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MAXIMAL INEQUALITIES FOR PARTIAL SUMS OF ρ-MIXING SEQUENCES

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A Rosenthal-type inequality for maximum partial sums of ρ-mixing sequences is obtained. Applications to the complete convergence and almost sure summability of partial sums are also discussed.

1. Introduction. It is well known that the key step of various proofs in limit theory is usually the estimate of moments and/or probabilities for partial sums or for the maximum of partial sums. For independent random variables a lot of sharp and elegant estimates are available, such as the Lévy inequality, the Kolmogorov exponential inequality, the Marcinkiewicz–Zygmund inequality, the Burkholder–Davis–Gundy inequality and so forth. The main purpose of this note is to establish a Rosenthal-type inequality for a special class of dependent random variables, so-called ρ-mixing sequences, which was first introduced by Kolmogorov and Rozanov (1960).

A sequence of random variables $\{X_n,\ n \geq 1\}$ on a probability space (Ω, F, P) is called ρ-mixing if the maximal correlation coefficient

$$\rho(n) = \sup_{\substack{k\geq 1\\ X\in L^2(F_1^k)\\ Y\in L^2(F_{k+n}^\infty)}} |\operatorname{Cov}(X,Y)|/\|X\|_2\,\|Y\|_2 \longrightarrow 0$$

as $n \to \infty$, where F_n^m is the σ-field generated by the random variables $X_n, X_{n+1}, \dots, X_m$. Here and throughout this paper $\|X\|_p = (E|X|^p)^{1/p}$.

In what follows, we will always assume that $\{X_n,\ n \geq 1\}$ is a ρ-mixing sequence of random variables. Put $S_k(n) = \sum_{i=k+1}^{k+n} X_i$, $k \geq 0$, $n \geq 1$; $S(n) = S_0(n)$. Moreover, $[x]$ denotes the integer part of x, $\log x$ stands for the logarithm with base 2, $I(\cdot)$ is the indicator function and $a \wedge b = \min(a, b)$.

Our main results are as follows:

THEOREM 1.1. *Assume that $EX_i = 0$ and $\|X_i\|_q < \infty$ for some $q \geq 2$. Then there exists a positive constant $K = K(q, \rho(\cdot))$ depending only on q and $\rho(\cdot)$ such that for any $k \geq 0$, $n \geq 1$,*

$$\begin{aligned} E\max_{1\leq i\leq n}|S_k(i)|^q \leq K\Bigg(& n^{q/2}\exp\Bigg(K\sum_{i=0}^{[\log n]}\rho(2^i)\Bigg)\max_{k<i\leq k+n}\|X_i\|_2^q \\ &+ n\exp\Bigg(K\sum_{i=0}^{[\log n]}\rho^{2/q}(2^i)\Bigg)\max_{k<i\leq k+n}\|X_i\|_q^q\Bigg). \end{aligned} \tag{1.1}$$

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Applying Theorem 1.1 to the truncated random variables, we obtain the following probability inequality for the maximum of partial sums.

THEOREM 1.2. *Assume that $EX_i = 0$ for each $i \geq 1$. Then, for any $q \geq 2$, there exists $K = K(q, \rho(\cdot))$ depending only on q and $\rho(\cdot)$ such that*

$$\begin{aligned} P\Big(\max_{i\leq n}|S_k(i)| \geq x\Big) \\ &\leq \sum_{i=k+1}^{k+n} P(|X_i| \geq A) \\ &\quad + K\,x^{-q}\bigg(n^{q/2}\exp\bigg(K\sum_{i=0}^{[\log n]}\rho(2^i)\bigg)\max_{k<i\leq k+n}\|X_i I\{|X_i|\leq A\}\|_2^q \\ &\qquad + n\,\exp\bigg(K\sum_{i=0}^{[\log n]}\rho^{2/q}(2^i)\bigg)\max_{k<i\leq k+n} E|X_i|^q I\{|X_i|\leq A\}\bigg) \end{aligned} \tag{1.2}$$

for any $0 < A \leq \infty$ and $x > 0$ with

$$2n\max_{k<i\leq k+n} E|X_i| I\{|X_i| \geq A\} \leq x. \tag{1.3}$$

From Theorems 1.1 and 1.2, the next corollary follows immediately.

COROLLARY 1.1. *Let $q \geq 2$. Assume that $EX_i = 0$, $\|X_i\|_q < \infty$ and*

$$\sum_{n=1}^{\infty}\rho^{2/q}(2^n) < \infty. \tag{1.4}$$

Then there exists a positive constant $K = K(q, \rho(\cdot))$ depending only on q and $\rho(\cdot)$ such that for any $k \geq 0$, $n \geq 1$,

$$E\max_{i\leq n}|S_k(i)|^q \leq K\Big(\Big(n\max_{k<i\leq k+n} EX_i^2\Big)^{q/2} + n\max_{k<i\leq k+n} E|X_i|^q\Big)$$

and

$$\begin{aligned} P\Big(\max_{i\leq n}|S_k(i)| \geq x\Big) &\leq \sum_{i=k+1}^{k+n} P(|X_i| \geq A) \\ &\quad + K\,x^{-q}\Big(n^{q/2}\max_{k<i\leq k+n}\|X_i I\{|X_i|\leq A\}\|_2^q \\ &\qquad + n\max_{k<i\leq k+n} E|X_i|^q I\{|X_i|\leq A\}\Big) \end{aligned}$$

provided that (1.3) *is satisfied.*

REMARK 1.1. A similar result to (1.1) for ϕ-mixing sequences was obtained by Shao (1988), Peligrad (1989) and Utev (1991), independently.

REMARK 1.2. Under the condition of Theorem 1.1, it was proved in Shao (1989a) that (1.1) holds for $E|S_k(n)|^q$ rather than $E\max_{i\le n}|S_k(i)|^q$. Clearly, Theorem 1.1 as well as Theorem 1.2 here is much more useful, especially in the proof related to almost sure behaviour of partial sums, as we will see later on.

