

Markovian loop soups: permenantal processes and isomorphism theorems*

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Abstract

We construct loop soups for general Markov processes without transition densities and show that the associated permenantal process is equal in distribution to the loop soup local time. This is used to establish isomorphism theorems connecting the local time of the original process with the associated permenantal process. Further properties of the loop measure are studied.

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1 Introduction

A Markovian loop soup is a particular Poisson point process \mathcal{L} on paths associated to a Markov process X . It is determined by its intensity measure μ which we refer to as loop measure. Loop measure for Brownian motion was introduced by Symanzik in his seminal paper [25] on Euclidean quantum field theory, where it is referred to as ‘blob measure’, and is a basic building block in his construction of quantum fields. Brownian loop soup was introduced by Lawler and Werner [20], in their work on SLE and conformally invariant processes in the plane. Le Jan extended the notion of loop soups to other Markov processes [12], and this has been generalized further in [14, 15]. In all this work the loop measure is constructed using bridge measures for X . This requires that X have transition densities. The main point of this paper is to show how to construct loop measures and hence loop soups for Markov processes which have potential densities but not transition densities.

Our motivation in studying Markovian loop soups is to better understand the wonderful and mysterious Isomorphism Theorem of Dynkin, [7, 8], which connects the family of total local times $L = \{L_\infty^x, x \in S\}$ of a symmetric Markov process X in S with the Gaussian process $G = \{G_x, x \in S\}$ of covariance $u(x, y)$. (When X is symmetric, $u(x, y)$

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is positive definite.) Actually, in the Isomorphism Theorem it is the family of squares of G , that is $G^2 = \{G_x^2, x \in S\}$, which is connected with L . This theorem is not an isomorphism in the usual sense, but the connection between L and G^2 is sufficiently tight that it has been used to derive many new properties of the local times, as described in [23]. This is why we consider the Isomorphism Theorem to be wonderful. We call it mysterious because it is hard to see intuitively why there should be any connection between Markov local times and Gaussian processes.

As noted by Le Jan, [13, Theorem 9], loop soups offer a deep understanding of this connection. Recall that each realization of \mathcal{L} is a countable collection of paths ω . Set

$$\widehat{L}^x = \sum_{\omega \in \mathcal{L}} L_\infty^x(\omega). \quad (1.1)$$

We call \widehat{L}^x the loop soup local time at x . A simple application of the Palm formula for Poisson point processes provides a connection, an Isomorphism Theorem, between $L = \{L_\infty^x, x \in S\}$ and $\widehat{L} = \{\widehat{L}^x, x \in S\}$. Since \widehat{L} is defined in terms of local times of X this should not be surprising. What may be surprising is that when X is symmetric then $\widehat{L} = \{\widehat{L}^x, x \in S\}$ has the distribution of $G^2 = \{G_x^2, x \in S\}$! Furthermore, the definition of (1.1) of \widehat{L} does not require the symmetry of X , so we obtain an Isomorphism Theorem for non-symmetric X .

In 1997, D. Vere-Jones, [27], introduced the α -permanental process $\theta := \{\theta_x, x \in S\}$ with kernel $u(x, y)$, which is a real valued positive stochastic process with joint distributions that satisfy

$$E \left(\prod_{j=1}^n \theta_{x_j} \right) = \sum_{\pi \in \mathcal{P}_n} \alpha^{c(\pi)} \prod_{j=1}^n u(x_j, x_{\pi(j)}), \quad (1.2)$$

for any $x_1, \dots, x_n \in S$, where $c(\pi)$ is the number of cycles in the permutation π of $[1, n]$. In addition, by [27, p. 128], the joint moment generating function of $(\theta_{x_1}, \dots, \theta_{x_n})$ has a non-zero radius of convergence. Consequently, an α -permanental process is determined by its moments. It is not hard to show that in the symmetric case $G^2/2 = \{G_x^2/2, x \in S\}$ is a $1/2$ -permanental process with kernel $u(x, y)$, the covariance of G .

In [9], Eisenbaum and Kaspi were able to show the existence of an α -permanental process with kernel $u(x, y)$ whenever $u(x, y)$ is the potential density of a transient Markov process X , and use this to obtain an Isomorphism Theorem for non-symmetric X , where the role played in the symmetric case by the Gaussian squares G^2 is now played by a permanental process. In this paper we will see that the loop soup local time \widehat{L} is an α -permanental process with kernel $u(x, y)$.

The advantage of using loop soups to construct permanental processes and obtain Isomorphism Theorems is two-fold. First, as mentioned, loop soups provide an intuitive understanding of the connection between permanental processes and local times. Second, this approach is capable of great generalization. Recent work, [14, 15], uses loop soups for Markov processes with potential densities $u(x, y)$ which may be infinite on the diagonal. In this case there are no local times and no permanental processes. Rather, loop soups are used to prove the existence of permanental fields (indexed by measures rather than points in S) with which to establish Isomorphism Theorems: for continuous additive functionals in [14], and for intersection local times in [15]. We know of no way other than using loop soups to prove the existence of permanental fields associated with not necessarily symmetric X , and the Isomorphism Theorems contain constructs which seem inaccessible without the loop soup. For example, in the symmetric case the Isomorphism Theorems contain random variables which are not in the associated Gaussian sigma field.

Here is an outline of this paper. The loop measure is constructed and studied in Section 2. In the short sub-Section 2.1 we show that when transition densities exist, our definition of loop measure agrees with the standard definition using bridge measures. In Section 3 we introduce the loop soup and quickly show that the loop soup local time \hat{L} is an α -permanental process with kernel $u(x, y)$. In the short Section 4 we use the Palm formula to prove our Isomorphism Theorem. Further properties of the loop measure are derived in Sections 5-7. These include invariance under loop rotation, and the behavior of the loop measure under restriction and space-time transformations. Here again the novelty is in deriving these properties in great generality and without the assumption of transition densities.

A recent paper of Lupu, [22], uses the results of this paper to construct a loop soup for Brownian motion on a simplicial complex associated with a graph. This Brownian motion does not have transition densities.

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2 The loop measure

Let S be a locally compact topological space with a countable base. Let $X = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a transient Borel right process with state space S , and continuous potential densities $u(x, y)$ with respect to some σ -finite measure m on S . That is

$$P^x \left(\int_0^\infty f(X_t) dt \right) = \int_S u(x, y) f(y) m(dy), \quad \forall x \in S,$$

for each non-negative Borel function $f : S \rightarrow [0, \infty)$. We assume further that $u(x, y) > 0$ for all $x, y \in S$. This amounts to the assumption that each point is regular, and that the process is irreducible in the sense that $P^x(T_y < \infty) > 0$ for all x, y . Then there is a function $(\omega, s, y) \rightarrow L_s^y(\omega)$ from $\Omega \times [0, \infty) \times S$ to $[0, \infty)$ that is jointly progressively measurable in (ω, s) and Borel measurable in y , such that for each $y \in S$, $t \mapsto L_t^y$ is a CAF increasing only when X is in state y (i.e., a local time at y), and $E^x(L_\infty^y) = u(x, y)$ for all $x, y \in S$. This follows from the proofs in [23, Section 3.6] if we choose the approximate delta functions $f_{\epsilon, y}(x)$ used there to be of the form

$$f_{\epsilon, y}(x) = \frac{f_\epsilon(d(y, x))}{\int_S f_\epsilon(d(y, z)) m(dz)} \tag{2.1}$$

where d is a metric for the topology of S , and f_ϵ is a continuous function supported on $[0, \epsilon]$, and define $L_t^y(\omega) = \liminf_{n \rightarrow \infty} \int_0^t f_{n^{-1}, y}(X_s) ds$. (This is used to show measurability in y).

Under our assumption that $u(x, y)$ is continuous, it follows as in the proof of [23, Lemma 3.4.3], that uniformly in x , $u(x, y)$ as a function of y is locally bounded and continuous. This implies that for any $\beta > 0$, the same is true for

$$u^\beta(x, y) := u(x, y) - \beta \int U^\beta(x, dz) u(z, y), \tag{2.2}$$

and it follows from the resolvent equation that for each x , $u^\beta(x, y)$ is a density for $U^\beta(x, dy)$ with respect to $m(dy)$. It then follows from the resolvent equation that for any $\alpha, \beta > 0$ and all x, y

$$\frac{u^\alpha(x, y) - u^\beta(x, y)}{\beta - \alpha} = \int u^\alpha(x, z) u^\beta(z, y) dm(z) = \int u^\beta(x, z) u^\alpha(z, y) dm(z). \tag{2.3}$$

Using (2.2) and the resolvent equation for additive functionals we see that

$$u^\beta(x, y) = E^x \left(\int_0^\infty e^{-\beta t} d_t L_t^y \right). \tag{2.4}$$

We now show that $m(K) < \infty$ for each compact $K \subseteq S$. To see this note first that from (2.4) and our assumption that $u(x, y) > 0$ for all $x, y \in S$, that also $u^1(x, y) > 0$ for all $x, y \in S$. It then follows from the last paragraph that $y \mapsto u^1(x, y)$ is bounded below for $y \in K$ by a constant $C = C(x) > 0$. Consequently

$$C \cdot m(K) \leq \int_K u^1(x, y) m(dy) \leq \int_S u^1(x, y) m(dy) = U^1 1(x) \leq 1. \tag{2.5}$$

We may take the canonical representation of X in which Ω is the set of right continuous paths ω in $S_\Delta = S \cup \Delta$ with $\Delta \notin S$, and is such that $\omega(t) = \Delta$ for all $t \geq \zeta = \inf\{t > 0 \mid \omega(t) = \Delta\}$. Then $X_t(\omega) = \omega(t)$. We define a σ -finite measure μ on (Ω, \mathcal{F}) by

$$\int F d\mu = \int_S P^x \left(\int_0^\infty \frac{1}{t} F \circ k_t d_t L_t^x \right) dm(x), \tag{2.6}$$

for all \mathcal{F} -measurable functions F on Ω . Here k_t is the killing operator defined by $k_t \omega(s) = \omega(s)$ if $s < t$ and $k_t \omega(s) = \Delta$ if $s \geq t$. We call μ the loop measure of X because, when X has continuous paths, μ is concentrated on the set of continuous loops. See also Lemma 2.4 below. Even if X is not assumed to have continuous paths we can verify that μ is concentrated on $\{X_0 = X_{\zeta-}, \zeta < \infty\}$. To see this, note first of all that since $1_{\{\zeta=\infty\}} \circ k_t = 0$ for each t , it is clear from (2.6) that $\mu(\zeta = \infty) = 0$. Then, since L_t^x is constant for $t \geq \zeta$, while on $t \leq \zeta$ we have $\zeta \circ k_t = t$,

$$P^x \left(\int_0^\infty \frac{1}{t} 1_{\{X_0 \neq X_{\zeta-}\}} \circ k_t d_t L_t^x \right) = P^x \left(\int_0^\infty \frac{1}{t} 1_{\{x \neq X_{t-}\}} d_t L_t^x \right). \tag{2.7}$$

But by right-continuity of paths, the set of times for which $X_{t-}(\omega)$ either fails to exist or exists but is different from $X_t(\omega)$ is at most countable, for each $\omega \in \Omega$, [4, IV, Theorem 88D], while L_t^x is continuous in t so that $d_t L_t^x$ has no atoms. Hence (2.7)

$$= P^x \left(\int_0^\infty \frac{1}{t} 1_{\{x \neq X_t\}} d_t L_t^x \right) = 0, \tag{2.8}$$

where the last equality used the fact that $d_t L_t^x$ is supported on $\{X_t = x\}$.

As usual, if F is a function, we often write $\mu(F)$ for $\int F d\mu$.

Lemma 2.1. *For any k , and any $y_1, \dots, y_k \in S$*

$$\mu \left(\prod_{j=1}^k L_\infty^{y_j} \right) = \sum_{\pi \in \mathcal{P}_{k-1}} u(y_k, y_{\pi(1)}) \cdots u(y_{\pi(k-2)}, y_{\pi(k-1)}) u(y_{\pi(k-1)}, y_k), \tag{2.9}$$

where \mathcal{P}_{k-1} denotes the set of permutations of $[1, k-1]$. When $k = 1$ this means $\mu(L_\infty^{y_1}) = u(y_1, y_1)$.

