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A MAXIMAL INEQUALITY AND DEPENDENT MARCINKIEWICZ–ZYGMUND STRONG LAWS

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This paper contains some extension of Kolmogorov's maximal inequality to dependent sequences. Next we derive dependent Marcinkiewicz–Zygmund type strong laws of large numbers from this inequality. In particular, for stationary strongly mixing sequences $(X_i)_{i\in\mathbb{Z}}$ with sequence of mixing coefficients $(\alpha_n)_{n\geq 0}$, the Marcinkiewicz–Zygmund SLLN of order p holds if $\int_0^1[\alpha^{-1}(t)]^{p-1}Q^p(t)\,dt<\infty$, where α^{-1} denotes the inverse function of the mixing rate function $t\to\alpha_{[t]}$ and Q denotes the quantile function of $|X_0|$. The condition is obtained by an interpolation between the condition of Doukhan, Massart and Rio implying the CLT ($p=2$) and the integrability of $|X_0|$ implying the usual SLLN ($p=1$). Moreover, we prove that this condition cannot be improved for stationary sequences and power-type rates of strong mixing.

0. Introduction. Among the most useful tools of probability theory are the many devices used to bound in probability or in L_p-norm the random variable $\sup_{p\leq n}|S_p|$, where $S_p=\sum_{i=1}^p X_i$. When the X_i are independent random variables, many inequalities such as those of Kolmogorov, Ottaviani and Bernstein are available. In the weakly dependent case, most of the extensions of these inequalities are mainly based on Doob's inequality for martingales [see, e.g., McLeish (1975)]: the weakly dependent sequence is approximated by some martingale difference sequence, and one can derive a maximal inequality for the weakly dependent sequence from the available inequalities for the approximating martingale [see Hall and Heyde (1980)]. Concerning the applications to strongly mixing sequences, this method leads to some loss concerning the conditions of decay of the mixing coefficients. In order to improve the previous results, we will prove in this paper a new maximal inequality. The proof is performed via a Lindeberg-type method. This method allows us to minimize the effect of long range interactions between the random variables. So we will obtain rates of convergence in the strong law under minimal conditions on the mixing coefficients and on the tail distributions of the random variables.

We are interested in two different types of results. First, we want to obtain some criteria on the mixing coefficients and on the tail distributions of the random variables implying the almost sure convergence of $n^{-1/p}S_n$ to 0 (such

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a result will be called a Marcinkiewicz–Zygmund strong law of order p). Second, we want to obtain some criteria implying the almost sure convergence of the series $\Sigma_{i>0} X_i$.

In order to give an outline of the results, we introduce the following notations.

NOTATIONS. For any nonincreasing cadlag function $H: \mathbb{R}^+ \to \mathbb{R}^+$, let H^{-1} denote the cadlag inverse function of H, which is defined by

$$H^{-1}(u) = \sup\{t \in \mathbb{R}^+ : H(t) > u\}, \tag{0.0}$$

with the convention that $\sup \varnothing = 0$. For any real-valued random variable X with distribution function F, we denote by Q_X or Q_F the inverse function of $t \to \mathbb{P}(|X| > t)$. We set $Q_i = Q_{X_i}$.

If $(\alpha_n)_{n \geq 0}$ is a nonincreasing sequence of nonnegative real numbers, we denote by $\alpha(\cdot)$ the cadlag rate function which is defined by $\alpha(t) = \alpha_{[t]}$. Throughout the sequel, α^{-1} denotes the inverse function of this rate function $\alpha(\cdot)$.

We first give the conditions implying the almost sure convergence of the series $\Sigma_{i>0} X_i$. Let us first explain the summability condition that we can expect in the strong mixing case. It follows from Theorem 1.2 in Rio (1993) that

$$\operatorname{Var} S_n \leq 4 \sum_{k=0}^{n-1} \sum_{i=1}^{n} \int_0^{2\alpha_k} Q_i^2(t)\, dt \leq 4 \sum_{i=1}^{n} \int_0^1 \alpha^{-1}(t/2) Q_i^2(t)\, dt. \tag{0.1}$$

The above upper bound for the variance improves the previous upper bounds based on Davydov's covariance inequality [Davydov (1968)]. In fact (0.1) is based on a more efficient covariance inequality than Davydov's [see Theorem 1.1 in Rio (1993)]. Hence, $(\operatorname{Var} S_n)_{n>0}$ is a bounded sequence as soon as

$$\sum_{i=1}^{\infty} \int_0^1 \alpha^{-1}(t/2) Q_i^2(t)\, dt < \infty. \tag{0.2}$$

Assume now that $\mathbb{E}(X_i) = 0$. In this paper, we obtain the almost sure convergence of the sequence $\Sigma_{i>0} X_i$ under condition (0.2) (we will compare this condition with the previous conditions in Section 1).

In the independent case, one can derive the Marcinkiewicz–Zygmund strong law from the classical Kolomogorov three series theorem. However, in the strong mixing case, such a method needs the mixing condition (0.2). Now, if there exists some positive i such that the random variable X_i is not a.s. a constant, condition (0.2) needs the summability condition

$$\int_0^1 \alpha^{-1}(u)\, du = \sum_{n \geq 0} \alpha_n < \infty. \tag{0.3}$$

Hence this method leads to the too restrictive summability condition $\Sigma_{n \geq 0} \alpha_n < \infty$, where $(\alpha_n)_{n \geq 0}$ denotes the strong mixing coefficients of the sequence $(X_i)_{i \in \mathbb{Z}}$.

By contrast, using the slightly stronger notion of β-mixing and a coupling method, Berbee (1987) proved that, for bounded mixing sequences, the Marcinkiewicz–Zygmund strong law of order p holds under the summability condition

$$\sum_{n>0} n^{p-2}\beta_n < \infty. \tag{0.4}$$

He also established that (0.4) is a minimal condition. His counterexample is a counterexample to Hipp's results (1979) on the convergence rates in the strong law [see Shao (1993) for a detailed discussion about the validity of Hipp's results]. Shao (1993) recently obtained some extensions of this result to unbounded variables and strong mixing. However, the blocking technique used by Shao [see the proof of Lemma 1 in Shao (1993)] does not lead to optimal results, as we shall see in Section 1.

In this paper, we obtain the natural extension of (0.4) to strongly mixing sequences of unbounded random variables. We prove that, for p in $]1,2[$ and sequences of identically distributed zero mean real-valued random variables, the Marcinkiewicz–Zygmund strong law of order p holds if

$$\int_0^1 [\alpha^{-1}(t/2)]^{p-1} Q^p(t)\,dt < \infty, \tag{0.5}$$

where $Q = Q_0$. In the stationary case, this condition is the natural condition obtained by an interpolation between the condition of Doukhan, Massart and Rio (1994) implying the CLT [condition (0.5) with p = 2] for strong mixing sequences and the integrability of $|X_0|$ implying the usual SLLN ($p = 1$) [note that for bounded sequences, (0.5) holds if and only if $\sum_{n>0} n^{p-2}\alpha_n < \infty$]. Since the proof cannot be done via a three series type theorem, we use both our maximal inequality for maxima of partial sums and adequate truncations of the random variables to prove this result. Moreover, we prove that this condition cannot be improved for power-type rates of strong mixing. We also apply this method to the strong law of large numbers. For example, we prove that if $(X_i)_{i\in\mathbb{Z}}$ is a strongly mixing sequence of real-valued identically distributed random variables such that

$$\mathbb{E}(|X_0|\log^+|X_0|) < \infty, \tag{0.6}$$

the strong law of large numbers holds as soon as $\alpha_n = O(n^{-\varepsilon})$ for some positive ε. As far as we know, this result is new. Moreover, we prove that this condition cannot be improved.