We will give the proofs in the next section. In Sections 3 and 4, we consider the complete convergence and almost sure summability of partial sums, respectively.

2. Proofs. We first need the following lemmas.

LEMMA 2.1. *Let $p, q > 1$ with $1/p + 1/q = 1$. Suppose $X \in L_p(F_1^k)$, $Y \in L_q(F_{k+n}^{\infty})$. Then we have*

$$|EXY - EX\,EY| \le 10\rho(n)^{2(1/p\wedge 1/q)}||X||_p\,||Y||_q.$$

PROOF. This is a theorem of Bradley and Bryc (1985) [see also Shao (1989b)]. □

LEMMA 2.2. *Assume that $EX_i = 0$ and $EX_i^2 < \infty$. Then there exists an absolute constant K such that for any $k \ge 0$, $n \ge 1$,*

$$E|S_k(n)|^2 \le K\,n\exp\biggl(2\sum_{i=0}^{[\log n]}\rho(2^i)\biggr)\max_{k<i\le k+n}EX_i^2.$$

PROOF. The proof is completely similar to that of Lemma 1 of Peligrad (1987) and so is omitted here [see also Shao (1989c) or Utev (1991)]. □

LEMMA 2.3. *Assume that $EX_i = 0$ and $E|X_i|^3 < \infty$. Then there exists a constant $K = K(\rho(\cdot))$ such that for any $k \ge 0$, $n \ge 1$,*

$$E|S_k(n)|^3 \le K\Biggl(n^{3/2}\exp\biggl(3\sum_{i=0}^{[\log n]}\rho(2^i)\biggr)\max_{k<i\le k+n}||X_i||_2^3 + n\exp\biggl(30\sum_{i=0}^{[\log n]}\rho^{2/3}(2^i)\biggr)\max_{k<i\le k+n}||X_i||_3^3\Biggr).$$

PROOF. This is a special case of Lemma 2.3 of Shao (1989a) [cf. Shao (1993)]. □

Our next lemma shows that Theorem 1.1 holds for $q = 2$, which, in turn, enables us to prove Theorem 1.1 for general $q \ge 2$.

LEMMA 2.4. *Assume that* $EX_i = 0$ *and* $EX_i^2 < \infty$. *Then there exists* $K = K(\rho(\cdot))$ *such that for any* $k \geq 0$, $n \geq 1$,

$$E \max_{i \leq n} |S_k(i)|^2 \leq K\, n \exp\left(6 \sum_{i=0}^{[\log n]} \rho(2^i)\right) \max_{k < i \leq k+n} EX_i^2. \tag{2.1}$$

PROOF. For the sake of convenience of statement, we assume that $\{X, X_n, n \geq 1\}$ is a strictly stationary sequence of ρ-mixing random variables. Without loss of generality, we also assume that

$$\rho(n) \geq 1/(4 \log(2n) \log^2 \log(4n)). \tag{2.2}$$

Otherwise, just put $\rho^*(n) = \max(\rho(n), 1/(4 \log(2n) \log^2 \log(4n)))$.

We shall prove that

$$E \max_{i \leq n} |S(i)|^2 \leq K\, n \exp\left(6 \sum_{i=0}^{[\log n]} \rho(2^i)\right) EX^2. \tag{2.3}$$

Let

$$\begin{aligned}
T &= \exp\left(40 \sum_{i=0}^{[\log n]} \rho^{2/3}(2^i)\right),\\
X_{i1} &= X_i I\{|X_i| \leq n^{1/2}||X_i||_2/T\} - EX_i I\{|X_i| \leq n^{1/2}||X_i||_2/T\},\\
X_{i2} &= X_i I\{|X_i| > n^{1/2}||X_i||_2/T\} - EX_i I\{|X_i| > n^{1/2}||X_i||_2/T\},\\
S_{n1}(k) &= \sum_{i=1}^{k} X_{i1}, \qquad S_{n2}(k) = \sum_{i=1}^{k} X_{i2}.
\end{aligned}$$

It is easy to see that $S(i) = S_{n1}(i) + S_{n2}(i)$ and

$$E \max_{i \leq n} |S(i)|^2 \leq 2E \max_{i \leq n} |S_{n1}(i)|^2 + 2E \max_{i \leq n} |S_{n2}(i)|^2. \tag{2.4}$$

From Lemma 2.3 it follows that for $0 \leq l < m \leq n$,

$$\begin{aligned}
&E|S_{n1}(m) - S_{n1}(l)|^3\\
&\quad \leq K \left((m-l)^{3/2} \exp\left(3 \sum_{i=0}^{[\log n]} \rho(2^i)\right) ||X||_2^3\right.\\
&\qquad \left. + (m-l) \exp\left(30 \sum_{i=0}^{[\log n]} \rho^{2/3}(2^i)\right) ||XI\{|X| \leq n^{1/2}||X||_2/T\}||_3^3\right)
\end{aligned}$$

and hence, by Corollary 3 of Moricz (1982),

$$
\begin{aligned}
&E\max_{i\le n}|S_{n1}(i)|^3\\
&\le K\Bigg(n^{3/2}\exp\bigg(3\sum_{i=0}^{[\log n]}\rho(2^i)\bigg)\|X\|_2^3+n\log^3(2n)\\
&\qquad\times\exp\bigg(30\sum_{i=0}^{[\log n]}\rho^{2/3}(2^i)\bigg)\|XI\{|X|\le n^{1/2}\|X\|_2/T\}\|_3^3\Bigg)\\
&\le K\Bigg(n^{3/2}\exp\bigg(3\sum_{i=0}^{[\log n]}\rho(2^i)\bigg)\|X\|_2^3\\
&\qquad+n\log^3(2n)\exp\bigg(30\sum_{i=0}^{[\log n]}\rho^{2/3}(2^i)\bigg)n^{1/2}\|X\|_2^3/T\Bigg)\\
&\le K\Bigg(n^{3/2}\exp\bigg(3\sum_{i=0}^{[\log n]}\rho(2^i)\bigg)\|X\|_2^3\\
&\qquad+n^{3/2}\|X\|_2^3\log^3(2n)\exp\bigg(-10\sum_{i=0}^{[\log n]}\rho^{2/3}(2^i)\bigg)\Bigg)\\
&\le K\,n^{3/2}\exp\bigg(3\sum_{i=0}^{[\log n]}\rho(2^i)\bigg)\|X\|_2^3.
\end{aligned}
\tag{2.5}
$$

Here the last inequality comes from the fact that (2.2) implies $\log n = o\big(\exp(\sum_{i=0}^{[\log n]}\rho^{2/3}(2^i))\big)$. Here and in the sequel, K denotes a constant depending only on $\rho(\cdot)$, but whose value may be different at each appearance.

