting Theorem 2 in Klebanov et al (1984) to pgf's, $H(z) = \exp\{1 - \frac{1}{P(z)}\}$ is an i.d. pgf. It follows by Lemma 2.2. that for any $p \in (0, 1)$ there exists $\alpha(p) \in (0, 1)$ such that $H(z) = (H(F_t(z)))^{1/p}$, $t = -\ln \alpha(p)$. This implies that $H(z)$ is the pgf of an F-stable distribution in the sense of van Harn et al (1982). By their Theorem 7.1., there exist $0 < \gamma \leq \delta_F$, and $d > 0$ such that $H(z) = \exp\{-dA_F(z)^\gamma\}$, from which (2.4) is easily obtained.

(iii) $\Rightarrow$ (i). If $P(z)$ has the form (2.4), then by using (1.3) it can be shown that (2.2) holds with $\alpha(p) = p^{1/\gamma}$ for any $p \in (0, 1)$. □

COROLLARY 2.4. *Any F-gss distribution is F-self-decomposable.*

Proof. Let $P(z)$ be as in (2.4). We need to show that for any $t > 0$, $P_t(z) = P(z)/P(F_t(z))$ is a pgf (cf. van Harn et al (1982).) We have by (1.3)

$$P_t(z) = \eta + (1-\eta)(1 + dA_F(z)^\gamma)^{-1}, \tag{2.5}$$

where $\eta = e^{-\gamma t} \in (0, 1)$. It can be easily shown that $P_t(z)$ is the pgf of a rv with the representation $Y \stackrel{d}{=} I\,E$ where I and E are independent, I is Bernoulli $(1-\eta)$ and E has pgf (2.4). □

As in classical stability, a theory of attraction can be developed in connection with the F-gss property. Let $\{X_n\}$ be a sequence of $\mathbf{Z}_+$-valued iid rv's and let $\{N_n\}$ be a sequence of rv's independent of $\{X_n\}$ such that for each $n \geq 1$, N_n has a geometric distribution with parameter $0 < p_n < 1$ and $\lim_{n\to\infty} p_n = 0$. We will say that the distribution of the X_i's belongs to the domain of F-geometric attraction of a distribution with pgf P if there exist constants a_n, $a_n \geq 1$, such that the pgf of $a_n^{-1} \odot_F \sum_{i=1}^{N_n} X_i$ converges to P. In this case, the sequence $\{a_n\}$ necessarily satisfies $\lim_{n\to\infty} a_n = \infty$.

This concept of geometric attraction has been studied in the case where the X_i's are nonnegative rv's and the operation is the ordinary multiplication (for example, see Gnedenko and Korolev (1996), chapter 2.) It was shown that in this context the only distributions on $\mathbf{R}_+$ with a nonempty domain of geometric attraction are those distributions with LST

$$\phi(\tau) = (1 + c\tau^\gamma)^{-1}, \quad \tau \geq 0 \tag{2.6}$$

for some $c > 0$ and $0 < \gamma \leq 1$ (see Theorem 2.5.2. and its consequences in Gnedenko and Korolev (1996).) This results extends as follows to the discrete case.

PROPOSITION 2.5. *Assume $\delta_F = 1$. A distribution on $\mathbf{Z}_+$ has a nonempty domain of F-geometric attraction if and only if it is F-gss.*

Proof. By definition, a gss distribution is in its own domain of attraction. Conversely, assume there exist $\{X_n\}$ and $\{N_n\}$ (as defined above) such that the pgf of $a_n^{-1} \odot_F \sum_{i=1}^{N_n} X_i$ converges to a pgf P. By Theorem 8.4.(i) in van Harn et al (1982), $a_n^{-1} \sum_{i=1}^{N_n} X_i$ (note we are back to ordinary multiplication) converges in distribution to an $\mathbf{R}_+$-valued rv with LST ϕ. Since the X_i's are obviously also $\mathbf{R}_+$-valued, this implies that ϕ has the form (2.6). By Theorem 8.4.(i) in van Harn et al (1982), $P(z) = \phi(\theta A_F(z))$ for some $\theta > 0$. The conclusion follows then from Prop 2.3. □

Remarks. (1) The assumption $\delta_F = 1$ in Prop. 2.5. can be achieved for any semigroup of pgf's F_t such that $\delta_F > 0$ by a change of time scale (see Remark 3.1. in van Harn et al (1982).)