Proof We present a derivation of Lemma 2.1 suggested by Symanzik, [25]. We first show that for any $k, \alpha \geq 0$ and $x, y \in S$

$$\begin{aligned} \mathcal{V} &:= P^x \left(\int_0^\infty e^{-\alpha t} \prod_{j=1}^k L_t^{z_j} d_t L_t^y \right) \\ &= \sum_{\pi \in \mathcal{P}_k} u^\alpha(x, z_{\pi(1)}) u^\alpha(z_{\pi(1)}, z_{\pi(2)}) \cdots u^\alpha(z_{\pi(k-1)}, z_{\pi(k)}) u^\alpha(z_{\pi(k)}, y). \end{aligned} \tag{2.10}$$

To see this, note that

$$\mathcal{V} = \sum_{\pi \in \mathcal{P}_k} P^x \left(\int_0^\infty e^{-\alpha t} \int_{\{0 < s_1 < \dots < s_k < t\}} \prod_{j=1}^k dL_{s_j}^{z_{\pi(j)}} d_t L_t^y \right), \quad (2.11)$$

and for each $\pi \in \mathcal{P}_k$ we have

$$\begin{aligned} \mathcal{J}_\pi &:= P^x \left(\int_0^\infty e^{-\alpha t} \int_{\{0 < s_1 < \dots < s_k < t\}} \prod_{j=1}^k dL_{s_j}^{z_{\pi(j)}} d_t L_t^y \right) \\ &= P^x \left(\int_{\{0 < s_1 < \dots < s_k < \infty\}} \left(\int_{s_k}^\infty e^{-\alpha t} d_t L_t^y \right) \prod_{j=1}^k d_{s_j} L_{s_j}^{y_{\pi(j)}} \right) \\ &= P^x \left(\int_{\{0 < s_1 < \dots < s_k < \infty\}} e^{-\alpha s_k} \left(\int_0^\infty e^{-\alpha t} d_t L_t^y \right) \circ \theta_{s_k} \prod_{j=1}^k d_{s_j} L_{s_j}^{z_{\pi(j)}} \right). \end{aligned} \quad (2.12)$$

Using the Markov property, see for example Theorems 28.7 and 22.8 of [24], and (2.4), we have

$$\begin{aligned} \mathcal{J}_\pi &= P^x \left(\int_{\{0 < s_1 < \dots < s_k < \infty\}} e^{-\alpha s_k} E^{X_{s_k}} \left(\int_0^\infty e^{-\alpha t} d_t L_t^y \right) \prod_{j=1}^k d_{s_j} L_{s_j}^{z_{\pi(j)}} \right) \\ &= P^x \left(\int_{\{0 < s_1 < \dots < s_k < \infty\}} e^{-\alpha s_k} \prod_{j=1}^k dL_{s_j}^{z_{\pi(j)}} \right) u^\alpha(z_{\pi(k)}, y). \end{aligned} \quad (2.13)$$

It then follows by induction that

$$\mathcal{J}_\pi = u^\alpha(x, z_{\pi(1)}) u^\alpha(z_{\pi(1)}, z_{\pi(2)}) \cdots u^\alpha(z_{\pi(k-1)}, z_{\pi(k)}) u^\alpha(z_{\pi(k)}, y). \quad (2.14)$$

Using (2.11) then proves (2.10).

It follows from (2.10) that

$$\begin{aligned} &\int_S P^x \left(\int_0^\infty e^{-\alpha t} \prod_{j=1}^k L_t^{z_j} d_t L_t^x \right) dm(x) \\ &= \sum_{\pi \in \mathcal{P}_k} \int_S u^\alpha(x, z_{\pi(1)}) u^\alpha(z_{\pi(1)}, z_{\pi(2)}) \cdots u^\alpha(z_{\pi(k-1)}, z_{\pi(k)}) u^\alpha(z_{\pi(k)}, x) dm(x). \end{aligned} \quad (2.15)$$

By (2.4) we have that $u^\beta(x, y) \uparrow u^\alpha(x, y)$ as $\beta \downarrow \alpha$. Hence by (2.3) and the monotone convergence theorem

$$\int_S u^\alpha(y, x) u^\alpha(x, z) dm(x) = -\frac{d}{d\alpha} u^\alpha(y, z). \quad (2.16)$$

Hence the right hand side of (2.15)

$$= - \sum_{\pi \in \mathcal{P}_k} u^\alpha(z_{\pi(1)}, z_{\pi(2)}) \cdots u^\alpha(z_{\pi(k-1)}, z_{\pi(k)}) \frac{d}{d\alpha} u^\alpha(z_{\pi(k)}, z_{\pi(1)}). \quad (2.17)$$

The sum is over all permutations of the ‘labels’ of the points z_1, \dots, z_k which in this expression appear in a circle. By fixing z_k and considering permutations $\pi \in \mathcal{P}_{k-1}$, we

can rewrite (2.17) as

$$\begin{aligned}
 & - \sum_{\pi \in \mathcal{P}_{k-1}} \sum_{j=1}^k u^\alpha(z_k, z_{\pi(1)}) \cdots \frac{d}{d\alpha} u^\alpha(z_{\pi(j-1)}, z_{\pi(j)}) \cdots u^\alpha(z_{\pi(k-1)}, z_k) \\
 & = - \sum_{\pi \in \mathcal{P}_{k-1}} \frac{d}{d\alpha} (u^\alpha(z_k, z_{\pi(1)}) \cdots u^\alpha(z_{\pi(j-1)}, z_{\pi(j)}) \cdots u^\alpha(z_{\pi(k-1)}, z_k)),
 \end{aligned} \tag{2.18}$$

where for notational convenience we set $\pi(0) = \pi(k) = k$ for $\pi \in \mathcal{P}_{k-1}$ in the first line. The second line is the product rule for differentiation.

By (2.4)

$$\lim_{\alpha \rightarrow \infty} u^\alpha(x, y) = 0. \tag{2.19}$$

Hence by what we have shown about the right hand side of (2.15)

$$\begin{aligned}
 & \int_0^\infty \left(\sum_{\pi \in \mathcal{P}_k} \int_S u^\alpha(x, z_{\pi(1)}) u^\alpha(z_{\pi(1)}, z_{\pi(2)}) \cdots u^\alpha(z_{\pi(k)}, x) dm(x) \right) d\alpha \\
 & = - \sum_{\pi \in \mathcal{P}_{k-1}} \int_0^\infty \frac{d}{d\alpha} (u^\alpha(z_k, z_{\pi(1)}) \cdots u^\alpha(z_{\pi(j-1)}, z_{\pi(j)}) \cdots u^\alpha(z_{\pi(k-1)}, z_k)) d\alpha \\
 & = \sum_{\pi \in \mathcal{P}_{k-1}} u(z_k, z_{\pi(1)}) \cdots u(z_{\pi(j-1)}, z_{\pi(j)}) \cdots u(z_{\pi(k-1)}, z_k).
 \end{aligned} \tag{2.20}$$

Thus by (2.15)

$$\begin{aligned}
 & \int_0^\infty \left\{ \int_S P^x \left(\int_0^\infty e^{-\alpha t} \prod_{j=1}^k L_t^{z_j} d_t L_t^x \right) dm(x) \right\} d\alpha \\
 & = \sum_{\pi \in \mathcal{P}_{k-1}} u(z_k, z_{\pi(1)}) \cdots u(z_{\pi(j-1)}, z_{\pi(j)}) \cdots u(z_{\pi(k-1)}, z_k),
 \end{aligned} \tag{2.21}$$

and Lemma 2.1 follows by applying Fubini's theorem to interchange the order of integration. □

Let $Q^{x,y}$ denote the measure defined on (Ω, \mathcal{F}) by

$$Q^{x,y}(F) = P^x \left(\int_0^\infty F \circ k_t d_t L_t^y \right), \tag{2.22}$$

for all \mathcal{F} measurable functions F on Ω . Since $\zeta \circ k_t = \zeta \wedge t$, it follows that if $F_s \in b\mathcal{F}_s^0$ where \mathcal{F}_s^0 is the σ -algebra generated by $\{X_r, 0 \leq r \leq s\}$ then $(1_{\{\zeta > s\}} F_s) \circ k_t = 1_{\{\zeta \wedge t > s\}} F_s$. Hence

$$\begin{aligned}
 Q^{x,y}(1_{\{\zeta > s\}} F_s) & = P^x \left(F_s \int_s^\infty d_t L_t^y \right) \\
 & = P^x (F_s L_\infty^y \circ \theta_s) = P^x(F_s u(X_s, y)).
 \end{aligned} \tag{2.23}$$

We remark that under the measures $P^{x/h} = \frac{1}{u(x,y)} Q^{x,y}$, the paths of X are conditioned to hit y and die on their last exit from y . $P^{x/h}$ is the h -transform of P^x , with $h(x) = u(x, y)/u(y, y) = P^x(T_y < \infty)$.

In the proof of the Isomorphism Theorem we will need the following Lemma.

Lemma 2.2. For any $k, x, x_j \in S, j = 1, \dots, k$, and any bounded measurable function H on R^k we have

$$\mu(L_\infty^x H(L_\infty^{x_1}, \dots, L_\infty^{x_k})) = Q^{x,x}(H(L_\infty^{x_1}, \dots, L_\infty^{x_k})). \tag{2.24}$$

Proof of Lemma 2.2: Let $x, y, x_j \in S, j = 1, \dots, k$. Since $L_\infty^{x_j} \circ k_t = L_t^{x_j}$, it follows from (2.22) that

$$Q^{x,y} \left(\prod_{j=1}^k L_\infty^{x_j} \right) = P^x \left(\int_0^\infty \prod_{j=1}^k L_t^{x_j} d_t L_t^y \right). \tag{2.25}$$

It then follows from (2.10) that

$$Q^{x,y} \left(\prod_{j=1}^k L_\infty^{x_j} \right) = \sum_{\pi \in \mathcal{P}_k} u(x, x_{\pi(1)}) u(x_{\pi(1)}, x_{\pi(2)}) \cdots \cdots u(x_{\pi(k-1)}, x_{\pi(k)}) u(x_{\pi(k)}, y). \tag{2.26}$$

Comparing (2.26) with $y = x$ and (2.9) we see that (2.24) holds for all polynomial H . But it is easily seen from (2.9) and (2.26) that the random variables L_∞^z are exponentially integrable both under $Q^{x,x}$ and $\mu(L_\infty^x \cdot)$, hence finite dimensional distributions are determined by their moments. □

Since $\zeta \circ k_t = \zeta \wedge t$, we note for future reference that

$$\begin{aligned} \mu(F) &= \int_S P^x \left(\int_0^\infty \frac{1}{t} F \circ k_t d_t L_t^x \right) dm(x) \\ &= \int_S P^x \left(\int_0^\infty \left(\frac{F}{\zeta} \right) \circ k_t d_t L_t^x \right) dm(x) \\ &= \int_S Q^{x,x} \left(\frac{F}{\zeta} \right) dm(x). \end{aligned} \tag{2.27}$$

In the sequel we will use the fact that $(t, x) \mapsto L_t^x(\omega)$ is an occupation density with respect to m :

$$\int_0^t f(X_s) ds = \int_S f(x) L_t^x m(dx), \tag{2.28}$$

for all $t \geq 0$ and all non-negative Borel functions f , almost surely. It suffices to prove this for $f \geq 0$ which are continuous and compactly supported. This case follows from the proof of [23, Theorem 3.7.1], with one change. That theorem assumed the joint continuity of L_t^x in order to show that the right hand side of (2.28), which we denote by A_t , is a CAF. But this can be seen directly. A_t is obviously monotone increasing in t and constant for $t \geq \zeta$. Also, using (2.5)

$$E^y(A_\infty) = \int_S u(y, x) f(x) m(dx) < \infty, \tag{2.29}$$

hence $A_\infty < \infty$ a.s. Hence the a.s. continuity of A_t follows from the dominated convergence theorem after applying Fubini to the fact for each $x \in S$, a.s. in ω , $L_t^x(\omega)$ is continuous in t . Finally, fix $s, t > 0$. We have $L_{s+t}^x(\omega) = L_s^x(\omega) + L_t^x \circ \theta_s(\omega)$ for each $x \in S$, a.s. in ω . Hence by Fubini this holds a.s. in ω for a.e. $x \in S$. From the right hand side of (2.28) we then see that a.s. in ω , $A_{s+t}(\omega) = A_s(\omega) + A_t \circ \theta_s(\omega)$, which completes the proof that A_t is a CAF, and hence the proof of (2.28).

2.1 Transition densities

For this sub-Section only, we assume that $P_t(x, dy) \ll dm(y)$ for each $t > 0$ and $x \in S$; in other words, $P_t(x, dy)$ has transition densities with respect to m . Under this assumption we give an alternate description of the loop measure. This is the description found in the literature. Using this description we give a simple proof of the fact that

the loop measure is invariant under loop rotation. A proof of this fact without assuming transition densities is given in Section 5. The material in this sub-section will not be used in the following sections of the paper.