By (2.5) and the Hölder inequality, we have

$$
E\max_{i\le n}|S_{n1}(i)|^2\le K\,n\exp\bigg(2\sum_{i=0}^{[\log n]}\rho(2^i)\bigg)\|X\|_2^2. \tag{2.6}
$$

To estimate the second term on the right-hand side of (2.4), let

$$
p=\exp\bigg(80\sum_{i=0}^{[\log n]}\rho^{2/3}(2^i)\bigg),\qquad r=[n/p],\qquad p_1=[p/2],
$$

$$
Y_{i1}=\sum_{j=1+(2i-1)r}^{2ir}X_{j2},\qquad Y_{i2}=\sum_{j=1+2(i-1)r}^{(2i-1)r}X_{j2},
$$

$$
T_1(i)=\sum_{j=1}^{i}Y_{j1},\qquad T_2(i)=\sum_{j=1}^{i}Y_{j2}.
$$

Since $\exp(80\sum_{i=0}^{[\log n]}\rho^{2/3}(2^i))$ is slowly varying as $n\to\infty$, there is n_0 such that $r\ge 1$ for every $n\ge n_0$. When $n\le n_0$, (2.1) is trivial. When $n\ge n_0$, noting that

$$\max_{i\le n}|S_{n1}(i)| \le \max_{i\le p_1}|T_1(i)| + \max_{i\le p_1}|T_2(i)| + \max_{i\le p+2}\max_{(i-1)r\le j\le ir}|S_{n2}(j) - S_{n2}((i-1)r)|,$$

we have

$$\begin{aligned}
E\max_{i\le n}|S_{n1}(i)|^2 &\le 3E\max_{i\le p_1}|T_1(i)|^2 + 3E\max_{i\le p_1}|T_2(i)|^2 \\
&\quad + 3E\max_{i\le p+2}\max_{(i-1)r\le j\le ir}|S_{n2}(j) - S_{n2}((i-1)r)|^2 \\
&\le 3E\max_{i\le p_1}|T_1(i)|^2 + 3E\max_{i\le p_1}|T_2(i)|^2 \\
&\quad + 3(p+2)E\max_{i\le r}|S_{n2}(i)|^2 \\
&\le 3E\max_{i\le p_1}|T_1(i)|^2 + 3E\max_{i\le p_1}|T_2(i)|^2 \\
&\quad + 3(p+2)E\left(\sum_{i=1}^{r}|X_{i2}|\right)^2 \\
&:= 3I_1 + 3I_2 + 3I_3.
\end{aligned} \tag{2.7}$$

Since

$$\begin{aligned}
\left(\sum_{i=1}^{r}E|X_{i2}|\right)^2 &\le 4\left(\sum_{i=1}^{r}E|X_i|I\{|X_i| > n^{1/2}||X_i||_2/T\}\right)^2 \\
&\le 4\left(\sum_{i=1}^{r}T\,E|X_i|^2/(n^{1/2}||X_i||_2)\right)^2 \\
&\le 4r^2T^2EX^2/n \le 4n\,T^2\,EX^2/p^2 = 4n\,EX^2/p,
\end{aligned}$$

it follows from Lemma 2.2 that

$$\begin{aligned}
I_3 &\le 4\,n\,(p+2)EX^2/p + K(p+2)\,r\,\exp\left(2\sum_{i=0}^{[\log r]}\rho(2^i)\right)EX^2 \\
&\le K\,n\,\exp\left(2\sum_{i=0}^{[\log n]}\rho(2^i)\right)EX^2.
\end{aligned} \tag{2.8}$$

We now turn to estimate I_1. Put

$$G_0 = (\Omega, \varnothing), \qquad G_i = \sigma(X_j, j \le 2ir),$$

$$u_i = E(Y_{i1} \mid G_{i-1}), \qquad U(i) = \sum_{j=1}^{i}u_j,$$

$$T^*(i) = T_1(i) - U(i) = \sum_{j=1}^{i}(Y_{j1} - E(Y_{j1} \mid G_{j-1})).$$

Obviously,

$$I_1 \le 2E \max_{i \le p_1} |T^*(i)|^2 + 2E \max_{i \le p_1} |U(i)|^2. \tag{2.9}$$

Noting that $\{T^*(i), G_i,\ i \ge 1\}$ is a martingale sequence and applying the maximum inequality [cf. Hall and Heyde (1980)] and Lemma 2.2 again, we get

$$\begin{aligned} E \max_{i \le p_1} |T^*(i)|^2 &\le K \sum_{i=1}^{p_1} EY_{i1}^2 \\ &\le K p_1 r \exp\left(2 \sum_{i=0}^{[\log r]} \rho(2^i)\right) EX^2 \\ &\le K n \exp\left(2 \sum_{i=0}^{[\log n]} \rho(2^i)\right) EX^2. \end{aligned} \tag{2.10}$$

From the definition of ρ-mixing it is easy to see that

$$\begin{aligned} Eu_i^2 &= E(Y_{i1} \mid G_{i-1})^2 = E(Y_{i1} E(Y_{i1} \mid G_{i-1})) \\ &\le \rho(r) ||Y_{i1}||_2 \, ||E(Y_{i1} \mid G_{i-1})||_2 = \rho(r) ||Y_{i1}||_2 \, ||u_i||_2. \end{aligned}$$