(2) The distribution with pgf (2.4) has been studied extensively by van Harn and Steutel (1993) and Prop. 2.3. and Prop. 2.5. add to other characterizations obtained by these authors.

(3) An F-gss distribution for the standard semigroup $F_t(z) = 1 - e^{-t} + e^{-t}z$ (for which $\delta_F = 1$ and $A_F(z) = 1 - z$) has a pgf of the form $P(z) = (1 + d(1-z)^\gamma)^{-1}$, $0 < \gamma \leq 1$, and $d > 0$. This distribution is also known as the discrete Mittag-Leffler distribution (cf. Pillai and Jayakumar (1995).) We refer to van Harn et al (1982) for a more general class of semigroups.

3. The $\mathbf{R}_+$-valued case via Poisson mixtures

Definition 3.1. An $\mathbf{R}_+$-valued rv X is said to have a C-gss distribution if for any $p \in (0, 1)$, there exists $\alpha(p) \in (0, 1)$ such that

$$X \stackrel{d}{=} \alpha(p) \odot_C \sum_{i=1}^{N_p} X_i, \tag{3.1}$$

where $\{X_i\}$ is a sequence of iid rv's, $X_i \stackrel{d}{=} X$, N_p has the geometric distribution with parameter p, and $\{X_i\}$ and N_p are independent.

Similarly to the discrete case (see Lemma 2.2.), a $\mathbf{R}_+$-valued rv X with LST ϕ has a C-gss distribution if and only if for any $p \in (0, 1)$ there exists α $(= \alpha(p)) \in (0, 1)$ such that

$$\phi(\tau) = \frac{p\phi(C_t(\tau))}{1 - q\phi(C_t(\tau))}, \qquad t = -\ln\alpha. \tag{3.2}$$

Equation (3.2) implies that any C-gss distribution on $\mathbf{R}_+$ is geometrically infinitely divisible and hence, infinitely divisible. In general, one can use the same techniques of proof as in the previous section to derive characterizations of the C-gss property. Rather than doing this, we will use the Poisson mixtures approach of van Harn and Steutel (1993) to deduce results for distributions on $\mathbf{R}_+$ from their $\mathbf{Z}_+$-counterparts.

Let $N_\lambda(\cdot)$ be a Poisson process of intensity λ and T be an $\mathbf{R}_+$-valued rv independent of $N_\lambda(\cdot)$. The $\mathbf{Z}_+$-valued rv $N_\lambda(T)$ is called a Poisson mixture. Its pgf is given by

$$P_{N_\lambda(T)}(z) = \phi_T(\lambda(1-z)), \tag{3.3}$$

where ϕ is the LST of T. For every $\lambda > 0$, $F^{(\lambda)} = (F_t^{(\lambda)};\ t \geq 0)$ with

$$F_t^{(\lambda)}(z) = 1 - \lambda^{-1}C_t(\lambda(1-z)) \tag{3.4}$$

is a continuous composition semigroup of pgf's with $\delta_{F^{(\lambda)}} = \delta_C$ (cf. van Harn and Steutel (1993).) Moreover, the A-function, A_λ, of $F^{(\lambda)}$ satisfies

$$A_\lambda(z) = A_C(\lambda(1-z))/A_C(\lambda). \tag{3.5}$$

PROPOSITION 3.2. *Let X be an $\mathbf{R}_+$-valued rv. Then X has a C-gss distribution if and only if for any $\lambda > 0$, $N_\lambda(X)$ has an $F^{(\lambda)}$-gss distribution.*

Proof. Assume that X is C-gss with LST ϕ. It follows by (3.2) and (3.4) that for all $\lambda > 0$ and $p \in (0, 1)$, and some $\alpha = (\alpha(p) \in (0, 1)$,

$$\phi(\lambda(1-z)) = \frac{p\phi\left(\lambda\left(1 - F_t^{(\lambda)}(z)\right)\right)}{1 - q\phi\left(\lambda\left(1 - F_t^{(\lambda)}(z)\right)\right)}, \qquad t = -\ln\alpha. \tag{3.6}$$