1. Definitions and results. Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued random variables. In order to state our main inequality, we introduce the following coefficients of dependence, which can easily be compared with the strong mixing coefficients of Rosenblatt (1956) [see Lemma 1 in this section].

Measures of weak dependence. For any positive x, let $\mathscr{L}_p(x)$ denote the family of functions from $\mathbb{R}^p$ into $[-1, 1]$ such that, for any $z = (z_1, \dots, z_p)$ and any $y = (y_1, \dots, y_p)$ in $\mathbb{R}^p$,

$$|f(z) - f(y)| \le \frac{1}{x} \sum_{i=1}^{p} |z_i - y_i|. \tag{1.0}$$

Let $\mathscr{L}_p^0(x)$ be the set of functions f in $\mathscr{L}_p(x)$ such that $f(z_1, \dots, z_{p-1}, 0) = 0$. For any positive integer n, we set

$$\gamma_{p,n}(x) = \sup_{f \in \mathscr{L}_p(x)} \left|\operatorname{Cov}\big(f(X_1, \dots, X_p), X_{p+n}\big)\right|, \tag{1.1a}$$

$$\delta_{p,n}(x) = \sup_{f \in \mathscr{L}_p^0(x)} x\left|\operatorname{Cov}\big(f(X_1, \dots, X_p), X_{p+n}\big)\right|. \tag{1.1b}$$

Now we recall some basic definitions. For any two σ-algebras $\mathscr{A}$ and $\mathscr{B}$ in $(\Omega, \mathscr{T}, \mathbb{P})$, let

$$\begin{aligned}\alpha(\mathscr{A}, \mathscr{B}) &= \sup_{(A,B) \in \mathscr{A} \times \mathscr{B}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \\ &= \sup_{(A,B) \in \mathscr{A} \times \mathscr{B}} |\operatorname{Cov}(\mathbb{1}_A, \mathbb{1}_B)|\end{aligned}$$

denote the strong mixing coefficient introduced by Rosenblatt (1956) [note that $\alpha(\mathscr{A}, \mathscr{B}) \le 1/4$]. The strong mixing coefficients α_n of the sequence $(X_i)_{i \in \mathbb{Z}}$ are defined by

$$\alpha_n = \sup_{k \in \mathbb{Z}} \alpha(\mathscr{F}_{k-n}, \mathscr{G}_k) \tag{1.2}$$

for any positive n, where $\mathscr{F}_l = \sigma(X_i\colon i \le l)$ and $\mathscr{G}_l = \sigma(X_l)$. We make the convention that $\alpha_0 = 1/4$. Let us also recall the definition of the β-mixing coefficients of $(X_i)_{i \in \mathbb{Z}}$ [see Rozanov and Volkonskii (1959)]. Given two σ-fields $\mathscr{A}$ and $\mathscr{B}$ in $(\Omega, \mathscr{T}, \mathbb{P})$, the β-mixing coefficient $\beta(\mathscr{A}, \mathscr{B})$ between $\mathscr{A}$ and $\mathscr{B}$ is defined by

$$\beta(\mathscr{A}, \mathscr{B}) = \tfrac{1}{2} \sup\left\{ \sum_{i \in I} \sum_{j \in J} \left|\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)\right| \right\},$$

where the supremum is taken over finite partitions $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$, respectively, $\mathscr{A}$ and $\mathscr{B}$ measurable, and the β-mixing coefficients β_n of the sequence $(X_i)_{i \in \mathbb{Z}}$ are defined by $\beta_n = \sup_{k \in \mathbb{Z}} \beta(\mathscr{F}_{k-n}, \mathscr{G}_k)$. The following inequalities between these coefficients hold:

$$1 \ge \beta_n \ge 2\alpha_n. \tag{1.3}$$

Rates of convergence in the strong law for mixing sequences. In this subsection, we state the extension of the Marcinkiewicz–Zygmund strong law of large numbers to strongly mixing sequences.

THEOREM 1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strongly mixing sequence of real-valued integrable random variables. We set $S_n = \sum_{i=1}^n (X_i - \mathbb{E}(X_i))$. Let F denote the unique distribution function of a nonnegative random variable X such that*

$$\mathbb{P}(X > t) = \sup_{i>0} \mathbb{P}(|X_i| > t) \quad \text{for all } t \in \mathbb{R}.$$

(i) *Let $p \in]1,2[$ and $S_n = \sum_{i=1}^n X_i$. Assume that* (0.5) *holds with $Q = Q_X = Q_F$. Then $n^{-1/p}S_n$ converges a.s. to* 0.

(ii) *Assume that $Q = Q_X$ satisfies*

$$\int_0^1 Q(t)\log(1 + \alpha^{-1}(t/2))\,dt < \infty. \tag{1.4}$$

Then S_n/n converges a.s. to 0.

COMMENTS. Let U be a r.v. with uniform distribution over $[0,1]$. Then $Q(U)$ has the same law as X. When the sequence $(X_i)_{i\in\mathbb{Z}}$ is m-dependent, $\alpha_n = 0$ for any $n \geq m$. Then $\alpha^{-1}(t/2) \leq m$. Hence (0.5) holds as soon as

$$\int_0^1 Q^p(t)\,dt = \mathbb{E}(X^p) < \infty.$$

Let us give another formulation of (0.5). Since $\alpha^{-1}(u) = n$ for any u in $[\alpha_n, \alpha_{n-1}[$,

$$[\alpha^{-1}(t/2)]^{p-1} \leq c_p \sum_{n\geq 0} (n+1)^{p-2}\mathbb{1}_{t<2\alpha_n}$$

for any p in $]1,2]$. It follows that (0.5) holds true if and only if

$$\sum_{n\geq 0} (n+1)^{p-2} \int_0^{2\alpha_n} Q^p(t)\,dt < \infty. \tag{1.5}$$

In the same way, one can prove that (1.4) holds true if

$$\sum_{n\geq 0} (n+1)^{-1} \int_0^{2\alpha_n} Q(t)\,dt < \infty. \tag{1.6}$$

We now compare Theorem 1 with earlier results. As we will see below, the main advantages of conditions (0.5) and (1.4) is that they give an unified approach of complete convergence for strongly mixing sequences. Furthermore these conditions cannot be improved, as shown by Theorems 2 and 3.

Applications of Theorem 1. Here we discuss the scope of conditions (0.5) and (1.4). We first treat the case of uniformly bounded random variables.

1. *Bounded random variables*. Assume that $\|X\|_\infty = M < \infty$. Then Q takes its values in $[0, M]$, and (1.5) holds if and only

$$\sum_{n>0} n^{p-2}\alpha_n < \infty.$$

Consequently, Theorem 1(i) generalizes Theorem 1.1 of Berbee (1987) to strongly mixing sequences.