Hence, by Lemma 2.2,

$$Eu_i^2 \le \rho^2(r) ||Y_{i1}||_2^2 \le K r \rho^2(r) \exp\left(2 \sum_{i=0}^{[\log n]} \rho(2^i)\right) EX^2.$$

By induction [cf. (2.14) in Shao (1989b)], one can obtain that there exists a constant K such that for any $0 \le l < m \le p_1$,

$$E(U(m) - U(l))^2 \le K (m-l) r \rho^2(r) \log^2(2(m-l)) \exp\left(2 \sum_{i=0}^{[\log n]} \rho(2^i)\right) EX^2.$$

Therefore, by Corollary 4 of Moricz (1982),

$$\begin{aligned} E \max_{i \le p_1} |U(i)|^2 &\le K p_1 r \rho^2(r) \log^4(2p_1) \exp\left(2 \sum_{i=0}^{[\log n]} \rho(2^i)\right) EX^2 \\ &\le K n \rho^2(r) \log^4 p \cdot \exp\left(2 \sum_{i=0}^{[\log n]} \rho(2^i)\right) EX^2. \end{aligned} \tag{2.11}$$

From the definition of r and p we get

$$\begin{aligned} \log^4 p &= 80^4 \left(\sum_{i=0}^{[\log n]} \rho^{2/3}(2^i) \right)^4 \\ &= 80^4 \left(\left(\sum_{0 \le i \le [\log r]} + \sum_{[\log r] < i \le [\log n]} \right) (\cdot) \right)^4 \end{aligned}$$

$$\le 80^4\left(\rho^{-1/3}(r)\sum_{i=0}^{[\log r]}\rho(2^i)+\rho^{2/3}(r)([\log n]-[\log r])\right)^4$$

$$\le 80^4\left(\rho^{-1/3}(r)\sum_{i=0}^{[\log n]}\rho(2^i)+\rho^{2/3}(r)(1+\log(n/r))\right)^4$$

$$\le 80^4\left(\rho^{-1/3}(r)\sum_{i=0}^{[\log n]}\rho(2^i)+\rho^{2/3}(r)(1+\log p)\right)^4.$$

Since $\rho^{2/3}(r)\to 0$ as $n\to\infty$, we have

$$\log^4 p\le K\left(\rho^{-1/3}(r)\sum_{i=0}^{[\log n]}\rho(2^i)\right)^4$$

and hence

$$\rho^2(r)\log^4 p\le K\rho^{2/3}(r)\left(\sum_{i=0}^{[\log n]}\rho(2^i)\right)^4,$$

which together with (2.11) yields

$$E\max_{i\le p_1}|U(i)|^2\le K\,n\,\exp\left(6\sum_{i=0}^{[\log n]}\rho(2^i)\right)EX^2. \tag{2.12}$$

From (2.9), (2.10) and (2.12) it follows that

$$I_1\le K\,n\,\exp\left(6\sum_{i=0}^{[\log n]}\rho(2^i)\right)EX^2. \tag{2.13}$$

Similarly,

$$I_2\le K\,n\,\exp\left(6\sum_{i=0}^{[\log n]}\rho(2^i)\right)EX^2. \tag{2.14}$$

This completes the proof of (2.3), by (2.4), (2.6)–(2.8), (2.13) and (2.14). □

We are now ready to prove Theorem 1.1. We proceed with the proof along the same line as that of Lemma 2.3 in Shao (1989a) [cf. Shao (1993)].

PROOF OF THEOREM 1.1. To simplify the statement, we assume again that $\{X, X_n, n\ge 1\}$ is a strictly stationary sequence of ρ-mixing random variables. Put

$$M_k(n)=\max_{i\le n}|S_k(i)|,\qquad M(n)=M_0(n).$$

It suffices to show that

$$EM(n)^q \le K\biggl(n^{q/2}\exp\biggl(K\sum_{i=0}^{[\log n]}\rho(2^i)\biggr)\|X\|_2^q + n\exp\biggl(K\sum_{i=0}^{[\log n]}\rho^{2/q}(2^i)\biggr)\|X\|_q^q\biggr). \tag{2.15}$$

We prove (2.15) by induction on q. When $q = 2$, (2.15) follows from Lemma 2.4 immediately.

When q (> 2) is not an integer, assuming that (2.15) holds for $[q]$, that is, there is $K_1 \ge 2$ such that for every $n \ge 1$,

$$EM(n)^{[q]} \le K_1\biggl(n^{[q]/2}\exp\biggl(K_1\sum_{i=0}^{[\log n]}\rho(2^i)\biggr)\|X\|_2^{[q]} + n\exp\biggl(K_1\sum_{i=0}^{[\log n]}\rho^{2/[q]}(2^i)\biggr)\|X\|_{[q]}^{[q]}\biggr) \tag{2.16}$$

and

$$EM(n)^2 \le K_1 n\exp\biggl(K_1\sum_{i=0}^{[\log n]}\rho(2^i)\biggr)EX^2, \tag{2.17}$$

we will show that (2.15) remains valid for q. Put

$$\theta = \frac{1}{2} + \left(\frac{1}{2}\right)^{1/2+(q-1)/[q]}.$$

It is easy to see that

$$2(1+4^{3q}(\rho^{2/q}(n^{1/2q}) + n^{-1/3q}))\left(1 - \frac{[n/2]}{n}\right)^{q/2} \to \left(\frac{1}{2}\right)^{(q-2)/2} \quad \text{as } n \to \infty,$$

$$2(1+4^{3q}(\rho^{2/q}(n^{1/2q}) + n^{-1/3q}))\left(1 - \frac{[n/2]}{n}\right)^{1/2+(q-1)/[q]} \to \left(\frac{1}{2}\right)^{(q-1)/[q]-1/2} \quad \text{as } n \to \infty,$$

and

$$0 < \frac{q-1}{[q]} - \frac{1}{2} \le \frac{q-2}{2}.$$

Hence, we can take an m_0 such that for every $n > m_0$,

$$\left.\begin{aligned} &2(1+4^{3q}(\rho^{2/q}(n^{1/2q}) + n^{-1/3q}))\left(1 - \frac{[n/2]}{n}\right)^{q/2} \\ &2(1+4^{3q}(\rho^{2/q}(n^{1/2q}) + n^{-1/3q}))\left(1 - \frac{[n/2]}{n}\right)^{1/2+(q-1)/[q]} \end{aligned}\right\} \le \theta. \tag{2.18}$$