Combining (3.3), (3.6), and (2.2) shows that $N_\lambda(X)$ is $F^{(\lambda)}$-*gss*. Conversely, let $0 < \tau < \lambda_1 < \lambda_2$. Since $N_{\lambda_1}(X)$ and $N_{\lambda_2}(X)$ are $F^{(\lambda_1)}$-gss and $F^{(\lambda_2)}$-*gss* respectively, for any $p \in (0, 1)$, there exist $t_1,\ t_2 > 0$ such that $\phi(\tau) = \frac{p\phi(C_{t_1}(\tau))}{1-q\phi(C_{t_1}(\tau))} = \frac{p\phi(C_{t_2}(\tau))}{1-q\phi(C_{t_2}(\tau))}$, or equivalently, $\phi(C_{t_1}(\tau)) = \phi(C_{t_2}(\tau)) = \frac{\phi(\tau)}{p+q\phi(\tau)}$. This implies that $C_{t_1}(\tau) = C_{t_2}(\tau)$, and hence by (1.3), $e^{-t_1} = e^{-t_2}$, or $t_1 = t_2$. Since λ_1 and λ_2 are arbitrary, we conclude that X is C-gss. □

COROLLARY 3.3. *Any C-gss distribution is C-self-decomposable.*

Proof. If X is C-*gss*, then by Prop. 3.2., for any $\lambda > 0$, $N_\lambda(X)$ is $F^{(\lambda)}$-gss, and hence $F^{(\lambda)}$-self-decomposable (by Corollary 2.4). The conclusion follows by applying Theorem 5.2. of van Harn and Steutel (1993). □

PROPOSITION 3.4. *Let X be an $\mathbf{R}_+$-valued rv with LST ϕ. The following assertions are equivalent.*
(i) *X has a C-gss distribution*;
(ii) $\psi(\tau) = \exp\{1 - \frac{1}{\phi(\tau)}\}$ *is the LST of a C-stable distribution*;
(iii) *There exist* $0 < \gamma \le \delta_C$ *and* $d > 0$ *such that*

$$\phi(\tau) = (1 + dA_C(\tau)^\gamma)^{-1}. \tag{3.7}$$

Proof. (i) $\Rightarrow$ (ii). If X is C-gss, then by Prop. 2.3. and Prop. 3.2., for any $\lambda > 0$, $\psi(\lambda(1-z)) = \exp\{1 - \frac{1}{\phi(\lambda(1-z)}\}$ is $F^{(\lambda)}$-stable, and (ii) then follows from Theorem 5.2. in van Harn and Steutel (1993).

(ii) $\Rightarrow$ (iii). Again by Theorem 5.2. in van Harn and Steutel (1993) and Prop. 2.3., for any $\lambda > 0$ we have

$$P_{N_\lambda(X)}(z) = (1 + d_\lambda A_\lambda(z)^{\gamma_\lambda})^{-1} \tag{3.8}$$

for some $0 < \gamma_\lambda \le \delta_{F^{(\lambda)}} (= \delta_C)$ and $d_\lambda > 0$. From (3.3) and (3.5), for any $0 \le \tau \le \lambda$ we have

$$\phi(\tau) = \left(1 + d_\lambda A_C(\lambda)^{-\gamma_\lambda} A_C(\tau)^{\gamma_\lambda}\right)^{-1}. \tag{3.9}$$

This implies that for any $0 \le \tau \le \lambda_1 \le \lambda_2$,

$$d_{\lambda_1} A_C(\lambda_1)^{-\gamma_{\lambda_1}} A_C(\tau)^{\gamma_{\lambda_1}} = d_{\lambda_2} A_C(\lambda_2)^{-\gamma_{\lambda_2}} A_C(\tau)^{\gamma_{\lambda_2}}. \tag{3.10}$$

Letting $\tau = \lambda_1$ in (3.10), we have $d_{\lambda_1} = d_{\lambda_2} A_C(\lambda_2)^{-\gamma_{\lambda_2}} A_C(\lambda_1)^{\gamma_{\lambda_2}}$. This in turn implies that $(A_C(\tau))^{\gamma_{\lambda_2}-\gamma_{\lambda_1}}$ is a constant function of τ, $0 \le \tau \le \lambda_1 \le \lambda_2$. Now $A_C(\tau)$ is strictly increasing. Hence $\gamma_{\lambda_1} = \gamma_{\lambda_2}$, and since λ_1 and λ_2 are arbitrary, $\gamma_\lambda (= \gamma)$ is constant. Using (3.10), we obtain that $d_\lambda A_C(\lambda)^{-\gamma} (= d)$ is also constant. The conclusion follows from (3.9).