Under the above assumption on the distribution of the random variables, (1.4) is equivalent to the mixing condition $\sum_{n>0} n^{-1}\alpha_n < \infty$, which gives another proof of Theorem 1.2 in Berbee (1987). However, in this case the strong law of large numbers is a direct consequence of Theorems 3 and 7 of Gál and Koksma (1950) via Ibragimov's covariance inequality for bounded random variables.

2. *Tail conditions.* Suppose that for some r in $]p, \infty]$,

$$\mathbb{P}(X > t) = O(t^{-r}) \quad \text{as } t \to \infty. \tag{1.7}$$

Then $Q(t) = O(t^{-1/r})$ as u tends to 0. So (1.5) and (0.5) hold if

$$\sum_{n>0} n^{p-2}\alpha_n^{1-p/r} < \infty. \tag{1.8}$$

Let us compare this result with Theorem 1 in Shao (1993): (1.8) holds as soon as $\alpha_n = O(n^{-r(p-1)/(r-p)}(\log n)^{-\beta})$ for some $\beta > r/(r-p)$, while Shao's result (applied with $\alpha p = 1$) needs $\beta > rp/(r-p)$ [note that Shao's moment assumption $\sup_{i>0}\|X_i\|^r < \infty$ implies (1.7)]. Moreover, it follows from Example 1 in Shao (1993) that the strong law of order p does not remain valid if $\beta = r/(r-p)$.

Now, by Theorem 1(ii) and (1.6), the SLLN holds if

$$\sum_{n>0} n^{-1}\alpha_n^{(r-1)/r} < \infty.$$

For example, this condition holds if $\alpha_n = O((\log n(\log\log n)^{\theta})^{-r/(r-1)})$ for some $\theta > 1$, which improves on Corollary 1 in Shao (1993). Moreover, it follows from Example 2 in Shao (1993) that the power of $\log n$ appearing here cannot be improved.

3. *Hölder spaces.* Assume now that $\mathbb{E}(X^r) < \infty$. Since this condition is equivalent to the integrability condition

$$\int_0^1 Q^r(t)\,dt < \infty, \tag{1.9}$$

it follows from Hölder's inequality that (0.5) holds as soon as

$$\sum_{k>0} k^{(r(p-2)+p)/(r-p)}\alpha_k < \infty. \tag{1.10}$$

For example, if (1.9) holds with $r = p/(2-p)$, (1.10) is equivalent to (0.3). Let us compare this result with Theorem 1 in Shao (1993): (1.10) holds if $\alpha_n = O(n^{-r(p-1)/(r-p)}(\log n)^{-\beta})$ for some $\beta > 1$, while Shao's result (applied with $\alpha p = 1$) needs $\beta > rp/(r-p)$. However, his moment assumption $\sup_{i>0}\|X_i\|_r < \infty$ is weaker than (1.9) in the nonstationary case.

Now, by the Hölder inequality, (1.4) holds if

$$\int_0^1 \left(\log(1 + \alpha^{-1}(t/2))\right)^{r/(r-1)} < \infty.$$

This condition is equivalent to the summability condition

$$\sum_{n>0} n^{-1}(\log n)^{1/(r-1)}\alpha_n < \infty.$$

For example the SLLN holds as soon as $\alpha_n = O((\log n)^{-r/(r-1)}(\log\log n)^{-\theta})$ for some $\theta > 1$.

4. *Exponential mixing rates.* Assume now that $\alpha_n = O(a^n)$ for some a in $]0, 1[$. Using the Young inequality in Orlicz spaces [see Dellacherie and Meyer (1975) and Rio (1993), pages 596 and 593, for its application in the context of this paper], we get that (0.5) holds if

$$\mathbb{E}\left(X^p(\log^+ X)^{p-1}\right) < \infty. \tag{1.11}$$

In the same way, one can prove that (1.4) holds as soon as

$$\mathbb{E}(X \log^+ \log^+ X) < \infty.$$

5. *Power-type mixing rates.* Suppose that $\alpha_n = O(n^{-a})$ for some positive a. Then (0.5) holds if

$$\int_0^1 t^{(1-p)/a} Q^p(t)\, dt < \infty. \tag{1.12}$$

This condition differs in a fundamental way from a moment condition on the random variable X. So we believe that the integral conditions proposed in this paper give a more intrinsic approach to limit theorems for weakly dependent random variables than Ibragimov's approach (1962).

By contrast, the Young inequality [see Rio (1993), pages 596 and 593, for its application in this context] ensures that condition (1.4) implying the strong law holds true if $\mathbb{E}(X \log^+ X) < \infty$, as noted in the Introduction. Let us compare this result with Theorems 2 and 3 of Birkel (1992) when the random variables are identically distributed. Set $G(s) = \mathbb{P}(X > t)$. Then

$$\begin{aligned}&\left|\mathbb{P}(X_i > s, X_j > t) - G(s)G(t)\right| \\ &\qquad \le \alpha_{|i-j|} \wedge G(s) \wedge G(t) \wedge (1 - G(s)) \wedge (1 - G(t)).\end{aligned}$$

Moreover, this inequality is optimal, up to some constant factor [see (b) of Theorem 1.1. in Rio (1993)]. Hence Birkel's results need the integrability of the upper bound, which is equivalent to $\mathbb{E}(X^2) < \infty$ [see Fréchet (1951, 1957) or Rio (1993)]. However, under the stronger assumption of ψ-mixing, Theorem 2 of Birkel applied to the random variables $\bar{X}_i = X_i \mathbb{1}_{|X_i| \le i}$ yields the strong law of large under the minimal moment assumption $\mathbb{E}(X) < \infty$ [see Example 3, page 360, in Birkel (1992)].

6. *Logarithmic mixing rates and the strong law.* Suppose that

$$\alpha_n = O(\log^{-b} n)$$

for some $b > 1$. Then (1.4) holds if

$$\int_0^1 t^{-1/b} Q(t)\, dt < \infty. \tag{1.13}$$

For example, (1.13) holds if

$$\mathbb{P}(X > t) = O\big(t^{-b/(b-1)} \log^{-\beta} t\big) \quad \text{as } t \to \infty$$

for some $\beta > b/(b-1)$.

Now, using a counterexample introduced in Doukhan, Massart and Rio (1994), we can prove the following partial converse of Theorem 1 (i) for stationary sequences and power-type rates of mixing.

THEOREM 2. *Let p be any real in $]1,2[$. Let $a > 0$ and $b \in \mathbb{R}$ be given. Let F be any continuous distribution function of a zero-mean real-valued random variable such that*

$$\text{(a)} \qquad \int_0^{1/2} \big(t \log^b(1/t)\big)^{(1-p)/a} Q_F^p(t)\, dt = +\infty.$$

Then there exists a stationary Markov chain $(X_i)_{i\in\mathbb{Z}}$ of r.v.'s with d.f. F such that

$$0 < \liminf_{n\to+\infty} n^a (\log n)^b \alpha_n \le \limsup_{n\to+\infty} n^a (\log n)^b \beta_n < \infty$$

and

$$\text{(b)} \qquad \limsup_{n\to+\infty} \frac{|\sum_{i=1}^n X_i|}{n^{1/p}} = +\infty \quad a.s.$$

COMMENTS. In Section 4, we will sketch the proof of Theorem 2 in the case $b = 0$. The proof in the case $b \neq 0$ uses exactly the same arguments.