Under our assumption that $P_t(x, dy)$ has transition densities with respect to m , it follows from [6] that we can find jointly measurable transition densities $p_t(x, y)$ with respect to m which satisfy the Chapman-Kolmogorov equation

$$\int p_s(x, y)p_t(y, z) dm(y) = p_{s+t}(x, z). \tag{2.30}$$

Assume that $p_t(x, y) < \infty$, for all $0 < t < \infty$ and $x, y \in S$. It then follows as in [10] that for all $0 < t < \infty$ and $x, y \in S$, there exists a finite measure $Q_t^{x,y}$ on \mathcal{F}_{t-} , of total mass $p_t(x, y)$, such that

$$Q_t^{x,y}(F_s) = P^x(F_s p_{t-s}(X_s, y)), \tag{2.31}$$

for all $F_s \in \mathcal{F}_s$ with $s < t$. In particular, for any $0 < t_1 \leq \dots \leq t_{k-1} \leq t_k < t$ and bounded Borel measurable functions f_1, \dots, f_k

$$\begin{aligned} & Q_t^{x,y} \left(\prod_{j=1}^k f_j(X_{t_j}) \right) \\ &= \int_{S^k} p_{t_1}(x, y_1) f_1(y_1) p_{t_2-t_1}(y_1, y_2) f_2(y_2) \cdots \\ & \quad \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) f_k(y_k) p_{t-t_k}(y_k, y) dm(y_1) \cdots dm(y_k). \end{aligned} \tag{2.32}$$

Lemma 2.3.

$$\mu(F) = \int_0^\infty \frac{1}{t} \int_S Q_t^{x,x}(F \circ k_t) dm(x) dt \tag{2.33}$$

for all \mathcal{F} measurable functions F on Ω .

Proof of Lemma 2.3 Let us temporarily use the notation $\tilde{\mu}(F)$ to denote the right hand side of (2.33). It suffices to show that $\mu(F) = \tilde{\mu}(F)$ for all F of the form $F = \prod_{j=1}^k f_j(X_{t_j})$, for all $0 < t_1 \leq \dots \leq t_{k-1} \leq t_k < \infty$ and bounded Borel measurable functions f_1, \dots, f_k on $S \cup \Delta$ with $f_j(\Delta) = 0$, $j = 1, \dots, k$. Note that this last condition implies that

$$\prod_{j=1}^k f_j(X_{t_j} \circ k_t) = 1_{\{t_k < t\}} \prod_{j=1}^k f_j(X_{t_j}). \tag{2.34}$$

Using (2.32)

$$\begin{aligned} \tilde{\mu}(F) &= \int_0^\infty \frac{1}{t} \int_S Q_t^{x,x} \left(\left(\prod_{j=1}^k f_j(X_{t_j}) \right) \circ k_t \right) dm(x) dt \\ &= \int_{t_k}^\infty \frac{1}{t} \int_S Q_t^{x,x} \left(\prod_{j=1}^k f_j(X_{t_j}) \right) dm(x) dt \\ &= \int_{t_k}^\infty \frac{1}{t} \int_{S^{k+1}} p_{t_1}(x, y_1) f_1(y_1) p_{t_2-t_1}(y_1, y_2) f_2(y_2) \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) \\ & \quad \cdots f_k(y_k) p_{t-t_k}(y_k, x) dm(y_1) \cdots dm(y_k) dm(x) dt. \end{aligned} \tag{2.35}$$

Similarly, using the Markov property

$$\begin{aligned}
 \mu(F) &= \int_S P^x \left(\int_0^\infty \frac{1}{t} \prod_{j=1}^k f_j(X_{t_j}) \circ k_t dL_t^x \right) dm(x) \\
 &= \int_S P^x \left(\int_{t_k}^\infty \frac{1}{t} \prod_{j=1}^k f_j(X_{t_j}) dL_t^x \right) dm(x) \\
 &= \int_S P^x \left(\prod_{j=1}^k f_j(X_{t_j}) \left(\int_{t_k}^\infty \frac{1}{t} d_t L_{t-t_k}^x \right) \circ \theta_{t_k} \right) dm(x) \\
 &= \int_S P^x \left(\prod_{j=1}^k f_j(X_{t_j}) E^{X_{t_k}} \left(\int_{t_k}^\infty \frac{1}{t} d_t L_{t-t_k}^x \right) \right) dm(x) \\
 &= \int_{S^{k+1}} p_{t_1}(x, y_1) f_1(y_1) p_{t_2-t_1}(y_1, y_2) f_2(y_2) \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) \\
 &\quad \cdots f_k(y_k) E^{y_k} \left(\int_{t_k}^\infty \frac{1}{t} d_t L_{t-t_k}^x \right) dm(x) dm(y_1) \cdots dm(y_k).
 \end{aligned}
 \tag{2.36}$$

For $y, x \in S$, we define the measure $\Gamma_{y,x}(\cdot)$ on $[0, \infty)$ with cdf:

$$\Gamma_{y,x}([0, t]) := E^y(L_t^x),
 \tag{2.37}$$

so that for all bounded measurable functions g on $[0, \infty)$

$$\int_0^\infty g(t) \Gamma_{y,x}(dt) = E^y \left(\int_0^\infty g(t) d_t L_t^x \right).
 \tag{2.38}$$

We claim that for each y , we can find a set $S_y \subset S$ with $m(S_y) = 0$ such that

$$E^y(L_t^x) = \int_0^t p_s(y, x) ds
 \tag{2.39}$$

for all t and $x \in S_y^c$. Then by (2.38), for any t_k and all $x \in S_y^c$

$$\int_{t_k}^\infty \frac{1}{t} p_{t-t_k}(y_k, x) dt = E^{y_k} \left(\int_{t_k}^\infty \frac{1}{t} d_t L_{t-t_k}^x \right).
 \tag{2.40}$$

Since the right hand side of (2.36) involves a $dm(x)$ integral, by Fubini we can replace the term $E^{y_k} \left(\int_{t_k}^\infty \frac{1}{t} d_t L_{t-t_k}^x \right)$ which appears there with the left hand side of (2.40). Thus we will obtain

$$\begin{aligned}
 \mu(F) &= \int_{t_k}^\infty \frac{1}{t} \int_{S^{k+1}} p_{t_1}(x, y_1) f_1(y_1) p_{t_2-t_1}(y_1, y_2) f_2(y_2) \cdots \\
 &\quad \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) f_k(y_k) p_{t-t_k}(y_k, x) dm(y_1) \cdots dm(y_k) dm(x) dt.
 \end{aligned}
 \tag{2.41}$$

Comparing with (2.35) then shows that $\mu(F) = \tilde{\mu}(F)$.

It only remains to verify our claim concerning (2.39). Note that since the left hand side of (2.39) is continuous in t and the right hand side is monotone, it suffices to find a set S_y which works for all rational t , hence for each fixed t . By the occupation density formula (2.28)

$$\begin{aligned}
 \int_S f(x) \left(\int_0^t p_s(y, x) ds \right) m(dx) &= E^y \left(\int_0^t f(X_s) ds \right) \\
 &= \int_S f(x) (E^y(L_t^x)) m(dx).
 \end{aligned}
 \tag{2.42}$$

Since this holds for all bounded measurable f , our claim for fixed t is established. \square

For later use we note that applying the Chapman-Kolmogorov equation (2.30) for the $dm(x)$ integral in (2.41) shows that

$$\begin{aligned} \mu \left(\prod_{j=1}^k f_j(X_{t_j}) \right) &= \int_{t_k}^{\infty} \frac{1}{t} \int_{S^k} f_1(y_1) p_{t_2-t_1}(y_1, y_2) f_2(y_2) \cdots \\ &\quad \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) f_k(y_k) p_{t_1+t-t_k}(y_k, y_1) dm(y_1) \cdots dm(y_k) dt. \end{aligned} \tag{2.43}$$

The next result justifies our calling μ the loop measure even for a process with discontinuous paths. This result will be proved in full generality in section 5. Define the loop rotation ρ_u by

$$\rho_u \omega(s) = \begin{cases} \omega(s + u \pmod{\zeta(\omega)}), & \text{if } 0 \leq s < \zeta(\omega) \\ \Delta, & \text{otherwise.} \end{cases} \tag{2.44}$$

Here, for two positive numbers a, b we define $a \pmod b = a - mb$ for the unique positive integer m such that $0 \leq a - mb < b$. Set $(a)_b = a \pmod b$

Lemma 2.4. μ is invariant under ρ_u , for any $u > 0$.

Proof of Lemma 2.4 Let $0 < t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty$ and let f_1, \dots, f_k be bounded Borel measurable functions on $S \cup \Delta$ with $f_j(\Delta) = 0, j = 1, \dots, k$. Fix some t and u .

Since $f_j(\Delta) = 0, j = 1, \dots, k$,

$$\prod_{j=1}^k f_j(X_{t_j} \circ \rho_u \circ k_t) = 1_{\{t_k < t\}} \prod_{j=1}^k f_j(X_{(t_j+u)_t}). \tag{2.45}$$

Set $s_j = t_j + u$. Since $0 < t_1 \leq \cdots \leq t_{k-1} \leq t_k < t$, it is clear that for some i and some l

$$0 \leq s_l - it \leq \cdots \leq s_k - it \leq s_1 - (i-1)t \leq \cdots \leq s_{l-1} - (i-1)t < t \tag{2.46}$$

Therefore, by (2.32)

$$\begin{aligned} Q_t^{x,x} \left(\prod_{j=1}^k f_j(X_{t_j}) \circ \rho_u \circ k_t \right) &= 1_{\{t_k < t\}} \int_{S^k} p_{s_l-it}(x, y_l) f_l(y_l) p_{t_{l+1}-t_l}(y_l, y_{l+1}) f_{l+1}(y_2) \cdots \\ &\quad \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) f_k(y_k) p_{t_1+t-t_k}(y_k, y_1) f_1(y_1) \cdots \\ &\quad \cdots p_{t_{l-1}-t_{l-2}}(y_{l-2}, y_{l-1}) f_{l-1}(y_{l-1}) p_{it-s_{l-1}}(y_{l-1}, x) dm(y_1) \cdots dm(y_k). \end{aligned} \tag{2.47}$$

Integrating both sides with respect to $dm(x)$ and applying the Chapman-Kolmogorov equation (2.30) we obtain

$$\begin{aligned} \int_S Q_t^{x,x} \left(\prod_{j=1}^k f_j(X_{t_j}) \circ \rho_u \circ k_t \right) dm(x) &= 1_{\{t_k < t\}} \int_{S^k} f_l(y_l) p_{t_{l+1}-t_l}(y_l, y_{l+1}) f_{l+1}(y_2) \cdots \\ &\quad \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) f_k(y_k) p_{t_1+t-t_k}(y_k, y_1) f_1(y_1) \cdots \\ &\quad \cdots p_{t_{l-1}-t_{l-2}}(y_{l-2}, y_{l-1}) f_{l-1}(y_{l-1}) p_{t_l-t_{l-1}}(y_{l-1}, y_l) dm(y_1) \cdots dm(y_k). \end{aligned} \tag{2.48}$$

where in the last line we used

$$it - s_{l-1} + s_l - it = s_l - s_{l-1} = t_l - t_{l-1}.$$

Comparing with (2.43) we obtain our Lemma. □

3 The loop soup

Let Ω be the path space for X described after (2.5). For any $\alpha > 0$, let \mathcal{L}_α be a Poisson point process on Ω with intensity measure $\alpha\mu$. Note that \mathcal{L}_α is a random variable; each realization of \mathcal{L}_α is countable subset of Ω . To be more specific, let

$$N(A) := \#\{\mathcal{L}_\alpha \cap A\}, \quad A \subseteq \Omega. \tag{3.1}$$

Then for any disjoint measurable subsets A_1, \dots, A_n of Ω , the random variables $N(A_1), \dots, N(A_n)$, are independent, and $N(A)$ is a Poisson random variable with parameter $\alpha\mu(A)$, i.e.

$$P(N(A) = k) = \frac{(\alpha\mu(A))^k}{k!} e^{-\alpha\mu(A)}. \tag{3.2}$$

(When $\mu(A) = \infty$, this means that $P(N(A) = \infty) = 1$.) The Poisson point process \mathcal{L}_α is called the ‘loop soup’ of the Markov process X . The term ‘loop soup’ is used in [20], [19] and [17, Chapter 9]. In [12] \mathcal{L}_α is referred to, less colorfully albeit more descriptively, as Poissonian ensembles of Markov loops. See also [26] and [21].

We define the ‘loop soup local time’, \widehat{L}^x , of X , by

$$\widehat{L}^x = \sum_{\omega \in \mathcal{L}_\alpha} L_\infty^x(\omega). \tag{3.3}$$

The next theorem is given for associated Gaussian squares in [13, Theorem 9].