For $n \geq 2$, let

$$n_1 = [n/2], \qquad n_2 = n - n_1, \qquad n_3 = [n^{1/(2q)}] + 1.$$

Applying an elementary inequality

$$\forall\, x \geq 0,\ \forall\, q > 1, \qquad (1+x)^q \leq 1 + x^q + 4^q(x + x^{q-1})$$

and the fact that for any $k \geq 0$, $0 \leq m \leq n$,

$$M_k(n) \leq M_k(m) + M_{k+m}(n-m), \tag{2.19}$$

we have

$$\begin{aligned} EM(n)^q &\leq E(M(n_1) + M_{n_1}(n_2))^q \\ &\leq EM(n_1)^q + EM_{n_1}(n_2)^q \\ &\quad + 4^q(EM(n_1)\,M_{n_1}(n_2)^{q-1} + EM_{n_1}(n_2)\,M(n_1)^{q-1}). \end{aligned} \tag{2.20}$$

By Lemma 2.1, (2.19) and the Hölder inequality, we obtain

$$\begin{aligned} &EM(n_1)\,M_{n_1}(n_2)^{q-1} \\ &\quad\leq EM(n_1-n_3)\,M_{n_1}(n_2)^{q-1} + EM_{n_1-n_3}(n_3)\,M_{n_1}(n_2)^{q-1} \\ &\quad\leq EM(n_1-n_3)EM_{n_1}(n_2)^{q-1} \\ &\qquad + 10\rho^{2/q}(n_3)||M(n_1-n_3)||_q||M_{n_1}(n_2)||_q^{q-1} \\ &\qquad + ||M_{n_1-n_3}(n_3)||_q\,||M_{n_1}(n_2)||_q^{q-1} \\ &\quad\leq EM(n_1)EM_{n_1}(n_2)^{q-1} + 10\rho^{2/q}(n_3)||M(n_1)||_q||M_{n_1}(n_2)||_q^{q-1} \\ &\qquad + 12||M_{n_1-n_3}(n_3)||_q\,||M_{n_1}(n_2)||_q^{q-1} \\ &\quad\leq ||M(n_1)||_2\,||M_{n_1}(n_2)||_{[q]}^{q-1} \\ &\qquad + 10\rho^{2/q}(n_3).(EM(n_1)^q + EM_{n_1}(n_2)^q) \\ &\qquad + 12n^{1/2+1/3}E|X|^q + 12\,n^{-1/(3q)}EM_{n_1}(n_2)^q. \end{aligned} \tag{2.21}$$

Here, in the last inequality, we have used the Minkowski inequality and an elementary inequality $a^\alpha b^\beta \leq a + b$ for nonnegatives a, b, α, β with $\alpha + \beta = 1$. Similarly,

$$\begin{aligned} &EM(n_1)^{q-1}\,M_{n_1}(n_2) \\ &\quad\leq ||M_{n_1}(n_2)||_2\,||M(n_1)||_{[q]}^{q-1} \\ &\qquad + 10\rho^{2/q}(n_3)(EM(n_1)^q + EM_{n_1}(n_2)^q) \\ &\qquad + 12n^{5/6}E|X|^q + 12\,n^{-1/(3q)}EM(n_1)^q. \end{aligned} \tag{2.22}$$

Inserting (2.21) and (2.22) into (2.20) and noting that $24 \cdot 4^q \le 4^{3q}$ for $q \ge 2$, we have

$$\begin{aligned} EM(n)^q &\le \big(1 + 4^{3q}(\rho^{2/q}(n_3) + n^{-1/(3q)})\big)(EM(n_1)^q + EM_{n_1}(n_2)^q) \\ &\quad + 4^{3q}\Big(n^{5/6}E|X|^q + ||M(n_1)||_2\, ||M_{n_1}(n_2)||_{[q]}^{q-1} \\ &\qquad + ||M_{n_1}(n_2)||_2\, ||M(n_1)||_{[q]}^{q-1}\Big). \end{aligned} \tag{2.23}$$

Let

$$K_2 = 2 \cdot 4^{3q} \cdot K_1^2/(1-\theta). \tag{2.24}$$

Define $J_n = J_0 := m_0^q$ for $1 \le n \le m_0$ and

$$J_n = J_{n_2}\big(1 + 4^{3q}\big(\rho^{2/q}(n_2^{1/(3q)}) + n_2^{-1/(3q)} + 2\,n_2^{-1/6}\big)\big) \quad \text{for } n > m_0. \tag{2.25}$$

First we show below that for every $n \ge 1$,

$$\begin{aligned} EM(n)^q &\le K_2\, n^{q/2} \exp\Big(2K_1 \sum_{i=0}^{[\log n]} \rho(2^i)\Big)||X||_2^q + K_2\, n^{1/2+(q-1)/[q]} \\ &\qquad \times \exp\Big(K_1 \sum_{i=0}^{[\log n]} \big(\rho(2^i) + \rho^{2/[q]}(2^i)\big)\Big)||X||_2\, ||X||_{[q]}^{q-1} \\ &\quad + J_n\, n\, E|X|^q. \end{aligned} \tag{2.26}$$

By the Minkowski inequality, (2.26) obviously holds for $1 \le n \le m_0$. When $n > m_0$, assuming that (2.26) holds for any integer less than n, we will prove that (2.26) remains valid for n itself.