Let us apply Theorem 2 with $a = r(p-1)/(r-p)$ and $b = 1$. In this case, the function $t \to (\alpha^{-1}(t))^{p-1}$ does not belong to the Hölder space $L^{r/(r-p)}([0,1])$. Hence, by the Fischer–Riesz theorem and a classical result of Fréchet [see Fréchet (1957), page 691, lines 6–14] there exists some continuous d.f. F satisfying (a) and such that $\int_0^1 |x|^r\, dF(x) = 1$, which gives a positive answer to a conjecture of Shao (1993), page 287] and a counterexample to Hipp's result in the unbounded case.

Now we state a theorem, which proves that Theorem 1(ii) cannot be improved in the case of moderate rates of mixing.

THEOREM 3. *Let $(\varphi_n)_{n>0}$ be a sequence of reals in $]0,1]$ decreasing to 0 and let φ^{-1} be the cadlag inverse function of $t \to \varphi_{[t]}$. Assume that $n \to n^\theta \varphi_n$ is nondecreasing for some large enough θ. We set $\varphi_0 = 1$. Let F be any* *continuous distribution function of an integrable random variable with mean zero such that*

$$\text{(a)} \qquad \int_0^1 Q_F(t)\log(1+\varphi^{-1}(t))\,dt = +\infty.$$

Then there exists a sequence $(X_i)_{i\in\mathbb{Z}}$ *of random variables with d.f. F such that*

$$\text{(b)} \qquad 0 < \liminf_{n\to+\infty}(\alpha_n/\varphi_n) \le \limsup_{n\to+\infty}(\alpha_n/\varphi_n) < \infty$$

and

$$\text{(c)} \qquad \limsup_{n\to+\infty}\frac{|\sum_{i=1}^n X_i|}{n} > 0 \quad a.s.$$

COMMENTS. (a) is equivalent to $\sum_{n\ge 0}(n+1)^{-1}\int_0^{\varphi_n}Q(t)\,dt = +\infty$. Since Q is nonincreasing, it follows that (a) holds if and only if (1.6) and (1.4) are violated. Hence Theorem 3 is a converse of Theorem 1(ii) for power-type mixing rates or logarithmic mixing rates.

Maximal inequalities, convergence of series of dependent r.v.'s. We start with the key inequality of the paper, which is Theorem 4.

THEOREM 4. *Let* $(X_i)_{i\in\mathbb{Z}}$ *be a sequence of real-valued random variables with finite variance. Let* $S_n = \sum_{i=1}^n(X_i - \mathbb{E}(X_i))$ *and* $S_n^* = \sup_{p\le n}|S_p|$. *For any positive* x,

$$\text{(a)} \qquad \mathbb{P}(S_n^* \ge 2x) \le \frac{2}{x^2}\sum_{1=1}^{n}\mathbb{E}(X_i^2) + \frac{4}{x}\sum_{i=1}^{n-1}\gamma_{i,1}(x).$$

Moreover, for any positive p and k,

$$\text{(b)} \qquad \gamma_{p,1}(x) \le \gamma_{p+1-k,k}(x) + \frac{2}{x}\sum_{i=1}^{k-1}\delta_{p+1-i,i}(x)$$

with the conventions that $\gamma_{l,p}(x) = 0$ *and* $\delta_{l,p}(x) = 0$ *for any* $l \le 0$.

Now we quote a lemma, which allows us to estimate the dependence coefficients $\delta_{p,n}(x)$ and $\gamma_{p,n}(x)$. The proof is mainly based on Theorem 1.1 in Rio (1993) and will be carried out in the Appendix.

LEMMA 1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of square-integrable random variables. Then*

$$\text{(a)} \qquad \delta_{p,n}(x) \le 2\int_0^{2\alpha_n} Q_p(t)Q_{p+n}(t)\,dt,$$

$$\text{(b)} \qquad \gamma_{p,n}(x) \le 2\int_0^{2\alpha_n} Q_{p+n}(t)\,dt.$$

Next we can derive from Theorem 4 the following criterion implying the almost sure convergence of the series $\sum_{i>0} X_i$.

COROLLARY 1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued zero-mean random variables with finite variance. Then the series $\sum_{i=1}^{\infty} X_i$ converges if*

$$\text{(a)} \qquad \sum_{i=1}^{\infty} \int_0^1 \alpha^{-1}(t/2)Q_i^2(t)\,dt < +\infty.$$

COMMENT. If $(X_i)_{i\in\mathbb{Z}}$ is an m-dependent sequence, (a) is equivalent to Kolmogorov's condition $\sum_{i>0}\mathbb{E}(X_i^2) < \infty$.

Applications of Corollary 1. Let us discuss the scope of (a). If the random variables X_i are defined in such a way that $X_i = c_i Z_i$ for some stationary sequence $(Z_i)_{i\in\mathbb{Z}}$ and if $\sum_{i>0} c_i^2 < \infty$, (a) holds as soon as

$$\int_0^1 \alpha^{-1}(t/2)Q_{Z_0}^2(t)\,dt < \infty. \tag{1.14}$$

For example, when $\mathbb{E}(|Z_0|^r) < \infty$, (1.14) holds if

$$\sum_{n>0} n^{2/(r-2)}\alpha_n < \infty,$$

which is a weaker condition than condition (b) in Remark (2.6) of McLeish (1975) (note also that this condition is weaker than the classical condition of Ibragimov implying the CLT).

The organization of the paper is the following: in Section 2, we prove the maximal inequality. Sections 3 and 4 are devoted to the strong laws for strongly mixing sequences. In Section 5, we apply the maximal inequality to series of dependent random variables. We prove Lemma 1 in the Appendix.

2. The maximal inequality. In this section, we prove Theorem 4. We start by the proof of (a). The guideline of the proof is Section 2 in Sakhanenko (1986). Set

$$S_{0,x} = 0, \qquad S_{p,x} = S_p/x \quad\text{and}\quad S_{n,x}^* = \sup_{p\le n} S_{p,x}.$$

Let the function g be defined by $g(y) = y - 1$ for any y in $[1, 2]$, $g(y) = 0$ for any $y \leq 1$ and $g(y) = 1$ otherwise:

$$\mathbb{P}\Big(\sup_{p \leq n} S_p \geq 2x\Big) \leq \mathbb{E}(g(S^*_{n,x})). \tag{2.1}$$

Let the nonnegative differentiable function f be defined by $f(y) = 0$ if $y \leq 0$, $f(y) = y^2$ if $y \in [0, 1]$ and $f(y) = 2y - 1$ if $y \geq 1$. If $g(S^*_{i,x}) - g(S^*_{i-1,x}) > 0$, $S_{i,x} \geq 1$, which implies that $2S_{i,x} - 1 \geq 1$. Hence

$$\begin{aligned} g(S^*_{n,x}) &= \sum_{i=1}^{n} \big(g(S^*_{i,x}) - g(S^*_{i-1,x})\big) \\ &\leq \sum_{i=1}^{n} (2S_{i,x} - 1)\big(g(S^*_{i,x}) - g(S^*_{i-1,x})\big) \\ &\leq f(S_{n,x}) - \frac{2}{x} \sum_{i=1}^{n} g(S^*_{i-1,x})(X_i - \mathbb{E}(X_i)). \end{aligned}$$