Theorem 3.1. *Let X be a transient Borel right process with state space S , as described in the beginning of this section, and let $u(x, y)$, $x, y \in S$ denote its potential density. Let $\{\widehat{L}^x, x \in S\}$ be the loop soup local time of X . Then $\{\widehat{L}^x, x \in S\}$, is an α -permanental process with kernel $u(x, y)$.*

Proof By the master formula for Poisson processes, [16, (3.6)],

$$E \left(e^{\sum_{j=1}^n z_j \widehat{L}^{x_j}} \right) = \exp \left(\alpha \left(\int_{\Omega} \left(e^{\sum_{j=1}^n z_j L_\infty^{x_j}(\omega)} - 1 \right) d\mu(\omega) \right) \right), \tag{3.4}$$

for $|z_1|, \dots, |z_n|$ sufficiently small. Differentiating each side of (3.4) with respect to z_1, \dots, z_n and then setting z_1, \dots, z_n equal to zero, we get

$$E \left(\prod_{j=1}^n \widehat{L}^{x_j} \right) = \sum_{l=1}^n \sum_{\cup_{i=1}^l B_i = [1, n]} \alpha^l \prod_{i=1}^l \mu \left(\prod_{j \in B_i} L_\infty^{x_j} \right), \tag{3.5}$$

where the second sum is over all partitions B_1, \dots, B_l of $[1, n]$. Using (2.9) it is easily seen that this is

$$E \left(\prod_{j=1}^n \widehat{L}^{x_j} \right) = \sum_{\pi \in \mathcal{P}_n} \alpha^{c(\pi)} \prod_{j=1}^n u(x_j, x_{\pi(j)}). \tag{3.6}$$

□

Let $\theta = \{\theta_x, x \in S\}$ be an α -permanental process with kernel $u(x, y)$, $x, y \in S$, as considered in Theorem 3.1. Clearly, by our loop soup construction, θ is infinitely divisible. In [9, Corollary 3.4], Eisenbaum and Kaspi show that the Lévy measure of $\{\theta_x, x \in S\}$ is given by the law of $\{L_\infty^x, x \in S\}$ under the σ -finite measure

$$\frac{\alpha}{L_\infty^y} Q^{y,y} \tag{3.7}$$

for any $y \in S$. However it follows from Theorem 3.1 that the loop measure $\alpha\mu$ is the Lévy measure of $\{\theta_x, x \in S\}$. Therefore

$$\{L_\infty^x, x \in S; \mu\} \stackrel{law}{=} \{L_\infty^x, x \in S; \frac{1}{L_\infty^y} Q^{y,y}\}, \tag{3.8}$$

for any $y \in S$. This fact is also an immediate consequence of Lemma 2.2.

4 The Isomorphism Theorem via loop soup

For our Isomorphism Theorem we will need a special case of the Palm formula for Poisson processes \mathcal{L} with intensity measure n on a measurable space S , see [2, Lemma 2.3]. This says that for any positive function f on S and any measurable functional G of \mathcal{L}

$$E_{\mathcal{L}} \left(\left(\sum_{\omega \in \mathcal{L}} f(\omega) \right) G(\mathcal{L}) \right) = \int E_{\mathcal{L}} (G(\omega' \cup \mathcal{L})) f(\omega') dn(\omega'). \tag{4.1}$$

Theorem 4.1 (Isomorphism Theorem). *For any $x, x_1, x_2, \dots \in S$ and any bounded measurable function F on R_+^∞ ,*

$$E_{\mathcal{L}_\alpha} Q^{x,x} \left(F \left(\widehat{L}^{x_j} + L^{x_j} \right) \right) = \frac{1}{\alpha} E_{\mathcal{L}_\alpha} \left(\widehat{L}^x F \left(\widehat{L}^{x_j} \right) \right). \tag{4.2}$$

(Here we use the notation $F(f(x_j)) := F(f(x_1), f(x_2), \dots)$.)

Proof We apply the Palm formula with intensity measure $\alpha\mu$,

$$f(\omega) = L^x(\omega) \tag{4.3}$$

and

$$G(\mathcal{L}) = F \left(\widehat{L}^{x_j} \right). \tag{4.4}$$

Note that

$$\sum_{\omega \in \mathcal{L}} f(\omega) = \widehat{L}^x. \tag{4.5}$$

Also

$$\begin{aligned} \widehat{L}^{x_j}(\omega' \cup \mathcal{L}_\alpha) &= \sum_{\omega \in \omega' \cup \mathcal{L}_\alpha} L^{x_j}(\omega) \\ &= \widehat{L}^{x_j}(\mathcal{L}_\alpha) + L^{x_j}(\omega'), \end{aligned} \tag{4.6}$$

so that

$$G(\omega' \cup \mathcal{L}_\alpha) = F \left(\widehat{L}^{x_j}(\mathcal{L}_\alpha) + L^{x_j}(\omega') \right). \tag{4.7}$$

Then by (4.1)

$$E_{\mathcal{L}_\alpha} \left(\widehat{L}^x F \left(\widehat{L}^{x_j} \right) \right) = \alpha E_{\mathcal{L}_\alpha} \mu \left(L^x F \left(\widehat{L}^{x_j} + L^{x_j} \right) \right), \tag{4.8}$$

so that our theorem follows from Lemma 2.2. □

5 Invariance under loop rotation

In sub-Section 2.1, assuming the existence of transition densities, we gave a simple proof of the fact that the loop measure is invariant under loop rotation. In this section we give a proof of this fact without assuming the existence of transition densities. This proof is considerably more complicated.

Because the lifetime ζ is rotation invariant ($\zeta(\rho_v\omega) = \zeta(\omega)$ so long as $\zeta(\omega) < \infty$), the rotation invariance of the loop measure μ is equivalent to that of the measure ν defined by $\nu(F) := \mu(\zeta F)$. By (2.27) and (2.22) we have

$$\nu(F) = \int_S Q^{x,x}(F) dm(x) = \int_S P^x \left(\int_0^\infty F \circ k_t d_t L_t^x \right) dm(x). \tag{5.1}$$

The measure ν is more convenient for the calculations that follow, because of the following formula, where $\Gamma_{x,y}$ is defined in (2.37):

Lemma 5.1. For a.e. $0 < t_1 < t_2 < \dots < t_k$,

$$\begin{aligned} \nu(X_{t_1} \in dy_1, \dots, X_{t_k} \in dy_k, \zeta \in dt) \\ = m(dy_1) \prod_{j=2}^k P_{t_j-t_{j-1}}(y_{j-1}, dy_j) \Gamma_{y_k, y_1}(dt - t_k + t_1) \end{aligned} \tag{5.2}$$

as measures on the product space $S^k \times (t_k, \infty)$. Furthermore, with $t_1 = 0$, (5.2) holds for all $0 < t_2 < t_3 < \dots < t_k$.

Proof of Lemma 5.1: Using (5.1) we see that

$$\nu \left(\prod_{j=1}^k f_j(X_{t_j}) e^{-\beta\zeta} \right) = \int_S P^x \left(\prod_{j=1}^k f_j(X_{t_j}) \int_{t_k}^\infty e^{-\beta t} d_t L_t^x \right) dm(x). \tag{5.3}$$

Using the Markov property and (2.4) we see that

$$\begin{aligned} & P^x \left(\prod_{j=1}^k f_j(X_{t_j}) \int_{t_k}^\infty e^{-\beta t} d_t L_t^x \right) \\ &= P^x \left(\prod_{j=1}^k f_j(X_{t_j}) e^{-\beta t_k} \left(\int_0^\infty e^{-\beta t} d_t L_t^x \right) \circ \theta_{t_k} \right) \\ &= P^x \left(\prod_{j=1}^k f_j(X_{t_j}) e^{-\beta t_k} E^{X_{t_k}} \left(\int_0^\infty e^{-\beta t} d_t L_t^x \right) \right) \\ &= P^x \left(\prod_{j=1}^k f_j(X_{t_j}) e^{-\beta t_k} u^\beta(X_{t_k}, x) \right) \\ &= \int_{S^k} P_{t_1}^\beta(x, dy_1) f_1(y_1) \left(\prod_{j=2}^k P_{t_j-t_{j-1}}^\beta(y_{j-1}, dy_j) f_j(y_j) \right) u^\beta(y_k, x). \end{aligned} \tag{5.4}$$

Here $P_t^\beta(x, dy) = e^{-\beta t} P_t(x, dy)$. Using (2.38) and then the Markov property as in the

previous display

$$\begin{aligned}
 & \int_{t_1}^{\infty} e^{-\beta t} \Gamma_{y_k, y_1}(dt) & (5.5) \\
 &= E^{y_k} \left(\int_{t_1}^{\infty} e^{-\beta t} d_t L_t^{y_1} \right) \\
 &= e^{-\beta t_1} E^{y_k} \left(\left(\int_0^{\infty} e^{-\beta t} d_t L_t^{y_1} \right) \circ \theta_{t_1} \right) \\
 &= e^{-\beta t_1} E^{y_k} (u^\beta (X_{t_1}, y_1)) \\
 &= \int_S P_{t_1}^\beta (y_k, dz) u^\beta (z, y_1).
 \end{aligned}$$

We claim that for a.e. t_1 , as measures in y_1 ,

$$\int_{x \in S} u^\beta (y_k, x) P_{t_1}^\beta (x, dy_1) dm(x) = \int_{z \in S} P_{t_1}^\beta (y_k, dz) u^\beta (z, y_1) dm(y_1). \quad (5.6)$$

To see this, it suffices to integrate both sides with respect to $e^{-\alpha t_1} dt_1$, use (2.3) with α replaced by $\alpha + \beta$, and the fact that S has a countable base. (It is important to note that we allow $y_k = y_1$).

Combining (5.3)-(5.6) we obtain for a.e. t_1

$$\begin{aligned}
 & \nu \left(\prod_{j=1}^k f_j(X_{t_j}) e^{-\beta \zeta} \right) & (5.7) \\
 &= \int f_1(y_1) \prod_{j=2}^k P_{t_j - t_{j-1}}^\beta (y_{j-1}, dy_j) f_j(y_j) \int_{t_1}^{\infty} e^{-\beta t} \Gamma_{y_k, y_1}(dt) dm(y_1).
 \end{aligned}$$

This agrees with what we obtain from the right hand side of (5.2):

$$\begin{aligned}
 & \int_{S^k \times (t_k, \infty)} m(dy_1) \prod_{j=2}^k P_{t_j - t_{j-1}}^\beta (y_{j-1}, dy_j) \Gamma_{y_k, y_1}(dt - t_k + t_1) & (5.8) \\
 & \left(\prod_{j=1}^k f_j(y_j) e^{-\beta t} \right) \\
 &= \int_{S^k} m(dy_1) f_1(y_1) \prod_{j=2}^k P_{t_j - t_{j-1}}^\beta (y_{j-1}, dy_j) f_j(y_j) \\
 & \int_{t_k}^{\infty} e^{-\beta t} \Gamma_{y_k, y_1}(dt - t_k + t_1) \\
 &= \int_{S^k} m(dy_1) f_1(y_1) \prod_{j=2}^k P_{t_j - t_{j-1}}^\beta (y_{j-1}, dy_j) f_j(y_j) \\
 & e^{-\beta(t_k - t_1)} \int_{t_1}^{\infty} e^{-\beta t} \Gamma_{y_k, y_1}(dt).
 \end{aligned}$$

This completes the proof of our Lemma when $t_1 > 0$.

When $t_1 = 0$, it follows from (5.3)-(5.4), and then (2.4) and (2.38) that

$$\begin{aligned} & \nu \left(\prod_{j=1}^k f_j(X_{t_j}) e^{-\beta \zeta} \right) \\ &= \int_S f_1(x) \prod_{j=2}^k P_{t_j - t_{j-1}}^\beta(y_{j-1}, dy_j) f_j(y_j) u^\beta(y_k, x) dm(x) \\ &= \int_S f_1(x) \prod_{j=2}^k P_{t_j - t_{j-1}}^\beta(y_{j-1}, dy_j) f_j(y_j) \int_0^\infty e^{-\beta t} \Gamma_{y_k, x}(dt) dm(x). \end{aligned} \tag{5.9}$$

This agrees with (5.7) for $t_1 = 0$, and the rest of the proof follows as in (5.8). □

As a byproduct of our proof we now show that

$$\sup_{t_1 \geq 0} \nu (f_1(X_{t_1}) e^{-\beta \zeta}) < \infty, \tag{5.10}$$

for any continuous compactly supported f_1 . To see this, note that by (5.3)-(5.4)

$$\nu (f_1(X_{t_1}) e^{-\beta \zeta}) = \int_S \int_S P_{t_1}^\beta(x, dy_1) f_1(y_1) u^\beta(y_1, x) dm(x). \tag{5.11}$$

By (5.6), for a.e. t_1 this equals

$$\int_S \left(\int_S P_{t_1}^\beta(y_1, dz) u^\beta(z, y_1) \right) f_1(y_1) dm(y_1). \tag{5.12}$$

But as noted in the paragraph containing (2.2), $u^\beta(z, y_1)$ is bounded, uniformly in z for y_1 in the compact support of $f_1(y_1)$. Hence (5.12) is bounded by

$$C \int_S \left(\int_S P_{t_1}^\beta(y_1, dz) \right) f_1(y_1) dm(y_1) \leq C \int_S f_1(y_1) dm(y_1). \tag{5.13}$$

Thus we have shown that for some dense $D \subseteq R_+^1$

$$\sup_{t_1 \in D} \nu (f_1(X_{t_1}) e^{-\beta \zeta}) \leq C \int_S f_1(y_1) dm(y_1), \tag{5.14}$$

and the right hand side is finite by (2.5). (5.10) then follows using right continuity.