From (2.23), (2.25), (2.18), (2.16), (2.17), the hypothesis of induction and the fact that J_n is nondecreasing it follows that

$$\begin{aligned} EM(n)^q &\le \big(1 + 4^{3q}(\rho^{2/q}(n_3) + n^{-1/(3q)})\big) \\ &\qquad \times K_2\,(n_1^{q/2} + n_2^{q/2}) \exp\Big(2K_1 \sum_{i=0}^{[\log n]} \rho(2^i)\Big)||X||_2^q \\ &\quad + \big(1 + 4^{3q}(\rho^{2/q}(n_3) + n^{-1/(3q)})\big)K_2\,\big(n_1^{1/2+(q-1)/[q]} + n_2^{1/2+(q-1)/[q]}\big) \\ &\qquad \times \exp\Big(K_1 \sum_{i=0}^{[\log n]} \big(\rho(2^i) + \rho^{2/[q]}(2^i)\big)\Big)||X||_2\, ||X||_{[q]}^{q-1} \\ &\quad + \big(1 + 4^{3q}(\rho^{2/q}(n_3) + n^{-1/(3q)})\big)(J_{n_1} n_1 + J_{n_2} n_2)E|X|^q \\ &\quad + 2 \cdot 4^{3q}\Big(n^{5/6}E|X|^q + K_1^{1/2}\, n^{1/2} \exp\Big(K_1 \sum_{i=0}^{[\log n]} \rho(2^i)\Big)||X||_2\, K_1^{(q-1)/[q]} \end{aligned}$$

$$\times \left(n^{[q]/2} \exp\left(K_1 \sum_{i=0}^{[\log n]} \rho(2^i) \right) ||X||_2^{[q]} + n \exp\left(K_1 \sum_{i=0}^{[\log n]} \rho^{2/[q]}(2^i) \right) E|X|^{[q]} \right)^{(q-1)/[q]} \Bigg)$$

$$\le J_{n_2}\big(1 + 4^{3q}\big(\rho^{2/q}(n_3) + n^{-1/(3q)} + 2\,n^{-1/6}\big)\big) n\, E|X|^q$$

$$+ 2\,K_2\big(1 + 4^{3q}\big(\rho^{2/q}(n_3) + n^{-1/(3q)}\big)\big)(1 - [n/2]/n)^{q/2}\, n^{q/2}$$

$$\times \exp\left(2K_1 \sum_{i=0}^{[\log n]} \rho(2^i) \right) ||X||_2^q$$

$$+ 2 \cdot 4^{3q}\, K_1^2\, n^{q/2} \exp\left(2K_1 \sum_{i=0}^{[\log n]} \rho(2^i) \right) ||X||_2^q$$

$$+ 2\,K_2\big(1 + 4^{3q}\big(\rho^{2/q}(n_3) + n^{-1/(3q)}\big)\big)$$

$$\times \left(1 - \frac{[n/2]}{n} \right)^{1/2+(q-1)/[q]} n^{1/2+(q-1)/[q]}$$

$$\times \exp\left(K_1 \sum_{i=0}^{[\log n]} \big(\rho(2^i) + \rho^{2/[q]}(2^i)\big) \right) ||X||_2\, ||X||_{[q]}^{q-1}$$

$$+ 2 \cdot 4^{3q}\, K_1^2\, n^{1/2+(q-1)/[q]}$$

$$\times \exp\left(K_1 \sum_{i=0}^{[\log n]} \big(\rho(2^i) + \rho^{2/[q]}(2^i)\big) \right) ||X||_2\, ||X||_{[q]}^{q-1}$$

$$\le J_n\, n\, E|X|^q + \big(K_2\,\theta + 2 \cdot 4^{3q} K_1^2\big) n^{1/2+(q-1)/[q]}$$

$$\times \exp\left(K_1 \sum_{i=0}^{[\log n]} \big(\rho(2^i) + \rho^{2/[q]}(2^i)\big) \right) ||X||_2\, ||X||_{[q]}^{q-1}$$

$$+ \big(K_2\,\theta + 2 \cdot 4^{3q} K_1^2\big)\, n^{q/2} \exp\left(2K_1 \sum_{i=0}^{[\log n]} \rho(2^i) \right) ||X||_2^q$$

$$= J_n\, n\, E|X|^q + K_2\, n^{1/2+(q-1)/[q]}$$

$$\times \exp\left(K_1 \sum_{i=0}^{[\log n]} \big(\rho(2^i) + \rho^{2/[q]}(2^i)\big) \right) ||X||_2\, ||X||_{[q]}^{q-1}$$

$$+ K_2\, n^{q/2} \exp\left(2K_1 \sum_{i=0}^{[\log n]} \rho(2^i) \right) ||X||_2^q.$$

This proves that (2.26) holds for n, too.

We now estimate J_n. By (2.25), we have, for $2^n > m_0$,

$$\begin{aligned} J_{2^n} &\le J_{2^{n-1}}\big(1 + 4^{3q}\big(\rho^{2/q}\big(2^{(n-1)/(3q)}\big) + 2^{-(n-1)/(3q)} + 2^{1-(n-1)/6}\big)\big) \\ &\le J_{2^{n-1}} \exp\big(4^{3q}\big(\rho^{2/q}\big(2^{(n-1)/(3q)}\big) + 2^{-(n-1)/(3q)} + 2^{1-(n-1)/6}\big)\big). \end{aligned}$$

Thus, by recurrence,

$$\begin{aligned} J_{2^n} &\le J_0 \exp\bigg(4^{3q} \sum_{i=0}^{n} \big(\rho^{2/q}\big(2^{i/(3q)}\big) + 2^{-i/(3q)} + 2^{1-i/6}\big)\bigg) \\ &\le J_0\, C_1 \exp\bigg(C_1 \sum_{i=0}^{n} \rho^{2/q}(2^i)\bigg), \end{aligned} \tag{2.27}$$

where $C_1 \ge 1$ is a constant whose value depends only on q. Clearly, by the definition of J_n, (2.27) remains true when $2^n \le m_0$. For any given $m \ge 1$, take n such that $2^n \le m < 2^{n+1}$. Hence, by (2.27),

$$\begin{aligned} J_m \le J_{2^{n+1}} &\le J_0\, C_1 \exp\bigg(C_1 \sum_{i=0}^{n+1} \rho^{2/q}(2^i)\bigg) \\ &\le J_0\, C_1\, e^{C_1} \exp\bigg(C_1 \sum_{i=0}^{[\log m]} \rho^{2/q}(2^i)\bigg). \end{aligned} \tag{2.28}$$