Now, by the Taylor formula,

$$\begin{aligned} f(S_{n,x}) &= \sum_{i=1}^{n} \big(f(S_{i,x}) - f(S_{i-1,x})\big) \\ &\leq \frac{1}{x^2} \sum_{i=1}^{n} (X_i - \mathbb{E}(X_i))^2 + \frac{1}{x} \sum_{i=1}^{n} f'(S_{i-1,x})(X_i - \mathbb{E}(X_i)). \end{aligned}$$

Let $h(X_1, \dots, X_{i-1}) = f'(S_{i-1,x})/2 - g(S^*_{i-1,x})$. Collecting the two above inequalities, we get that

$$\mathbb{E}(g(S^*_{n,x})) \leq \frac{1}{x^2} \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \frac{2}{x} \sum_{i=1}^{n} \operatorname{Cov}(h(X_1, \dots, X_{i-1}), X_i). \tag{2.2}$$

Some elementary calculations show that h is in $\mathscr{L}_{i-1}(x)$. Hence (2.2) together with (2.1) yields Theorem 4(a). Theorem 4(b) follows from the elementary equality

$$\begin{aligned} f(X_1, \dots, X_p) = f(X_1, \dots, X_{p+1-k}) + \sum_{i=0}^{k-2} \big(f(X_1, \dots, X_{p-i}, 0, \dots, 0)\big) \\ + \big(f(X_1, \dots, X_{p-i-1}, 0, \dots, 0)\big), \end{aligned}$$

which shows that $\mathscr{L}_p(x) \subset \mathscr{L}_{p+1-k}(x) = (2/x)\Sigma_{i=0}^{k-2} \mathscr{L}^0_{p-i}(x)$.

3. Dependent strong laws. In this section, we prove Theorem 1. Theorem 1 follows from the proposition below via the Borel–Cantelli lemma.

PROPOSITION 1. *Under the assumptions of Theorem* 1(i), *for any positive* ε,

$$\sum_{n>0} n^{-1} \mathbb{P}(S^*_n \geq \varepsilon n^{1/p}) < \infty. \tag{a}$$

Under the assumptions of Theorem 1(ii), *for any positive* ε,

$$\sum_{n>0} n^{-1} \mathbb{P}(S^*_n \geq \varepsilon n) < \infty. \tag{b}$$

PROOF. Let $p \in [1, 2[$. By definition of the random variable X,

$$Q(t) = Q_X(t) = \sup_{i>0} Q_i(t). \tag{3.0}$$

Since the random variables X_i fail to have finite second moments, we need to use truncation arguments.

For any real u in $]0, 1[$, let the sequences $(\bar{X}_i)_{i \in \mathbb{Z}}$ and $(\tilde{X}_i)_{i \in \mathbb{Z}}$ be defined by

$$\bar{X}_i = (X_i \wedge Q(u)) \vee (-Q(u)) \quad \text{and} \quad \tilde{X}_i = X_i - \bar{X}_i.$$

Let U be a random variable with uniform distribution over $[0, 1]$. Then $|X_i|$ has the same distribution as $Q_i(U)$. Hence

$$Q_{\bar{X}_i}(t) = Q_i(t) \wedge Q(u) \quad \text{and} \quad Q_{\tilde{X}_i}(t) = (Q_i(t) - Q(u))^+,$$

where $x^+ = \max(0, x)$. Since $Q_i \le Q$, it follows that

$$|\mathbb{E}(\tilde{X}_i)| \le \mathbb{E}(|\tilde{X}_i|) \le \int_0^u (Q(t) - Q(u))\, dt. \tag{3.1}$$

Let $\bar{S}_p = \sum_{i=1}^p (\bar{X}_i - \mathbb{E}(\bar{X}_i))$ and $\bar{S}_n^* = \sup_{p \le n} |\bar{S}_p|$. Since

$$S_n^* \le \bar{S}_n^* + \sum_{i=1}^n \left(|\tilde{X}_i| + |\mathbb{E}(\tilde{X}_i)|\right),$$

it follows from (3.1) that

$$\mathbb{P}(S_n^* \ge 5x) \le \mathbb{P}(\bar{S}_n^* \ge 4x) + \frac{2n}{x} \int_0^u (Q(t) - Q(u))\, du. \tag{3.2}$$

In order to apply Theorem 4(a) to $\bar{S}_n^*$, we need to estimate the coefficients $\gamma_{i,1}(x)$. Let k be some positive integer. Theorem 4(b) and Lemma 1 yield

$$\gamma_{i,1}(x) \le \frac{4}{x} \sum_{i=1}^{k-1} \int_0^{2\alpha_i} Q^2(t \vee u)\, dt + 2\int_0^{2\alpha_k} Q(t \vee u)\, dt. \tag{3.3}$$

Combining inequality (3.3) with Theorem 4(a) and noting that

$$\mathbb{E}(\bar{X}_i^2) = \int_0^1 Q_i^2(t \vee u)\, dt \le 2\int_0^{2\alpha_0} Q^2(t \vee u)\, dt$$

(recall that $\alpha_0 = 1/4$), we get that

$$\begin{aligned} n^{-1}\mathbb{P}(\bar{S}_n^* \ge 4x) &\le \frac{4}{x^2} \sum_{i=0}^{k-1} \int_0^{2\alpha_i} Q^2(t \vee u)\, dt \\ &\quad + \frac{4}{x} \int_0^{2\alpha_k} Q(t \vee u)\, dt. \end{aligned} \tag{3.4}$$

The keystone of the proof is the choice of the parameters k and u. Let the weighted quantile function H be defined by

$$H(t) = \alpha^{-1}(t/2) Q(t). \tag{3.5}$$

In fact, in the mixing case, H plays the role that Q plays in the independent case. This remark leads to the following choice of k:

$$k = \alpha^{-1}(u/2). \tag{3.6}$$

With the above choice of k, $2\alpha_k \le u$. So noting that

$$\sum_{i=0}^{k-1} \int_0^{2\alpha_i} Q^2(t \vee u)\,dt = \int_0^1 \left[\alpha^{-1}(t/2) \wedge k\right] Q^2(t \vee u)\,dt$$
$$\le \int_0^1 H(t \vee u) Q(t)\,dt$$

and combining (3.2) and (3.4), we obtain

$$n^{-1}\mathbb{P}(S_n^* \ge 5x) \le \frac{6}{x}\int_0^u Q(t)\,dt + \frac{4}{x^2}\int_0^1 H(t \vee u) Q(t)\,dt. \tag{3.7}$$

Now we prove that the series appearing in Proposition 1 is bounded up by the integrals of (0.5) or (1.4), up to some constant. Let $x = x_n = \varepsilon n^{1/p}$. We take $u = u_n = H^{-1}(n^{1/p})$ in (3.7). It follows from the "cadlaguity" of H that

$$\left(H(t) \le n^{1/p}\right) \quad \Leftrightarrow \quad (t \ge u_n). \tag{3.8}$$