We will also need the following.

Lemma 5.2.

$$P_s(x, dy) ds = m(dy) \Gamma_{x,y}(ds). \tag{5.15}$$

Proof of Lemma 5.2: We have

$$\begin{aligned} & \int \Gamma_{x,y}([0, t]) f(y) m(dy) \\ &= \int E^x (L_t^y) f(y) m(dy) = \int E^x (L_\infty^y - L_\infty^y \circ \theta_t) f(y) m(dy) \\ &= \int (u(x, y) - E^x (u(X_t, y))) f(y) m(dy) \\ &= \int \int_0^t P_s(x, dy) f(y) ds. \end{aligned} \tag{5.16}$$

□

Let us define the process \bar{X} to be the periodic extension of X ; that is,

$$\bar{X}_t = \begin{cases} X_{t-q\zeta}, & \text{if } q\zeta \leq t < (q+1)\zeta, \quad q = 0, 1, 2, \dots \\ X_t, & \text{if } \zeta = \infty \end{cases} \quad (5.17)$$

It will be convenient to write

$$\bar{I}_\alpha(f) := \int_0^\infty e^{-\alpha t} f(\bar{X}_t) dt, \quad I_\alpha(f) := \int_0^\infty e^{-\alpha t} f(X_t) dt. \quad (5.18)$$

The key observation is that

$$\bar{I}_\alpha(f) = \frac{I_\alpha(f)}{1 - e^{-\alpha\zeta}}, \quad (5.19)$$

for all $\alpha > 0$. This follows from

$$\begin{aligned} \bar{I}_\alpha(f) &= \int_0^\infty e^{-\alpha t} f(\bar{X}_t) dt \\ &= \sum_{q=0}^\infty \int_{q\zeta}^{(q+1)\zeta} e^{-\alpha t} f(\bar{X}_t) dt \\ &= \sum_{q=0}^\infty e^{-\alpha q\zeta} \int_0^\zeta e^{-\alpha t} f(X_t) dt = \frac{I_\alpha(f)}{1 - e^{-\alpha\zeta}}. \end{aligned}$$

Hence for any continuous compactly supported f

$$\begin{aligned} &\int_0^\infty e^{-\alpha t} \nu((1 - e^{-\alpha\zeta}) f(\bar{X}_t) e^{-\beta\zeta}) dt \\ &= \nu((1 - e^{-\alpha\zeta}) \bar{I}_\alpha(f) e^{-\beta\zeta}) = \nu(I_\alpha(f) e^{-\beta\zeta}) \\ &= \int_0^\infty e^{-\alpha t} \nu(f(X_t) e^{-\beta\zeta}) dt < \infty \end{aligned} \quad (5.20)$$

by (5.10). It follows that for any α

$$\nu((1 - e^{-\alpha\zeta}) f(\bar{X}_t) e^{-\beta\zeta}) < \infty, \text{ for a.e. } t. \quad (5.21)$$

The rotation invariance of μ or ν is equivalent to the statement that

$$\nu\left(\prod_{j=1}^k f_j(\bar{X}_{t_j+r}) 1_{\{t_k < \zeta\}}\right) = \nu\left(\prod_{j=1}^k f_j(\bar{X}_{t_j}) 1_{\{t_k < \zeta\}}\right) \quad (5.22)$$

for all $0 < t_1 < \dots < t_k$ and $r > 0$ and all $f_j \geq 0$ continuous with compact support. This will follow once we show that the joint distribution of (\bar{X}, ζ) is invariant under time shifts. That is, $((\bar{X}_{t+v})_{t \geq 0}, \zeta)$ has the same distribution (under ν) as $((\bar{X}_t)_{t \geq 0}, \zeta)$ for all $v > 0$.

To prove this we will first show that for all k and all $\alpha_1, \dots, \alpha_k$,

$$\begin{aligned} &\int_{[0, \infty)^k} e^{-\sum_{j=1}^k \alpha_j t_j} \nu\left(\prod_{j=1}^k f_j(\bar{X}_{t_j}) g(\zeta)\right) \prod_{j=1}^k dt_j \\ &= \int_{[0, \infty)^k} e^{-\sum_{j=1}^k \alpha_j t_j} F_k(t_1, \dots, t_k) \prod_{j=1}^k dt_j \end{aligned} \quad (5.23)$$

for all g of the form $g(\zeta) = (1 - e^{-\alpha\zeta})e^{-\beta\zeta}$, and where

$$F_k(t_1, \dots, t_k) = \sum_{\sigma \in \mathcal{P}_k} 1_{\{0 \leq t_{\sigma(1)} \leq \dots \leq t_{\sigma(k)}\}} \nu \left(f_{\sigma(1)}(X_0) \prod_{j=2}^k f_{\sigma(j)}(\bar{X}_{t_{\sigma(j)} - t_{\sigma(1)}}) g(\zeta) \right). \tag{5.24}$$

By (5.20) the left hand side of (5.23) is finite for all $\alpha_1 \geq \alpha$, while the right hand side is finite since

$$\nu(f_j(X_0)) = \int_S Q^{x,x}(f_j(X_0)) dm(x) = \int_S u(x,x) f_j(x) dm(x) < \infty. \tag{5.25}$$

By uniqueness of Laplace transforms, it then follows that

$$\nu \left(\prod_{j=1}^k f_j(\bar{X}_{t_j}) g(\zeta) \right) = F_k(t_1, \dots, t_k) \tag{5.26}$$

for Lebesgue a.e. k -tuple (t_1, \dots, t_k) , and in particular, for any $r > 0$,

$$\begin{aligned} \int_{[r,\infty)^k} e^{-\sum_{j=1}^k \alpha_j t_j} \nu \left(\prod_{j=1}^k f_j(\bar{X}_{t_j}) g(\zeta) \right) \prod_{j=1}^k dt_j \\ = \int_{[r,\infty)^k} e^{-\sum_{j=1}^k \alpha_j t_j} F_k(t_1, \dots, t_k) \prod_{j=1}^k dt_j. \end{aligned} \tag{5.27}$$

It follows that for any $r > 0$,

$$\begin{aligned} \int_{[0,\infty)^k} e^{-\sum_{j=1}^k \alpha_j (t_j+r)} \nu \left(\prod_{j=1}^k f_j(\bar{X}_{t_j+r}) g(\zeta) \right) \prod_{j=1}^k dt_j \\ = \int_{[0,\infty)^k} e^{-\sum_{j=1}^k \alpha_j (t_j+r)} F_k(t_1+r, \dots, t_k+r) \prod_{j=1}^k dt_j. \end{aligned} \tag{5.28}$$

But it is easily seen that $F_k(t_1+r, \dots, t_k+r) = F_k(t_1, \dots, t_k)$ so that, canceling the common constant factor $e^{-\sum_{j=1}^k \alpha_j r}$, we obtain

$$\begin{aligned} \int_{[0,\infty)^k} e^{-\sum_{j=1}^k \alpha_j t_j} \nu \left(\prod_{j=1}^k f_j(\bar{X}_{t_j+r}) g(\zeta) \right) \prod_{j=1}^k dt_j \\ = \int_{[0,\infty)^k} e^{-\sum_{j=1}^k \alpha_j t_j} F_k(t_1, \dots, t_k) \prod_{j=1}^k dt_j, \end{aligned} \tag{5.29}$$

and thus comparing with (5.23) we have that for each $r > 0$

$$\begin{aligned} \int_{[0,\infty)^k} e^{-\sum_{j=1}^k \alpha_j t_j} \nu \left(\prod_{j=1}^k f_j(\bar{X}_{t_j+r}) g(\zeta) \right) \prod_{j=1}^k dt_j \\ = \int_{[0,\infty)^k} e^{-\sum_{j=1}^k \alpha_j t_j} \nu \left(\prod_{j=1}^k f_j(\bar{X}_{t_j}) g(\zeta) \right) \prod_{j=1}^k dt_j. \end{aligned} \tag{5.30}$$

It follows that

$$\nu \left(\prod_{j=1}^k f_j(\bar{X}_{t_j+r}) g(\zeta) \right) = \nu \left(\prod_{j=1}^k f_j(\bar{X}_{t_j}) g(\zeta) \right), \quad \text{for a.e. } t_1, \dots, t_k. \quad (5.31)$$

This holds for any k , in particular for $k = 1$, so that using (5.21) we have

$$\nu (f_1(\bar{X}_{t_1+r}) g(\zeta)) = \nu (f_1(\bar{X}_{t_1}) g(\zeta)) < \infty, \quad \text{for a.e. } t_1. \quad (5.32)$$

Thus by Fubini we can find a set $T \subseteq R_+$ with T^c of Lebesgue measure 0 such that for all $t_1 \in T$ we have (5.32), and (5.31) holds for a.e. t_2, \dots, t_k . Using the boundedness and continuity of the f_j and the right continuity of \bar{X}_t it follows from the Dominated Convergence Theorem that (5.31) holds for all $(t_1, t_2, \dots, t_k) \in T \times R_+^{k-1}$. Let now $f_{1,n}$ be a sequence of continuous functions with compact support with the property that $f_{1,n} \uparrow 1$. By the above, (5.31) with f_1 replaced by $f_{1,n}$ holds for all $(t_1, t_2, \dots, t_k) \in T_n \times R_+^{k-1}$ for an appropriate $T_n \subseteq R_+$ with T_n^c of Lebesgue measure 0. In particular $T_* = \bigcap_n T_n \neq \emptyset$, and we can apply the Monotone Convergence Theorem with $t_1 \in T_*$ to conclude, spelling out $g(\zeta)$, that

$$\nu \left((1 - e^{-\alpha\zeta}) \prod_{j=2}^k f_j(\bar{X}_{t_j+r}) e^{-\beta\zeta} \right) = \nu \left((1 - e^{-\alpha\zeta}) \prod_{j=2}^k f_j(\bar{X}_{t_j}) e^{-\beta\zeta} \right) \quad (5.33)$$

for all t_2, \dots, t_k . Applying once again the Monotone Convergence Theorem for $\alpha \rightarrow \infty$ we obtain

$$\nu \left(\prod_{j=2}^k f_j(\bar{X}_{t_j+r}) e^{-\beta\zeta} \right) = \nu \left(\prod_{j=2}^k f_j(\bar{X}_{t_j}) e^{-\beta\zeta} \right) \quad (5.34)$$

for all t_2, \dots, t_k . Fix a compact $K \subseteq S$. If we replace f_2 by a sequence $f_{2,n} \uparrow 1_K$ and then set $t_2 = 0$, we can conclude from (5.34) and (5.25) that the finite measures $1_K(X_0) \cdot \nu$ and $1_K(X_0) \cdot \rho_{r*} \nu$ agree on the σ -algebra generated by $\bar{X}_t, t \geq 0$ and ζ . Since this holds for any compact $K \subseteq S$, so do ν and $\rho_{r*} \nu$. Here and below we use the notation $f_* \nu(A) = \nu(f^{-1}(A))$.