A combination of (2.28) with (2.26) yields that there is a K_3 such that

$$\begin{aligned} EM(n)^q \le K_3\bigg(& n^{q/2} \exp\bigg(K_3 \sum_{i=0}^{[\log n]} \rho(2^i)\bigg) \|X\|_2^q + n^{1/2+(q-1)/[q]} \\ &\times \exp\bigg(K_3 \sum_{i=0}^{[\log n]} \big(\rho(2^i) + \rho^{2/[q]}(2^i)\big)\bigg) \|X\|_2\, \|X\|_{[q]}^{q-1} \\ &+ n \exp\bigg(K_3 \sum_{i=0}^{[\log n]} \rho(2^i)\bigg) E|X|^q\bigg). \end{aligned} \tag{2.29}$$

If $2 < q < 3$, then $[q] = 2$. In this case, (2.15) follows from (2.29) immediately.

If $q > 3$, we put

$$\alpha = \frac{(q-1)([q]-2)}{(q-2)[q]}.$$

It is easy to see that $0 < \alpha < 1$ and

$$\frac{2}{q}\bigg(\frac{1}{2} + \frac{q-[q]}{q-2}\frac{(q-1)}{[q]}\bigg) + \alpha = 1.$$

In terms of the Lyapunov inequality, we have

$$E|X|^{[q]} \le (E|X|^2)^{(q-[q])/(q-2)} (E|X|^q)^{([q]-2)/(q-2)}$$

and hence

$$n^{1/2+(q-1)/[q]}\exp\bigg(K_3\sum_{i=0}^{[\log n]}\big(\rho(2^i)+\rho^{2/[q]}(2^i)\big)\bigg)\|X\|_2\,\|X\|_{[q]}^{q-1}$$

$$\le n^{1/2+(q-1)/[q]}\exp\bigg(K_3\sum_{i=0}^{[\log n]}\big(\rho(2^i)+\rho^{2/q}(2^i)\big)\bigg)$$

$$\times\|X\|_2^{1+(2(q-1))/[q]\cdot(q-[q])/(q-2)}\,(E|X|^q)^{((q-1)([q]-2))/((q-2)[q])}$$

$$= n^{q(1-\alpha)/2}\exp\bigg(K_3\sum_{i=0}^{[\log n]}\rho(2^i)\bigg)\|X\|_2^{q(1-\alpha)} \tag{2.30}$$

$$\times\, n^{\alpha}\exp\bigg(K_3\sum_{i=0}^{[\log n]}\rho^{2/q}(2^i)\bigg)(E|X|^q)^{\alpha}$$

$$\le n^{q/2}\exp\bigg(\frac{K_3}{1-\alpha}\sum_{i=0}^{[\log n]}\rho(2^i)\bigg)\|X\|_2^q$$

$$+\, n\exp\bigg(\frac{K_3}{\alpha}\sum_{i=0}^{[\log n]}\big(\rho^{2/q}(2^i)\big)\bigg)E|X|^q.$$

This proves (2.15) by (2.30) and (2.29).

When $q\ge 3$ is an integer, along the same lines as the above proof, but with $q-1$ instead of $[q]$, we can deduce that (2.15) remains true. Now the proof of the theorem is complete. □

PROOF OF THEOREM 1.2. Clearly, we have

$$P\Big(\max_{i\le n}|S_k(i)|\ge x\Big)$$

$$\le\sum_{i=k+1}^{k+n}P(|X_i|\ge A)+P\bigg(\max_{i\le n}\bigg|\sum_{j=k+1}^{k+i}X_jI\{|X_j|<A\}\bigg|\ge x\bigg)$$

$$\le\sum_{i=k+1}^{k+n}P(|X_i|\ge A)$$

$$+\,P\bigg(\max_{i\le n}\bigg|\sum_{j=k+1}^{k+i}X_jI\{|X_j|<A\}-EX_jI\{|X_j|<A\}\bigg| \tag{2.31}$$

$$\ge x-\sum_{j=k+1}^{k+n}E|X_j|I\{|X_j|\ge A\}\bigg)$$

$$\le\sum_{i=k+1}^{k+n}P(|X_i|\ge A)$$

$$+\,P\bigg(\max_{i\le n}\bigg|\sum_{j=k+1}^{k+i}X_jI\{|X_j|<A\}-EX_jI\{|X_j|<A\}\bigg|\ge x/2\bigg),$$

by (1.3).

Now (1.2) follows from (2.31), Theorem 1.1 and the Chebychev inequality. □

REMARK 2.1. Along the same lines as the proofs of Theorem 1.1, one can easily deduce the following result, which is rather useful in some cases. The proof is left to the reader: *Assume that* $EX_i = 0$ *and* $\|X_i\|_q < \infty$ *for some* $2 < q \le 3$ *and that there is a slowly varying function* $h(n)$ *satisfying* $h(n) \ge \exp(-C_0 \sum_{i=0}^{[\log n]} \rho(2^i))$ *for some* C_0 *such that for any* $k \ge 0$, $n \ge 1$,

$$ES_k^2(n) \le n\, h(n) \max_{k<i\le k+n} EX_i^2.$$

Then there exists a constant K *such that for any* $k \ge 0$, $n \ge 1$,

$$E \max_{1\le i\le n} |S_k(i)|^q \le K\biggl(\Bigl(n\, h(n) \max_{k<i\le k+n} EX_i^2\Bigr)^{q/2} + n \exp\biggl(K \sum_{i=0}^{[\log n]} \rho^{2/q}(2^i)\biggr) \max_{k<i\le k+n} E|X_i|^q\biggr).$$

3. Complete convergence. Applying Theorem 1.2, one can obtain the following results on the complete convergence for ρ-mixing sequences, which is a kind of convergence rate with respect to the strong law of large numbers [cf. Hsu and Robbins (1947) and Baum and Katz (1965)].