Hence

$$\int_0^{u_n} H(u_n) Q(t)\,dt \le n^{1/p} \int_0^{u_n} Q(t)\,dt.$$

This implies that

$$\begin{aligned} &n^{-1}\mathbb{P}\left(S_n^* \ge 5\varepsilon n^{1/p}\right) \\ &\quad \le 10\varepsilon^{-2}\left(n^{-1/p}\int_0^{u_n} Q(t)\,dt + n^{-2/p}\int_{u_n}^1 H(t)Q(t)\,dt\right). \end{aligned} \tag{3.9}$$

We now finish the proof of Proposition 1(a). Let $p \in]1,2[$. We set $c_\varepsilon = \varepsilon^2/10$. Summing on n in (3.9), we get

$$c_\varepsilon \sum_{n>0} \frac{1}{n}\mathbb{P}(S_n^* \ge 5x_n)$$
$$\le \int_0^1 Q(t)\left(\sum_{n>0} \frac{\mathbb{1}_{t<u_n}}{n^{1/p}}\right)dt + \int_0^{t_0} H(t)Q(t)\left(\sum_{n>0}\frac{\mathbb{1}_{t\ge u_n}}{n^{2/p}}\right)dt,$$

where $x_n = \varepsilon n^{1/p}$ and $t_0 = \sup\{t \ge 0\colon H(t) > 0\}$. Now, by (3.8), $(t < u_n)$ if and only if $(n < H^p(t))$. Hence

$$\begin{aligned} c_\varepsilon \sum_{n>0}\frac{1}{n}\mathbb{P}(S_n^* \ge 5x_n) &\le \int_0^1 Q(t)\left(\sum_{0<n<H^p(t)} n^{-1/p}\right)dt \\ &\quad + \int_0^{t_0} H(t)Q(t)\left(\sum_{n \ge H^p(t)} n^{-2/p}\right)dt. \end{aligned} \tag{3.10}$$

If p belongs to $]1,2[$,

$$\sum_{0<n<H^p(t)} n^{-1/p} \le c_p H^{p-1}(t) \quad \text{and} \quad \sum_{n \ge H^p(t)} n^{-2/p} \le C_p H^{p-2}(t),$$

which together with (3.10), implies that

$$\sum_{n>0} \frac{1}{n} \mathbb{P}(S_n^* \ge 5\varepsilon n^{1/p}) \le C \int_0^1 H^{p-1}(t) Q(t)\, dt < \infty$$

under assumption (0.5). Hence Proposition 1(a) holds.

If $p = 1$, we need some more truncation arguments. For any $i \in [1, n]$, let

$$Y_i = (X_i \wedge n) \vee (-n) \quad \text{and} \quad \tilde{Y}_i = X_i - Y_i.$$

Let $T_n^* = \sup_{p \le n} \sum_{i=1}^{p} (Y_i - \mathbb{E}(Y_i))$. We may apply inequality (3.9) to the random variables Y_i. Since

$$\mathbb{P}(|Y_i| > t) \le \mathbb{P}(X \wedge n > t),$$

we have

$$\sup_{i \in [1,n]} Q_{Y_i} \le Q \wedge n.$$

Hence, by (3.9),

$$\begin{aligned} &n^{-1}\mathbb{P}(T_n^* \ge 5\varepsilon n) \\ &\quad \le c_\varepsilon^{-1}\left(n^{-1} \int_0^{u_n} (Q(t) \wedge n)\, dt + n^{-2} \int_{u_n}^1 H(t) Q(t)\, dt \right). \end{aligned} \tag{3.11}$$

Let $\Gamma = \bigcup_{i=1}^{n} (X_i \ne Y_i)$. For any $\omega \notin \Gamma$,

$$S_n^*(\omega) \le T_n^*(\omega) + \sum_{i=1}^{n} \mathbb{E}(|Y_i - X_i|). \tag{3.12}$$

Clearly,

$$\mathbb{P}(\Gamma) \le \sum_{i=1}^{n} \mathbb{P}(|X_i| > n) \le n\mathbb{P}(X > n)$$

and

$$\sum_{i=1}^{n} \mathbb{E}(|Y_i - X_i|) = \sum_{i=1}^{n} \int_n^\infty \mathbb{P}(|X_i| > t)\, dt \le n\mathbb{E}(\sup(0, X - n)).$$

Since $\mathbb{E}(\sup(0, X-n)) \le \varepsilon$ for n large enough, it follows that there exists some integer n_0 such that, for any $n \ge n_0$,

$$\begin{aligned} &n^{-1}\mathbb{P}(S_n^* \ge 6\varepsilon n) \le \mathbb{P}(X > n) \\ &\qquad + c_\varepsilon^{-1}\left(n^{-1} \int_0^{u_n} (Q(t) \wedge n)\, dt + n^{-2} \int_{u_n}^1 H(t) Q(t)\, dt \right). \end{aligned} \tag{3.13}$$

Let $v_n = Q^{-1}(n) = \mathbb{P}(X > n)$. Since $v_n \le t < u_n$ if and only if $Q(t) \le n < H(t)$,

$$n^{-1} \int_0^{u_n} (Q(t) \wedge n)\, dt = \mathbb{P}(X > n) + n^{-1} \int_0^1 Q(t)\, \mathbb{1}_{Q(t) \le n < H(t)}\, dt.$$

Hence there exists some $c > 0$ such that

$$\frac{c}{n}\mathbb{P}(S_n^* \geq 6\varepsilon n) \leq \mathbb{P}(X > n) + \frac{1}{n}\int_0^1 Q(t)\,\mathbb{1}_{Q(t)\leq n < H(t)}\,dt + \frac{1}{n^2}\int_0^{t_0} H(t)Q(t)\,\mathbb{1}_{n\geq H(t)}\,dt. \tag{3.14}$$

Since

$$\sum_{Q(t)\leq n < H(t)} n^{-1} \leq 1 + \log(1 + \alpha^{-1}(t/2))$$

and

$$\sum_{n\geq H(t)} n^{-2} \leq 2/H(t),$$

Proposition 1(b) follows. □

4. On the optimality of the strong laws. In this section, we prove Theorems 2 and 3. The proof of Theorem 2 is based on the counterexample introduced in Doukhan, Massart and Rio (1994). As in Doukhan, Massart and Rio, we start by proving a theorem which works for β-mixing sequences as well.

THEOREM 5. *For any positive a, there exists a stationary Markov chain $(U_i)_{i\in\mathbb{Z}}$ of r.v.'s with uniform distribution over $[0,1]$ and a sequence of β-mixing coefficients $(\beta_n)_{n>0}$, such that*:

(i) $0 < \liminf_{n\to+\infty} n^a\beta_n \leq \limsup_{n\to+\infty} n^a\beta_n < \infty$.