(iii) $\Rightarrow$ (i). By combining (3.3), (3.5), (3.7) and Prop. 2.3, we easily conclude that for any $\lambda > 0$, $N_\lambda(X)$ is $F^{(\lambda)}$-gss. □

C-geometric attraction is defined similarly to its F-counterpart introduced in the previous section (with $\odot_F$ and pgf changing respectively into $\odot_C$ and LST.)

PROPOSITION 3.5. *Assume $\delta_C = 1$. A distribution on $\mathbf{R}_+$ has a nonempty domain of C-geometric attraction if and only if it is C-gss.*

Proof. By definiton, any C-gss distribution is in its own domain of C-geometric distribution. To establish the converse, we note that if a distribution has a nonempty domain of C-geometric attraction, then by (3.3), (3.4), and (3.6), for any $\lambda > 0$, the corresponding Poisson mixture has a nonempty domain of $F^{(\lambda)}$-geometric attraction. Since $\delta_C = \delta_{F^{(\lambda)}} = 1$, the conclusion follows from Prop. 2.5. and Prop. 3.2. □

Remarks. (1) As in the discrete case, the assumption $\delta_C = 1$ in Prop. 3.5. can be achieved for any semigroup of cgf's C_t such that $\delta_C > 0$ by a change of time scale.

(2) The semigroup C of cgf's defined by $C_t(\tau) = e^{-t}\tau$, corresponds to the ordinary multiplication. In this case, $A_C(\tau) = \tau$ and $\delta_C = 1$ and a C-gss distribution has LST $\phi(\tau) = (1 + d\tau^\gamma)^{-1}$, $0 < \gamma \le 1$, $d > 0$. This distribution is known as the (continuous) Mittag-Leffler distribution (cf. Pillai (1990) and also Fujita (1993).)

REFERENCES

1. J. Bertoin, *Lévy processes*. Cambridge Univ. Press, New York, 1996.
2. W. Feller, *An introduction to probability theory and its applications*, vol. II, 2nd. ed., Wiley, New York, 1971.
3. Y. Fujita, *A generalization of the results of Pillai*, Ann. Inst. Statist. Math. **45** (1993), 361–365.
4. B. V. Gnedenko and V. Y. Korolev, *Random summation: Limit theorems and applications*. CRC Press, New York, 1996.
5. K. van Harn, F. W. Steutel, and W. Vervaat, *Self-decomposable discrete distributions and branching processes*, Z. Wahrsch. Verw. Gebiete **61** (1982), 97–108.
6. K. van Harn and F. W. Steutel, *Stability equations for processes with stationary independent increments using branching processes and Poisson mixtures*, Stochastic Process. Appl. **45** (1993), 209–230.
7. L. B. Klebanov, G. M. Maniya, and I. A. Melamed, *A problem of Zolotarev and analogs of infinitely divisible and stable distributions in a scheme for summing a random number of random variables*, Theory Probab. Appl. **29** (1984), 791–794.

8. M. Loève, *Probability theory*, Vol. I, 4th. ed., Springer-Verlag, New York, 1977.
9. R. N. Pillai, *On Mittag-Leffler functions and related distributions*, Ann. Inst. Statist. Math. **42** (1990), 157–161.
10. R. N. Pillai and K. Jayakumar, *Specialized class L property and stationary autoregressive process*, Stat. Prob. Letters **19** (1994), 51–56.
11. ———, *Discrete Mittag-Leffler distributions*, Statist. Probab. Letters **23** (1995), 271–274.
12. F. W. Steutel and K. van Harn, Discrete analogues of self-decomposability and stability. Ann. Probab. **7** (1979), 893–899.

Department of Mathematics, University of Indianapolis, Indianapolis, IN 46227
nbouzar@uindy.edu