THEOREM 3.1. *Let* $1 \ge \alpha > 1/2$, $p\alpha \ge 1$, $\{X_n, n \ge 1\}$ *be a* ρ*-mixing sequence of identically distributed random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$. *Assume that*

$$\sum_{n=1}^{\infty} \rho^{2/r}(2^n) < \infty, \tag{3.1}$$

where $r = 2$ *if* $1 \le p < 2$ *and* $r > p$ *if* $p \ge 2$. *Then*

$$\forall\, \varepsilon > 0, \quad \sum_{n=1}^{\infty} n^{p\alpha-2} P\Bigl(\max_{i\le n} |S(i)| > \varepsilon n^{\alpha}\Bigr) < \infty. \tag{3.2}$$

The proof is the same as that of Corollary 1 in Shao (1989a), by using Theorem 1.2 instead of Lemma 2.4 there, so it is omitted here.

An immediate consequence of the above complete convergence is the following Marcinkiewicz–Zygmund law of large numbers.

COROLLARY 3.1. *Let* $1 \le p < 2$, $\{X_n, n \ge 1\}$ *be a* ρ*-mixing sequence of identically distributed random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$. *Assume that*

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty.$$

Then

$$\lim_{n\to\infty} S(n)/n^{1/p} = 0 \quad a.s.$$

4. Almost sure summability of partial sums. Another way to describe the convergence rate in the strong law of large numbers is by studying the almost sure summability of partial sums [cf. Csörgő, Horváth and Shao (1991)]. Based on Theorem 1.2, we present the following result.

THEOREM 4.1. *Let $\{X_n,\ n \ge 1\}$ be a ρ-mixing sequence of identically distributed random variables with $EX_n = 0$ and $E|X_n|^2 < \infty$. Assume that $p > 0$ and*

$$\sum_{n=1}^{\infty} \rho^{1\wedge(2/p)}(2^n) < \infty. \tag{4.1}$$

Then, for any sequence $\{q(n), n \ge 1\}$ of positive numbers satisfying

$$\sum_{n=1}^{\infty} n^{p/2}/q(n) < \infty, \tag{4.2}$$

we have

$$\sum_{n=1}^{\infty} \max_{i\le n} |S(i)|^p/q(n) < \infty \quad a.s. \tag{4.3}$$

COROLLARY 4.1. *Let $p > 0$, $\{q(n), n \ge 1\}$ be a sequence of positive numbers, $\{X_n,\ n \ge 1\}$ i.i.d. random variables with $EX_1 = 0$, $0 < EX_1^2 < \infty$. Then the following statements are equivalent:*

$$\sum_{n=1}^{\infty} |S(n)|^p/q(n) < \infty \quad a.s. \tag{4.4}$$

$$\sum_{n=1}^{\infty} \max_{i\le n} |S(i)|^p/q(n) < \infty \quad a.s. \tag{4.5}$$

$$\sum_{n=1}^{\infty} n^{p/2}/q(n) < \infty. \tag{4.6}$$

REMARK 4.1. Corollary 4.1 was first obtained by Csörgő, Horváth and Shao (1991), but there it was assumed that $\max_{i\le 2n} q(i) \le C\,q(n)$ for some $C > 0$ and for any $n \ge 1$.

PROOF OF THEOREM 4.1. When $0 < p \le 2$, it follows from (4.1) and Corollary 1.1 used with $q = 2$ that

$$E \max_{i\le n} |S(i)|^p \le K n^{p/2}.$$

Therefore, by (4.2),

$$E\left(\sum_{n=1}^{\infty}\max_{i\le n}|S(i)|^p/q(n)\right)<\infty,$$

which yields (4.3) immediately.

If $p>2$, we put

$$X_{k,1}=X_kI\{|X_k|\le k^{1/2}\},\qquad S_{k,1}=\sum_{i=1}^{k}X_{k,1}.$$

Since $EX_1^2<\infty$ implies $P(X_k\ne X_{k,1},\text{ i.o.})=0$, it suffices to show that

$$E\left(\sum_{n=1}^{\infty}\max_{i\le n}|S_{i,1}|^p/q(n)\right)<\infty. \tag{4.7}$$

Using Corollary 1.1 again, we have

$$\begin{aligned}
&E\max_{i\le n}|S_{i,1}|^p\\
&\quad\le K\left(\left(\sum_{i=1}^{n}|EX_{i,1}|\right)^p+(n\,EX_1^2)^{p/2}+n\,E|X_1|^pI\{|X_1|\le n^{1/2}\}\right)\\
&\quad\le K\left(\left(\sum_{i=1}^{n}E|X_1|I\{|X_1|>i^{1/2}\right)^p\right.\\
&\qquad\qquad\left.+(n\,EX_1^2)^{p/2}+n\,E|X_1|^pI\{|X_1|\le n^{1/2}\}\right)\\
&\quad\le K\left(\left(\sum_{i=1}^{n}i^{-1/2}EX_1^2\right)^p+(n\,EX_1^2)^{p/2}+n^{p/2}\,E|X_1|^2\right)\\
&\quad\le Kn^{p/2}.
\end{aligned} \tag{4.8}$$

This proves (4.7), by (4.8) and (4.2), as desired.

PROOF OF COROLLARY 4.1. In terms of Lemma 2.1 of Csörgő, Horváth and Shao (1993), (4.4) implies

$$\sum_{n=1}^{\infty}(\mathrm{med}|S(n)|)^p/q(n)<\infty. \tag{4.9}$$

By the central limit theorem,

$$n^{1/2}=O(\mathrm{med}|S(n)|),\qquad \mathrm{med}|S(n)|=O(n^{1/2}).$$

Therefore, (4.4) implies (4.6). On the other hand, from Theorem 4.1 it follows that (4.6) $\Rightarrow$ (4.5) $\Rightarrow$ (4.4).

This proves the corollary. □

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