(ii) *For any $p > 1$ and any Borel measurable and integrable function $f\colon [0,1] \to \mathbb{R}$ with mean zero such that*

$$\int_0^1 u^{(1-p)/a}|f(u)|^p\,du = +\infty, \tag{a}$$

we have

$$\limsup_{n\to+\infty} n^{-1/p}\left|\sum_{i=1}^n f(U_i)\right| = +\infty \quad a.s.$$

PROOF. The sequence $(U_i)_{i\in\mathbb{Z}}$ will be defined by means of a strictly stationary Markov chain $(Z_i)_{i\in\mathbb{Z}}$. Let λ denote the Lebesgue measure on $[0,1]$. We define the probability laws μ and ν by

$$\mu = (1+a)x^a\lambda \quad \text{and} \quad \nu = ax^{a-1}\lambda. \tag{4.1}$$

The conditional distribution $\Pi(x,\cdot)$ of Z_{n+1}, given $(Z_n = x)$, is defined by

$$\Pi(x,\cdot) = \Pi(\delta_x,\cdot) = (1-x)\delta_x + x\mu. \tag{4.2}$$

Then ν is the unique invariant probability of the chain with transition probabilities $\Pi(x,\cdot)$ [see Doukhan, Massart and Rio (1994)]. Let $(Z_i)_{i\in\mathbb{Z}}$ be the stationary Markov chain with transition probabilities $\Pi(x,\cdot)$ and law ν:

we set $U_i = Z_i^a$. The random variables U_i have uniform distribution over $[0, 1]$. Moreover, by Lemma 2 in Doukhan, Massart and Rio (1994), Theorem 5(i) holds.

Let $S_n(f) = \sum_{i=1}^n f(U_i)$. The stopping times $(T_k)_{k \geq 0}$ are defined by

$$T_0 = \inf\{i > 0: Z_i \neq Z_{i-1}\} \text{ and } T_k = \inf\{i > T_{k-1}: Z_i \neq Z_{i-1}\}.$$

for $k > 0$. Let $\tau_k = T_{k+1} - T_k$. The r.v.'s $(Z_{T_k}, \tau_k)_{k>0}$ are i.i.d., Z_{T_k} has law μ and the conditional distribution of τ_k for given $Z_{T_k} = z$ is the geometric distribution $\mathscr{G}(1 - z)$. Hence τ_1 is integrable and $\lim_n T_n/n = \mathbb{E}(\tau_1) > 0$ a.s. [these assertions are proved in Doukhan, Massart and Rio (1994), page 78, lines 13–23].

Suppose that assumption (a) holds. Then $\mathbb{E}(|\tau_1 f(U_{T_1})|^p) = \infty$. Since the random variables $(\tau_n f(U_{T_n}))_{n>0}$ are i.i.d., we infer that

$$\limsup_{n\to\infty} T_n^{-1/p}|S_{T_n}(f)| = \left(\mathbb{E}(\tau_1)\right)^{-1/p} \limsup_{n\to\infty} n^{-1/p}|S_{T_n}(f)| = +\infty \quad \text{a.s.}$$

Hence Theorem 5 holds. □

PROOF OF THEOREM 2. Under the assumptions of Theorem 2, either

$$\int_0^{1/2} u^{(1-p)/a}|F^{-1}(u)|^p\, du = +\infty$$

or

$$\int_0^{1/2} u^{(1-p)/a}|F^{-1}(1-u)|^p\, du = +\infty.$$

So setting $X_i = F^{-1}(U_i)$ in the first case and $X_i = F^{-1}(1 - U_i)$ otherwise, we obtain the existence of a stationary Markov chain of real-valued random variables satisfying Theorem 2(a) and (b) with $b = 0$.

It remains to prove that $\liminf_{n\to+\infty} n^a \alpha_n > 0$. We do not detail the proof of this assertion, since it is sufficient to adapt the arguments of the proof of Corollary 1 in Doukhan, Massart and Rio (1994). □

PROOF OF THEOREM 3. Under assumption (a), either

$$\int_0^{1/2} |F^{-1}(1-t)| \log\left(1 + \varphi^{-1}(t)\right) dt = +\infty$$

or

$$\int_0^{1/2} |F^{-1}(t)| \log\left(1 + \varphi^{-1}(t)\right) dt = +\infty.$$

Hence there is no loss of generality in assuming that

$$\int_0^{1/2} |F^{-1}(t)| \log\left(1 + \varphi^{-1}(t)\right) dt = +\infty.$$

We first prove that (a) is equivalent to the divergence condition

$$\sum_{N>0} \int_{2^{-N}\varphi(2^N)}^{\varphi(2^N)} |F^{-1}(t)|\, dt = +\infty. \tag{4.3}$$

PROOF. Let $\psi(n) = n^{-1}\varphi(n)$. Clearly

$$\sum_{N>0} \int_{2^{-N}\varphi(2^N)}^{\varphi(2^N)} |F^{-1}(t)|\, dt = \int_0^1 \sum_{N>0} \left(\mathbb{1}_{\varphi(2^N)>t} - \mathbb{1}_{\psi(2^N)>t}\right)|F^{-1}(t)|\, dt.$$

Let $\log_2(x) = \log x/\log 2$.

$$\sum_{N>0} \mathbb{1}_{\varphi(2^N)>t} = \max\{N > 0 : 2^N \le \varphi^{-1}(t) - 1\} = \left[\log_2\left(\varphi^{-1}(t) - 1\right)\right],$$

where the square brackets designate the integer part. The same equality holds for $\sum_{N>0}\mathbb{1}_{\psi(2^N)>t}$. Hence the convergence of the series in (4.3) is equivalent to the convergence of the integral

$$\int_0^{1/2} |F^{-1}(t)| \log\left(\frac{1 + \varphi^{-1}(t)}{1 + \psi^{-1}(t)}\right) dt.$$

Since $n \to n^{\theta}\varphi_n$ is nondecreasing, it can be proven that, for some $\eta \in]0,1[$ and some $t_0 > 0$,

$$1 + \psi^{-1}(t) \le \left(1 + \varphi^{-1}(t)\right)^{\eta} \quad \text{for all } t \le t_0.$$

Hence (4.3) is equivalent to condition (a). □

In order to prove Theorem 3, we need some more notation.

NOTATIONS. We set $\varphi(2^N) = \theta_N$. Let the sequence $(k_N)_{N>0}$ be defined by

$$k_N = 1 + \left[\frac{1}{\theta_N} \sum_{2^{-N}\theta_N}^{\theta_N} |F^{-1}(t)|\, dt\right], \tag{4.4}$$

where the square brackets designate the integer part. Let the increasing sequence $(l_N)_{N>0}$ be defined by $l_0 = 0$ and $l_{N+1} = l_N + 2^N k_N$.

Exactly as in the proof of Theorem 2, we start by defining a sequence $(U_i)_{i>0}$ of random variables with the uniform distribution over $[0,1]$.

Let $(V_i)_{i>0}$ be a sequence of independent random variables with uniform distribution over $[0,1]$, and let $(W_{N,k})_{N>0,\, 0<k\le k_N}$ be an array of independent random variables with uniform distribution over $[0,1]$, independent of $(V_i)_{i>0}$.

The keystone of the proof is that we can choose the random variables U_i in such a way that the supremum of the partial sums on the subintervals of length 2^N included in $]k_N, k_{N+1}]$ will be strongly related to the integral defining k_N.