It remains to prove (5.23). Using (5.19)

$$\begin{aligned} & \int e^{-\sum_{j=1}^k \alpha_j t_j} \nu \left(\prod_{j=1}^k f_j(\bar{X}_{t_j}) g(\zeta) \right) \prod_{j=1}^k dt_j \\ &= \nu \left(\prod_{j=1}^k \bar{I}_{\alpha_j}(f_j) g(\zeta) \right) \\ &= \nu \left(\int e^{-\sum_{j=1}^k \alpha_j t_j} \prod_{j=1}^k \frac{f_j(X_{t_j})}{1 - e^{-\alpha_j \zeta}} \prod_{j=1}^k dt_j g(\zeta) \right) = \sum_{\sigma \in \mathcal{P}_k} J_k(\sigma), \end{aligned} \quad (5.35)$$

where

$$J_k(\sigma) := \nu \left(\int_{0 < t_1 < \dots < t_k} e^{-\sum_{j=1}^k \alpha_{\sigma(j)} t_j} \prod_{j=1}^k \frac{f_{\sigma(j)}(X_{t_j})}{1 - e^{-\alpha_{\sigma(j)} \zeta}} \prod_{j=1}^k dt_j g(\zeta) \right),$$

and using (5.2)

$$\begin{aligned}
 J_k(\sigma) &= \int_{0 < t_1 < \dots < t_k < t} g(t) \int_{S^k} \frac{e^{-\alpha_{\sigma(1)}t_1}}{1 - e^{-\alpha_{\sigma(1)}t}} m(dy_1) f_{\sigma(1)}(y_1) \\
 &\quad \prod_{j=2}^k \frac{e^{-\alpha_{\sigma(j)}t_j}}{1 - e^{-\alpha_{\sigma(j)}t}} P_{t_j - t_{j-1}}(y_{j-1}, dy_j) f_{\sigma(j)}(y_j) \\
 &\quad \Gamma_{y_k, y_1}(dt - t_k + t_1) dt_1 \cdots dt_k.
 \end{aligned} \tag{5.36}$$

We now make the change of variables $r = t_1$, $s_j = t_j - t_{j-1}$ ($j = 2, \dots, k$), $s_1 = t - t_k + t_1$ (with accompanying limits of integration $0 < r < s_1$, $s_j > 0$) and then integrate out r . Writing $\hat{s}_j := s_2 + \dots + s_j$ and $\bar{s} := \sum_{j=1}^k s_j$, the expression in (5.36) is thereby transformed to

$$\begin{aligned}
 J_k(\sigma) &= \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) \int_{S^k} \left(\int_0^{s_1} e^{-(\alpha_1 + \dots + \alpha_k)r} dr \right) m(dy_1) f_{\sigma(1)}(y_1) \\
 &\quad \frac{1}{1 - e^{-\alpha_{\sigma(1)}\bar{s}}} \prod_{j=2}^k \frac{e^{-\alpha_{\sigma(j)}\hat{s}_j}}{1 - e^{-\alpha_{\sigma(j)}\bar{s}}} P_{s_j}(y_{j-1}, dy_j) f_{\sigma(j)}(y_j) \Gamma_{y_k, y_1}(ds_1) ds_2 \cdots ds_k \\
 &= \left(\sum_{j=1}^k \alpha_j \right)^{-1} \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) \int_{S^k} m(dy_1) f_{\sigma(1)}(y_1) \frac{(1 - e^{-(\alpha_1 + \dots + \alpha_k)s_1})}{1 - e^{-\alpha_{\sigma(1)}\bar{s}}} \\
 &\quad \prod_{j=2}^k \frac{e^{-\alpha_{\sigma(j)}\hat{s}_j}}{1 - e^{-\alpha_{\sigma(j)}\bar{s}}} P_{s_j}(y_{j-1}, dy_j) f_{\sigma(j)}(y_j) \Gamma_{y_k, y_1}(ds_1) ds_2 \cdots ds_k.
 \end{aligned} \tag{5.37}$$

Using (5.15), we can write (5.37) as

$$\begin{aligned}
 J_k(\sigma) &= \left(\sum_{j=1}^k \alpha_j \right)^{-1} \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) (1 - e^{-(\alpha_1 + \dots + \alpha_k)s_1}) \\
 &\quad \prod_{j=1}^k \frac{e^{-\alpha_{\sigma(j)}\hat{s}_j}}{1 - e^{-\alpha_{\sigma(j)}\bar{s}}} \int_{S^k} \prod_{j=1}^k m(dy_j) f_{\sigma(j)}(y_j) \Gamma_{y_{j-1}, y_j}(ds_j),
 \end{aligned} \tag{5.38}$$

where $y_0 = y_k$ and $\hat{s}_1 := 0$.

We now turn to the right hand side of (5.23) and try to rewrite it in terms which are similar to our last expression for the $J_k(\sigma)$'s. Using $\sum_{j=1}^k \alpha_j t_j = \sum_{j=1}^k \alpha_{\sigma(j)} t_{\sigma(j)}$ we have

$$\begin{aligned}
 R(\alpha_1, \dots, \alpha_k) &:= \int e^{-\sum_{j=1}^k \alpha_j t_j} F_k(t_1, \dots, t_k) \prod_{j=1}^k dt_j \\
 &= \sum_{\sigma \in \mathcal{P}_k} \int_{\{0 \leq t_{\sigma(1)} \leq \dots \leq t_{\sigma(k)}\}} e^{-\sum_{j=1}^k \alpha_{\sigma(j)} t_{\sigma(j)}} \\
 &\quad \nu \left(f_{\sigma(1)}(X_0) \prod_{j=2}^k f_{\sigma(j)}(\bar{X}_{t_{\sigma(j)} - t_{\sigma(1)}}) g(\zeta) \right) \prod_{j=1}^k dt_j \\
 &= \sum_{\sigma \in \mathcal{P}_k} \int_{\{0 \leq t_1 \leq \dots \leq t_k\}} e^{-\sum_{j=1}^k \alpha_{\sigma(j)} t_j} \\
 &\quad \nu \left(f_{\sigma(1)}(X_0) \prod_{j=2}^k f_{\sigma(j)}(\bar{X}_{t_j - t_1}) g(\zeta) \right) \prod_{j=1}^k dt_j.
 \end{aligned} \tag{5.39}$$

Let us now fix $\sigma \in \mathcal{P}_k$ and consider the corresponding term in (5.39)

$$\int_{0 < t_1 < \dots < t_k} \prod_{j=1}^k e^{-\alpha_{\sigma(j)} t_j} dt_j \nu \left(f_{\sigma(1)}(\bar{X}_0) \prod_{j=2}^k f_{\sigma(j)}(\bar{X}_{t_j - t_1}) g(\zeta) \right). \quad (5.40)$$

Changing variables ($r_1 = t_1, r_j = t_j - t_1$ for $j = 2, \dots, k$) and integrating out r_1 , this can be rewritten as

$$\left(\sum_{j=1}^k \alpha_j \right)^{-1} \int_{0 < r_2 < \dots < r_k} \prod_{j=2}^k e^{-\alpha_{\sigma(j)} r_j} dr_j \nu \left(f_{\sigma(1)}(\bar{X}_0) \prod_{j=2}^k f_{\sigma(j)}(\bar{X}_{r_j}) g(\zeta) \right). \quad (5.41)$$

Summing first over all permutations $\sigma \in \mathcal{P}_k$ with $\sigma(1) = i$ and then over i we obtain

$$R(\alpha_1, \dots, \alpha_k) = \left(\sum_{j=1}^k \alpha_j \right)^{-1} \sum_{i=1}^k \nu \left(f_i(\bar{X}_0) \prod_{j \neq i} \bar{I}_{\alpha_j}(f_j) \cdot g(\zeta) \right). \quad (5.42)$$

Using (5.19) we can express this as

$$R(\alpha_1, \dots, \alpha_k) = \left(\sum_{j=1}^k \alpha_j \right)^{-1} \sum_{i=1}^k \nu \left(f_i(X_0) \prod_{j \neq i} \frac{I_{\alpha_j}(f_j)}{1 - e^{-\alpha_j \zeta}} \cdot g(\zeta) \right). \quad (5.43)$$

Using Lemma 5.1 we then see that

$$R(\alpha_1, \dots, \alpha_k) = \sum_{\sigma' \in \mathcal{P}_k} K_k(\sigma') \quad (5.44)$$

where

$$K_k(\sigma') := \left(\sum_{j=1}^k \alpha_j \right)^{-1} \int_{0 < t_2 < \dots < t_k < t} g(t) \int_{S^k} m(dy_1) f_{\sigma'(1)}(y_1) \prod_{j=2}^k P_{t_j - t_{j-1}}(y_{j-1}, dy_j) f_{\sigma'(j)}(y_j) \frac{e^{-\alpha_{\sigma'(j)} t_j}}{1 - e^{-\alpha_{\sigma'(j)} t}} \Gamma_{y_k, y_1}(dt - t_k) dt_2 \dots dt_k, \quad (5.45)$$

with the convention that $t_1 = 0$. Once more making the change of variables $s_1 = t - t_k, s_2 = t_2, s_j = t_j - t_{j-1}$ for $j = 3, \dots, k$, (5.45) becomes

$$K_k(\sigma') = \left(\sum_{j=1}^k \alpha_j \right)^{-1} \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) \int_{S^k} m(dy_1) f_{\sigma'(1)}(y_1) \prod_{j=2}^k P_{s_j}(y_{j-1}, dy_j) f_{\sigma'(j)}(y_j) \frac{e^{-\alpha_{\sigma'(j)} \hat{s}_j}}{1 - e^{-\alpha_{\sigma'(j)} \bar{s}}} \Gamma_{y_k, y_1}(ds_1) ds_2 \dots ds_k. \quad (5.46)$$

Using (5.15) again, we can write (5.46) as

$$K_k(\sigma') = \left(\sum_{j=1}^k \alpha_j \right)^{-1} \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) \int_{S^k} m(dy_1) f_{\sigma'(1)}(y_1) \quad (5.47)$$

$$\left(\prod_{j=2}^k m(dy_j) \Gamma_{y_{j-1}, y_j}(ds_j) \frac{e^{-\alpha_{\sigma'(j)} \hat{s}_j}}{1 - e^{-\alpha_{\sigma'(j)} \bar{s}}} \right) f_{\sigma'(j)}(y_j) \Gamma_{y_k, y_1}(ds_1) ds_2 \cdots ds_k$$

$$= \left(\sum_{j=1}^k \alpha_j \right)^{-1} \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) \int_{S^k} (1 - e^{-\alpha_{\sigma'(1)} \bar{s}}) \prod_{j=1}^k m(dy_j) \Gamma_{y_{j-1}, y_j}(ds_j) f_{\sigma'(j)}(y_j) \frac{e^{-\alpha_{\sigma'(j)} \hat{s}_j}}{1 - e^{-\alpha_{\sigma'(j)} \bar{s}}}, \quad (5.48)$$

where $y_0 = y_k$, and $\hat{s}_1 = 0$. We reorganize this as

$$K_k(\sigma') = \left(\sum_{j=1}^k \alpha_j \right)^{-1} \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) (1 - e^{-\alpha_{\sigma'(1)} \bar{s}}) \prod_{j=1}^k \frac{e^{-\alpha_{\sigma'(j)} \hat{s}_j}}{1 - e^{-\alpha_{\sigma'(j)} \bar{s}}} \int_{S^k} \prod_{j=1}^k m(dy_j) f_{\sigma'(j)}(y_j) \Gamma_{y_{j-1}, y_j}(ds_j). \quad (5.49)$$

In view of (5.35) and (5.44), to prove (5.23) we need to show that

$$\sum_{\sigma \in \mathcal{P}_k} J_k(\sigma) = \sum_{\sigma \in \mathcal{P}_k} K_k(\sigma), \quad (5.50)$$

and to this end it suffices to show that for each $\sigma^* \in \mathcal{P}_k$

$$\sum_{\sigma \sim \sigma^*} J_k(\sigma) = \sum_{\sigma \sim \sigma^*} K_k(\sigma), \quad (5.51)$$

where $\sigma \sim \sigma^*$ means that σ is a ‘rotation’ of σ^* . In other words, for some $1 \leq l \leq k$ we have

$$(\sigma(1), \sigma(2), \dots, \sigma(k)) = (\sigma^*(l), \sigma^*(l+1), \dots, \sigma^*(k), \sigma^*(1), \dots, \sigma^*(l-1)).$$

Comparing (5.38) and (5.49) with $\sigma' = \sigma$ we see that the only difference is the presence of $e^{-(\alpha_1 + \dots + \alpha_k) s_1}$ in (5.38) while in (5.49), with $\sigma' = \sigma$, this is replaced by $e^{-\alpha_{\sigma(1)} \bar{s}}$. Thus to prove (5.51) it suffices to show that

$$\int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) e^{-(\alpha_1 + \dots + \alpha_k) s_1} \prod_{j=1}^k \frac{e^{-\alpha_{\sigma(j)} \hat{s}_j}}{1 - e^{-\alpha_{\sigma(j)} \bar{s}}} \int_{S^k} \prod_{j=1}^k m(dy_j) f_{\sigma(j)}(y_j) \Gamma_{y_{j-1}, y_j}(ds_j) \quad (5.52)$$

$$= \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) e^{-\alpha_{\sigma'(1)} \bar{s}} \prod_{j=1}^k \frac{e^{-\alpha_{\sigma'(j)} \hat{s}_j}}{1 - e^{-\alpha_{\sigma'(j)} \bar{s}}} \int_{S^k} \prod_{j=1}^k m(dy_j) f_{\sigma'(j)}(y_j) \Gamma_{y_{j-1}, y_j}(ds_j)$$

whenever $(\sigma'(1), \sigma'(2), \dots, \sigma'(k)) = (\sigma(k), \sigma(1), \dots, \sigma(k-1))$.

Note that

$$\begin{aligned}
 M_{\sigma'}(ds_1, \dots, ds_k) &:= \int_{S^k} \prod_{j=1}^k m(dy_j) f_{\sigma'(j)}(y_j) \Gamma_{y_{j-1}, y_j}(ds_j) & (5.53) \\
 &= \int_{S^k} f_{\sigma'(1)}(y_1) \Gamma_{y_k, y_1}(ds_1) f_{\sigma'(2)}(y_2) \Gamma_{y_1, y_2}(ds_2) \cdots \\
 &\quad \cdots f_{\sigma'(k)}(y_k) \Gamma_{y_{k-1}, y_k}(ds_k) \prod_{j=1}^k m(dy_j) \\
 &= \int_{S^k} f_{\sigma(k)}(y_1) \Gamma_{y_k, y_1}(ds_1) f_{\sigma(1)}(y_2) \Gamma_{y_1, y_2}(ds_2) \cdots \\
 &\quad \cdots f_{\sigma(k-1)}(y_k) \Gamma_{y_{k-1}, y_k}(ds_k) \prod_{j=1}^k m(dy_j),
 \end{aligned}$$

and relabeling the y_j 's this is

$$\begin{aligned}
 &= \int_{S^k} f_{\sigma(k)}(y_k) \Gamma_{y_{k-1}, y_k}(ds_1) f_{\sigma(1)}(y_1) \Gamma_{y_k, y_1}(ds_2) \cdots & (5.54) \\
 &\quad \cdots f_{\sigma(k-1)}(y_{k-1}) \Gamma_{y_{k-2}, y_{k-1}}(ds_k) \prod_{j=1}^k m(dy_j) \\
 &= \int_{S^k} \prod_{j=1}^k m(dy_j) f_{\sigma(j)}(y_j) \Gamma_{y_{j-1}, y_j}(ds_{j+1}) = M_{\sigma}(ds_2, \dots, ds_k, ds_1),
 \end{aligned}$$

where $s_{k+1} = s_1$. Furthermore, (recall that $\hat{s}_1 = 0$),

$$\begin{aligned}
 &\alpha_{\sigma'(1)} \bar{s} + \sum_{j=2}^k \alpha_{\sigma'(j)} \hat{s}_j & (5.55) \\
 &= \alpha_{\sigma(k)} \bar{s} + \sum_{j=2}^k \alpha_{\sigma(j-1)} \left(\sum_{l=2}^j s_l \right) \\
 &= \alpha_{\sigma(k)} \bar{s} + \sum_{i=1}^{k-1} \alpha_{\sigma(i)} \left(\sum_{l=2}^{i+1} s_l \right) \\
 &= \alpha_{\sigma(k)} (s_1 + \cdots + s_k) + \sum_{i=1}^{k-1} \alpha_{\sigma(i)} \left(\sum_{l=1}^i s_{l+1} \right) = \sum_{i=1}^k \alpha_{\sigma(i)} \left(\sum_{l=1}^i s_{l+1} \right).
 \end{aligned}$$

But also, using $\sum_{j=1}^k \alpha_j = \sum_{j=1}^k \alpha_{\sigma(j)}$,

$$(\alpha_1 + \cdots + \alpha_k) s_1 + \sum_{j=2}^k \alpha_{\sigma(j)} \hat{s}_j = \sum_{j=1}^k \alpha_{\sigma(j)} \left(\sum_{l=1}^j s_l \right) \tag{5.56}$$

where $s_{k+1} = s_1$. Combining (5.53)-(5.56), we see that (5.52) is the claim that

$$\begin{aligned} & \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) \exp \left(- \sum_{j=1}^k \alpha_{\sigma(j)} \left(\sum_{l=1}^j s_l \right) \right) \\ & \qquad \qquad \qquad \prod_{j=1}^k \frac{1}{1 - e^{-\alpha_{\sigma(j)} \bar{s}}} M_{\sigma}(ds_1, \dots, ds_k) \\ = & \int_{s_1 > 0, \dots, s_k > 0} g(\bar{s}) \exp \left(- \sum_{j=1}^k \alpha_{\sigma(j)} \left(\sum_{l=1}^j s_{l+1} \right) \right) \\ & \qquad \qquad \qquad \prod_{j=1}^k \frac{1}{1 - e^{-\alpha_{\sigma'(j)} \bar{s}}} M_{\sigma}(ds_2, \dots, ds_k, ds_1) \end{aligned} \tag{5.57}$$

where $s_{k+1} = s_1$, and this claim follows immediately from the relabeling $(s_1, \dots, s_k) \rightarrow (s_2, \dots, s_k, s_1)$. This establishes (5.52) and hence (5.50). \square

6 The restriction property

Let $B \subseteq S$ be open and set

$$T_{B^c} = \inf\{t \geq 0 \mid X_t \in B^c\}. \tag{6.1}$$

Let

$$\tilde{X}_t(\omega) = \begin{cases} X_t(\omega) & \text{if } t < T_{B^c} \\ \Delta & \text{otherwise.} \end{cases} \tag{6.2}$$

Clearly, $t \mapsto \tilde{X}_t$ is right continuous. With

$$\tilde{P}_t f(x) = E^x(f(X_t)1_{\{t < T_{B^c}\}}), \tag{6.3}$$

and

$$\tilde{\theta}_t(\omega) = \begin{cases} \theta_t(\omega) & \text{if } t < T_{B^c}(\omega) \\ \Delta & \text{otherwise,} \end{cases} \tag{6.4}$$

we show in [23, Section 4.5] that $\tilde{X} = (\Omega, \mathcal{G}_t, \mathcal{G}, \tilde{X}_t, \tilde{\theta}_t, \tilde{P}^x)$ is a Borel right process with state space B and potential densities

$$\tilde{u}(x, y) = u(x, y) - E^x(u(X_{T_{B^c}}, y)), \quad x, y \in B, \tag{6.5}$$

with respect to the measure $m(dx)$ restricted to B . Here we have used the convention that $u(\Delta, y) = 0$ and that $X_t(\omega) = \Delta$ when $t = +\infty$. It follows as before that uniformly in x , $\tilde{u}(x, y)$ is locally bounded and continuous in y .

Let $\{L_t^x, (x, t) \in S \times R_+\}$ be the family of local times for X used in the construction of μ . Set $\tilde{L}_t^x = L_{t \wedge T_{B^c}}^x$ for $x \in B$. It is easy to see that \tilde{L}_t^x is a CAF for \tilde{X} and

$$\begin{aligned} \tilde{E}^x(\tilde{L}_\infty^y) &= E^x(L_{T_{B^c}}^y) \\ &= E^x(L_\infty^y) - E^x(L_\infty^y \circ \theta_{T_{B^c}}) \\ &= u(x, y) - E^x(u(X_{T_{B^c}}, y)) = \tilde{u}(x, y). \end{aligned}$$

It follows that $\{\tilde{L}_t^x, (x, t) \in B \times R_+\}$ are local times for \tilde{X} . We can then define the loop measure $\tilde{\mu}$ for \tilde{X} by the formula

$$\int F d\tilde{\mu} = \int_B \tilde{P}^x \left(\int_0^\infty \frac{1}{t} F \circ k_t d_t \tilde{L}_t^x \right) dm(x). \tag{6.6}$$

(In our definition (2.6) of μ we assumed that X had continuous potential densities. We do not know if $\tilde{u}(x, y)$ is continuous in x . However, the continuity of $u(x, y)$ was only used to guarantee a nice family of local times for X , and by the above this is inherited by $\{\tilde{L}_t^x, (x, t) \in B \times R_+\}$).

Theorem 6.1 (The Restriction Property).

$$\mu(F; T_{B^c} = \infty) = \tilde{\mu}(F). \tag{6.7}$$

Note that $B^c = S - B$ does not contain Δ .

Proof of Theorem 6.1: It suffices to prove this for F of the form $\prod_{j=1}^k f_j(X_{t_j})$ with $t_1 < \dots < t_k$. Since $f_k(\Delta) = 0$, and $1_{\{T_{B^c} = \infty\}} \circ k_t = 1_{\{t \leq T_{B^c}\}}$ we have

$$1_{\{T_{B^c} = \infty\}} \circ k_t \prod_{j=1}^k f_j(X_{t_j}) \circ k_t = 1_{\{t_k < t \leq T_{B^c}\}} \prod_{j=1}^k f_j(X_{t_j}). \tag{6.8}$$

Hence

$$\begin{aligned} & \mu \left(\prod_{j=1}^k f_j(X_{t_j}); T_{B^c} = \infty \right) \\ &= \int_S P^x \left(\int_{t_k}^{T_{B^c}} \frac{1}{t} \prod_{j=1}^k f_j(X_{t_j}) dL_t^x \right) dm(x) \\ &= \int_B P^x \left(\prod_{j=1}^k f_j(X_{t_j}) 1_{\{t_k < T_{B^c}\}} \int_{t_k}^{\infty} \frac{1}{t} d\tilde{L}_t^x \right) dm(x) \\ &= \int_B P^x \left(\prod_{j=1}^k f_j(X_{t_j}) 1_{\{t_k < T_{B^c}\}} \left(\int_{t_k}^{\infty} \frac{1}{t} d_t \tilde{L}_{t-t_k}^x \right) \circ \theta_{t_k} \right) dm(x) \\ &= \int_B P^x \left(\prod_{j=1}^k f_j(X_{t_j}) 1_{\{t_k < T_{B^c}\}} E^{X_{t_k}} \left(\int_{t_k}^{\infty} \frac{1}{t} d_t \tilde{L}_{t-t_k}^x \right) \right) dm(x). \end{aligned} \tag{6.9}$$

But this is clearly

$$\int_B \tilde{P}^x \left(\prod_{j=1}^k f_j(\tilde{X}_{t_j}) \tilde{E}^{\tilde{X}_{t_k}} \left(\int_{t_k}^{\infty} \frac{1}{t} d_t \tilde{L}_{t-t_k}^x \right) \right) dm(x) \tag{6.10}$$

which is precisely what we obtain from $\tilde{\mu} \left(\prod_{j=1}^k f_j(X_{t_j}) \right)$ by proceeding as in (6.9). \square

7 Transformations of the loop measure

7.1 Mappings of the state space

Let \bar{S} be another locally compact topological space with a countable base, and let $f : S \mapsto \bar{S}$ be a topological isomorphism. Then

$$\bar{P}_t(x, g) = P_t(f^{-1}(x), g \circ f). \tag{7.1}$$

forms a Borel transition semigroup on \bar{S} . Let $\bar{\Omega}$ be the set of right continuous paths ω in $\bar{S}_\Delta = \bar{S} \cup \Delta$ with $\Delta \notin \bar{S}$, and such that $\omega(t) = \Delta$ for all $t \geq \zeta = \inf\{t > 0 \mid \omega(t) = \Delta\}$.

Then with $\bar{X}_t(\omega) = \omega(t)$ it follows from [24, Section 13] that $\bar{X} = (\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{X}_t, \theta_t, \bar{P}^x)$ is a Borel right process. Furthermore,

$$\begin{aligned} \bar{U}(x, g) &= \int_0^\infty \bar{P}_t(x, g) dt = \int_0^\infty P_t(f^{-1}(x), g \circ f) dt \\ &= \int_S u(f^{-1}(x), y) g \circ f(y) dm(y) \\ &= \int_{\bar{S}} u(f^{-1}(x), f^{-1}(z)) g(z) df_* m(z). \end{aligned} \tag{7.2}$$

Thus \bar{X} has continuous potential densities $\bar{u}(x, y) = u(f^{-1}(x), f^{-1}(y))$ with respect to the measure $f_* m$.

If we let $\bar{f} : \Omega \mapsto \bar{\Omega}$ be defined as $\bar{f}(\omega)(t) = f(\omega(t))$, it follows that

$$\bar{P}^x(F) = P^{f^{-1}(x)}(F \circ \bar{f}). \tag{7.3}$$

Note further that $\bar{L}_t^y = L_t^{f^{-1}(y)} \circ \bar{f}^{-1}$ is a CAF for \bar{X} with

$$\bar{P}^x(\bar{L}_\infty^y) = P^{f^{-1}(x)}(L_\infty^{f^{-1}(y)}) = u(f^{-1}(x), f^{-1}(y)) = \bar{u}(x, y), \tag{7.4}$$

so that $\{\bar{L}_t^y, (y, t) \in S' \times R_+\}$ are local times for \bar{X} . Let $\bar{\mu}$ be the loop measure for \bar{X} .

Theorem 7.1.

$$\bar{f}_* \mu(F) = \bar{\mu}(F). \tag{7.5}$$

Proof of Theorem 7.1:

$$\begin{aligned} \bar{\mu}(F) &= \int_{S'} \bar{P}^x \left(\int_0^\infty \frac{1}{t} F \circ k_t d_t \bar{L}_t^x \right) df_* m(x) \\ &= \int_{S'} P^{f^{-1}(x)} \left(\int_0^\infty \frac{1}{t} F \circ k_t \circ \bar{f} d_t L_t^{f^{-1}(x)} \right) df_* m(x) \\ &= \int_S P^x \left(\int_0^\infty \frac{1}{t} F \circ k_t \circ \bar{f} d_t L_t^x \right) dm(x) \\ &= \mu(F \circ \bar{f}) = \bar{f}_* \mu(F). \end{aligned} \tag{7.6}$$

□

7.2 Unit Weights

We say that a random variable $T \geq 0$ is a unit weight if

$$\int_0^\zeta T \circ \rho_u du = 1. \tag{7.7}$$

Of course, since ζ is invariant under loop rotation, $1/\zeta$ is an example of a unit weight. (7.20) will provide another example, which is be used in the proof of Theorem 7.3 to determine how the loop measure transforms under a time change.