Let $R(x) = x - [x]$. For any $N > 0$, any $k \in]0, k_N]$ and any $i \in]l_N + (k-1)2^N, l_N + k2^N]$, we set

$$\begin{aligned} U_i = {} & \left(\theta_N + (1-\theta_N)V_i\right)\mathbb{1}_{W_{n,k}>\theta_N} \\ & + \theta_N R\left(2^{-N}(i - l_N) + \theta_N^{-1} W_{N,k}\right)\mathbb{1}_{W_{N,k}\le\theta_N}. \end{aligned} \tag{4.5}$$

Then the random variables U_i have uniform distribution over $[0,1]$. Moreover, it is not difficult to check that there exists some positive constants c_1 and c_2 such that

$$c_1\varphi(2^N) \le \alpha_n \le c_2\varphi(2^N) \quad \text{for all } n \in [2^{N-1}, 2^N[. \tag{4.6}$$

Hence Theorem 3(b) holds.

Now we prove (c). We set

$$X_i = F^{-1}(U_i).$$

It follows from (4.3) that the series $\sum_{N>0}\theta_N k_N$ is divergent. Since the random variables $W_{N,k}$ are independent, we may apply the converse Borel–Cantelli lemma, yielding

$$\mathbb{P}\big(\{\exists\, k \in]0, k_N] \text{ such that } W_{N,k} \le \theta_N\} \text{ i.o. } N\big) = 1. \tag{4.7}$$

Let $(N,k) = (N(\omega), k(\omega))$ be a bivariate integer such that $W_{N,k} \le \theta_N$. In order to prove (c), we have to estimate

$$\Delta_{N,k} = S_{l_N + k2^N} - S_{l_N+(k-1)2^N}. \tag{4.8}$$

Let us divide $W_{N,k}$ by $2^{-N}\theta_N$:

$$W_{N,k} = p2^{-N}\theta_N + r_{N,k} \quad \text{for some } p \in \mathbb{N} \text{ and some } r_{N,k} \in [\,0, 2^{-N}\theta_N[\,. \tag{4.9}$$

It follows from (4.5) that

$$\Delta_{N,k} = \sum_{j=0}^{2^N-1} F^{-1}\big(r_{N,k} + j2^{-N}\theta_N\big).$$

Since F^{-1} is a nondecreasing function, we infer that

$$\Delta_{N,k} \le \frac{2^N}{\theta_N}\int_{2^{-N}\theta_N}^{\theta_N + 2^{-N}\theta_N} F^{-1}(t)\,dt. \tag{4.10}$$

Let us discuss the sign of $F^{-1}(t)$. Since $E(X_1) = \int_0^1 F^{-1}(t)\,dt = 0$, there exists some $t_0 > 0$ such that $t \le t_0$ implies $F^{-1}(t) < 0$. Let N_0 be the first integer such that $2\theta_{N_0} \le t_0$. For any $N \ge N_0$, $F^{-1}(t) < 0$ for all $t \in]0, 2\theta_N]$, which implies that

$$|\Delta_{N,k}| = -\Delta_{N,k} \ge \frac{2^N}{\theta_N}\int_{2^{-N}\theta_N}^{\theta_N} |F^{-1}(t)|\,dt = 2^N D_N. \tag{4.11}$$

Taking (4.7) and (4.11) into account, it remains to prove that

$$\liminf_{N\to\infty} \frac{2^N D_N}{l_{N+1}} > 0. \tag{4.12}$$

Since $t \to |F^{-1}(t)|$ is a nonincreasing function on $]0, 2\theta_{N_0}]$, the sequence $(k_N)_{N \ge N_0}$ is nondecreasing. Hence

$$\limsup_{N\to\infty} \frac{l_{N+1}}{2^N k_N} \le 2. \tag{4.13}$$

Now, by (4.4), $k_N = 1 + [D_N]$. Hence

$$\liminf_{N\to\infty} \frac{2^N D_N}{l_{N+1}} \geq \frac{1}{2} \liminf_{N\to\infty} \frac{D_N}{1+[D_N]} \geq \lim_{t\to 0^+} \frac{|F^{-1}(t)|}{2(1+|F^{-1}(t)|)} > 0,$$

which completes the proof of (c). □

5. Convergence of series of strongly mixing random variables. In this section, we prove Corollary 1. By Theorem 4(a) and (b), applied with $p = i$ and $k = n + 1$,

$$\mathbb{P}(S_n^* \geq 2x) \leq \frac{2}{x^2} \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \frac{8}{x^2} \sum_{i=1}^{n-1} \sum_{j=1}^{i} \delta_{j,i+1-j}(x). \tag{5.0}$$

Now, by Lemma 1(a),

$$\delta_{j,i+1-j}(x) \leq \int_0^{2\alpha_{i+1-j}} \left(Q_{i+1}^2(t) + Q_j^2(t)\right) dt,$$

which, together with the above inequality, yields

$$\begin{aligned} \mathbb{P}(S_n^* \geq 2x) &\leq \frac{16}{x^2} \sum_{i=1}^{n} \sum_{j=0}^{n-1} \int_0^{2\alpha_j} Q_i^2(t)\, dt \\ &\leq \frac{16}{x^2} \sum_{i=1}^{n} \int_0^1 \alpha^{-1}\left(\frac{t}{2}\right) Q_i^2(t)\, dt. \end{aligned} \tag{5.1}$$

Since the sequence $(X_{i+m})_{i\in\mathbb{Z}}$ has the same strong mixing coefficients as $(X_i)_{i\in\mathbb{Z}}$, it follows from (5.1) that, for any positive x,

$$\mathbb{P}\left(\sup_{i\in[1,n]} |S_{m+i} - S_m| \geq 2x\right) \leq \frac{16}{x^2} \sum_{i=1}^{n} \int_0^1 \alpha^{-1}\left(\frac{t}{2}\right) Q_{i+m}^2(t)\, dt. \tag{5.2}$$

Corollary 1 follows then from (5.2) via some usual arguments [see Billingsley (1985), page 298]. □

APPENDIX

PROOF OF LEMMA 1. Let f be any element of $\mathscr{L}_p^0(x)$. Let $Y = xf(X_1, \dots, X_p)$. The strong mixing coefficient between $\sigma(Y)$ and $\sigma(X_{p+n})$ is less than α_n. Since $|Y| \leq |X_p|$, $Q_Y \leq Q_p$. Hence it follows from Theorem 1.1 in Rio (1993) that

$$|\mathrm{Cov}(xf(X_1, \dots, X_p), X_{p+n})| \leq 2\int_0^{2\alpha_n} Q_p(u) Q_{p+n}(u)\, du,$$

which concludes the proof of (a).

Let f be some element of $\mathscr{L}_p(x)$ and $Z = f(X_1, \dots, X_p)$. The strong mixing coefficient between $\sigma(Z)$ and $\sigma(X_{p+n})$ is less than α_n. Since $|Z| \leq 1$, it follows from Theorem 1.1 in Rio (1993) that

$$\gamma_{p,n}(x) \leq 2\int_0^{2\alpha_n} Q_{p+n}(u)\, du,$$

therefore establishing (b). □

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