Let $\mathcal{I}_\rho(X)$ be the collection of measurable functions on Ω which are invariant under loop rotation.

Lemma 7.2. *If T is a unit weight then for all $F \in \mathcal{I}_\rho(X)$*

$$\mu(F) = \int_S Q^{x,x}(TF) dm(x). \tag{7.8}$$

Proof: By invariance of μ we have that for each $u > 0$ and $F \in \mathcal{I}_\rho(X)$

$$\mu(T \circ \rho_u F) = \mu(T F). \tag{7.9}$$

Since ζ is invariant under loop rotation, this implies that for any $u > 0$

$$\mu(T \circ \rho_u 1_{\{u < \zeta\}} F) = \mu(T 1_{\{u < \zeta\}} F). \tag{7.10}$$

Hence

$$\int_0^\infty \mu(T \circ \rho_u 1_{\{u < \zeta\}} F) du = \int_0^\infty \mu(T 1_{\{u < \zeta\}} F) du. \tag{7.11}$$

This shows that

$$\mu\left(\int_0^\zeta T \circ \rho_u du F\right) = \mu\left(T F \int_0^\infty 1_{\{u < \zeta\}} du\right) = \mu(T F \zeta). \tag{7.12}$$

Using (7.7) and (2.27) our Lemma follows. □

7.3 Time change by the inverse of a CAF

Consider

$$A_t = \int_S L_t^x d\nu_A(x) \tag{7.13}$$

where ν_A is a Borel measure on S . We suppose that $P^x(A_t = \infty, t < \zeta) = 0$ for all $x \in S$ and $t > 0$. (This is the case, for instance, if $\nu_A(K) < \infty$ for each compact $K \subset S$.) By the argument at the beginning of the proof of Lemma 2.1, (7.13) defines a CAF of X . Let S_A denote the fine support of A ; that is, the set of $x \in S$ such that $P^x(R = 0) = 1$ where $R = \inf\{t > 0 \mid A_t > 0\}$, see [24, Section 64]. Because m is a reference measure and $v(x) := E^x(\exp(-R))$ is a 1-excessive function, $S_A = \{x \in S : v(x) = 1\}$ is a Borel subset of S ; see [1, Prop. V(1.4)].

Let τ_t be the right continuous inverse of A_t , and let $Y_t = X_{\tau_t}$. Then $Y = (\Omega, \mathcal{G}_t, Y_t, \hat{\theta}_t, P^x)$ is a Borel right process with state space S_A and lifetime A_ζ , see [24, Theorem 65.9] for details, noting that [24, (60.4)] applied to $\exp(-A_t)$ allows us to assume that A is a perfect CAF. Here $\hat{\theta}_t(\omega) = \theta_{\tau_t(\omega)}(\omega)$. Using the change of variables formula, [5, Chapter 6, (55.1)], we see that

$$\begin{aligned} E^x\left(\int_0^\infty f(Y_t) dt\right) &= E^x\left(\int_0^\infty f(X_{\tau_t}) dt\right) \\ &= E^x\left(\int_0^{A_\zeta} f(X_{\tau_t}) dt\right) \\ &= E^x\left(\int_0^\infty f(X_s) dA_s\right) \\ &= \int u(x, y) f(y) \nu_A(dy), \end{aligned} \tag{7.14}$$

so that Y_t has continuous potential densities $u(x, y)$ with respect to the measure $\nu_A(dy)$ on S_A . (In the last step we used the fact that for any measurable function h_s , we have $\int_0^\infty h_s dA_s = \int (\int_0^\infty h_s dL_s^y) \nu_A(dy)$. It suffices to verify this for functions of the form $h_s = 1_{[0, t]}(s)$, in which case it is obvious). Furthermore, since S_A is the fine support of A , $L_{\tau_t}^x$ is continuous in t for each $x \in S_A$, see [11, p. 1659], and of course $E^y(L_{\tau_\infty}^x) = u(y, x)$. It follows that $\{L_{\tau_t}^x, (x, t) \in S_A \times R_+\}$ is the family of local times for Y . See [24, Theorem 65.6] for additivity.

It will be convenient to use the canonical notation $X' = (\Omega, \mathcal{F}'_t, X'_t, \theta_t, P'^x)$ for Y . Thus $X'_t(\omega) = \omega(t)$, which is the same as $X_t(\omega)$, but we use the notation X'_t to emphasize that it is associated with the measures P'^x which we now define. If we set $g(\omega)(t) = \omega(\tau_t(\omega))$ we have $Y_t = X_t \circ g$ and put

$$P'^x(F) = P^x(F \circ g). \tag{7.15}$$

Using [24, (62.20)], compare (2.23), we see that if $t_1 < \dots < t_n$,

$$\begin{aligned} g_* Q^{x,x} \left(\prod_{j=1}^n f_j(X_{t_j}) \right) &= Q^{x,x} \left(\prod_{j=1}^n f_j(X_{\tau_{t_j}}) \right) \\ &= P^x \left(\prod_{j=1}^n f_j(X_{\tau_{t_j}}) u(X_{\tau_{t_n}}, x) \right) \\ &= P'^x \left(\prod_{j=1}^n f_j(X'_{t_j}) u(X'_{t_n}, x) \right). \end{aligned} \tag{7.16}$$

Let $Q'^{x,x}$ be the measure in (2.22) associated with X' . Using (2.23) and the fact that X' also has potential densities $u(x, y)$ we have that if $x \in S_A$

$$P'^x \left(\prod_{j=1}^n f_j(X'_{t_j}) u(X'_{t_n}, x) \right) = Q'^{x,x} \left(\prod_{j=1}^n f_j(X'_{t_j}) \right).$$

Hence for all measurable F

$$g_* Q^{x,x}(F) = Q'^{x,x}(F), \quad \forall x \in S_A. \tag{7.17}$$

Before considering general ν_A 's, we first study the special case where the measure ν_A is equivalent to m . Thus $\nu_A(dx) = h(x)m(dx)$ where h is a measurable function on S with $0 < h(x) < \infty$ for all x . It follows from (2.28) that

$$A_t = \int_0^t h(X_s) ds, \tag{7.18}$$

and thus $S_A = S$. Let μ, μ' be the loop measures for X, X' respectively.

Theorem 7.3. *If $\nu_A(dx) = h(x)m(dx)$ where h is a measurable function on S with $0 < h(x) < \infty$ for all x , then*

$$g_* \mu(F) = \mu'(F), \quad \forall F \in \mathcal{I}_\rho(X'). \tag{7.19}$$

Proof of Theorem 7.3: Define the unit weight

$$T(\omega) = \frac{h(\omega(0))}{A_\zeta(\omega)}. \tag{7.20}$$

By (7.8) we have $\mu(F) = \int_S Q^{x,x}(T F) m(dx)$ for all $F \in \mathcal{I}_\rho(X)$. It is easy to see that $F \in \mathcal{I}_\rho(X')$ implies that $F \circ g \in \mathcal{I}_\rho(X)$. Noting that $A_\zeta = \zeta \circ g$, and using (7.17)

$$\begin{aligned} g_* \mu(F) &= \mu(F \circ g) \\ &= \int_S Q^{x,x}(T F \circ g) m(dx) \\ &= \int_S Q^{x,x} \left(\frac{1}{\zeta \circ g} F \circ g \right) h(x) m(dx) \\ &= \int_S Q'^{x,x} \left(\frac{1}{\zeta} F \right) h(x) m(dx) = \mu'(F). \end{aligned} \tag{7.21}$$

The last equality used (2.27) and the fact that $\nu_A(dx) = h(x)m(dx)$. □

We next show how to combine Theorems 7.1 and 7.3. Let S' be another locally compact topological space with a countable base, and let $f : S \mapsto S'$ be a topological isomorphism. With h as above, let $m_{S'}$ be the measure on S' defined by

$$m_{S'}(dy) := f_*(h m_S)(dy). \tag{7.22}$$

It follows from the discussion in sub-section 7.1 and the present sub-Section that if we set $\bar{X}'_t := f(X_{\tau_t}) = f(Y_t)$ and $\{\bar{P}'^x, x \in S'\}$ the measures induced by $\{P^x, x \in S\}$, then $\bar{X}' = (\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{X}'_t, \theta_t, \bar{P}'^x)$ is a Borel right process with continuous potential densities

$$\bar{u}'(x, y) = u(f^{-1}(x), f^{-1}(y)) \tag{7.23}$$

with respect to the measure $f_*(h m_S) = m_{S'}$ on S' .

Set $f^\#(\omega)(t) = f(\omega(\tau_t))$ and let $\mu, \bar{\mu}'$ be the loop measures for X, \bar{X}' respectively. Combining Theorems 7.1 and 7.3 we obtain

Corollary 7.4.

$$f^\# \mu(F) = \bar{\mu}'(F), \quad \forall F \in \mathcal{I}_\rho(X'). \tag{7.24}$$

Remark 7.5. Let D, D' be two simply connected domains in the complex plane and let f be a conformal map from D onto D' . Let X be Brownian motion in D . Since the potential density for X with respect to λ_D , Lebesgue measure on D , is not continuous, (it has a logarithmic singularity on the diagonal), X does not fit into the framework of this paper. Nevertheless, we argue by analogy. Let $h(x) = |f'(x)|^2$. Then \bar{X}' is a Brownian motion in D' , and $f_*(h \lambda_D)(dy) = \lambda_{D'}(dy)$. It follows formally that (7.24) would yield [18, Proposition 5.27], the conformal invariance of Brownian loop measures.

We now turn to a general CAF as in (7.13), Our results are not as complete as (7.19), but see the Remark following the proof of Theorem 7.6.

For any $B \subseteq S$ let $\mathcal{L}_B(X)$ be the σ -algebra generated by the total local times $\{L_\infty^x, x \in B\}$ of X , and let μ' be the loop measure for X' .

Theorem 7.6.

$$g_* \mu(F) = \mu'(F), \quad \forall F \in \mathcal{L}_{S_A}(X'). \tag{7.25}$$

Proof of Theorem 7.6: By Lemma 2.2

$$\mu(L_\infty^x \cdot F) = Q^{x,x}(F), \quad \forall F \in \mathcal{L}_S(X), x \in S. \tag{7.26}$$

Recall that $L_\infty'^x = L_\infty^x \circ g$ so that

$$\mathcal{L}_{S_A}(X') = \mathcal{L}_{S_A}(X) \circ g. \tag{7.27}$$

Consider $F \in \mathcal{L}_{S_A}(X)$. Since $A_\zeta \in \mathcal{L}_{S_A}(X)$ and $A_\zeta > 0$, P_x a.s. for all $x \in S_A$, by replacing F in (7.26) by F/A_ζ and then integrating with respect to $d\nu_A(x)$ we can deduce immediately that

$$\mu(F) = \int_{S_A} Q^{x,x} \left(\frac{F}{A_\zeta} \right) \nu_A(dx), \quad \forall F \in \mathcal{L}_{S_A}(X). \tag{7.28}$$

Although S_A may not be locally compact, X' inherits from X all the properties needed to define μ' as in (2.6), and it then follows as in (2.22) that

$$\mu'(F) = \int_{S_A} Q'^{x,x} \left(\frac{1}{\zeta} F \right) d\nu_A(x). \tag{7.29}$$

By (7.17) this shows that

$$\mu'(F) = \int_{S_A} Q^{x,x} \left(\frac{1}{\zeta \circ g} F \circ g \right) d\nu_A(x). \quad (7.30)$$

Noting that $A_\zeta = \zeta \circ g$, (7.28) and (7.27) then imply our Theorem. \square

Remark 7.7. For $x_1, \dots, x_n \in S$ we define the multiple local time

$$L_t^{x_1, \dots, x_n} = \sum_{j=1}^n \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} dL_{s_1}^{x_j} dL_{s_2}^{x_{j+1}} \dots dL_{s_{n-j+1}}^{x_n} dL_{s_{n-j+2}}^{x_1} dL_{s_{n-j+3}}^{x_2} \dots dL_{s_n}^{x_{j-1}}, \quad (7.31)$$

that is, we measure n -tuples of times in which x_1, \dots, x_n are visited in cyclic order. If $n = 2$ and $x_1 \neq x_2$, then $L_t^{x_1, x_2} = L_t^{x_1} L_t^{x_2}$, but in general $L_t^{x_1, \dots, x_n}$ is not a product of the corresponding local times. Let $\mathcal{M}(X)$ denote the σ -algebra generated by the multiple local times. When $\text{Supp}(\nu_A) = S$ we can show that (7.25) holds for all $F \in \mathcal{M}(X) = \mathcal{M}(X')$. When S is finite, it is known that $\mathcal{M}(X) = \mathcal{I}_\rho(X)$, [13, p. 24]. For diffusions, see [21], especially Corollary 2.9, and for more general processes see [3].

We leave to the interested reader the task of combining Theorem 7.6 with spatial transformations as in Corollary 7.4.

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