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MEAN ERGODIC THEOREMS FOR ALMOST PERIODIC SEMIGROUPS

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. We show mean ergodic theorems for vector-valued weakly almost periodic functions (in the sense of Eberlein) defined on a semigroup which take values in a locally convex topological vector space. Next, motivated by Fréchet [10], we study the relationship between almost periodicity of semigroups of mappings and their equicontinuity, and also prove mean ergodic theorems for equicontinuous semigroups.

1. Introduction

Let C be a closed and convex subset of a real Banach space. Then a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 1975, Baillon [3] originally proved the first nonlinear ergodic theorem in the framework of Hilbert spaces: Let C be a closed and convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, we have that P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and Px is contained in the closure of convex hull of $\{T^n x : n = 1, 2, \dots\}$ for each $x \in C$. We call such a retraction an *"ergodic retraction"*. In 1981, Takahashi [27, 29] proved the existence of ergodic retractions for amenable semigroups of nonexpansive mappings on Hilbert spaces. Rodé [20] also found a

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sequence of means on a semigroup, generalizing the Cesàro means, and extended Baillon's theorem. These results were extended to a uniformly convex Banach space whose norm is Fréchet differentiable in the case of commutative semigroups of nonexpansive mappings by Hirano, Kido and Takahashi [12]. In 1999, Lau, Shioji and Takahashi [15] extended Takahashi's result and Rodé's result to amenable semigroups of nonexpansive mappings in the Banach space.

By using Rodé's method, Kido and Takahashi [14] also proved a mean ergodic theorem for noncommutative semigroups of bounded linear operators in Banach spaces.

On the other hand, by using results of Bruck [5], Atsushiba and Takahashi [1] proved a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset of a strictly convex Banach space. This result was extended to commutative semigroups of nonexpansive mappings by Atsushiba, Lau and Takahashi [2]. Miyake and Takahashi [16] proved a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset of a general Banach space. Later, these results were extended to amenable semigroups of nonexpansive mappings by Miyake and Takahashi [17].

Motivated by Kido and Takahashi [14], Hirano, Kido and Takahashi [12], Lau, Shioji and Takahashi [15], Atsushiba, Lau and Takahashi [2] and Miyake and Takahashi [17], Miyake and Takahashi [18] proved weak and strong mean ergodic theorems for vector-valued weakly almost periodic functions (in the sense of Eberlein) defined on a semigroup which take values in a Banach space. Using these results, they obtained well-known and new mean ergodic theorems for commutative and noncommutative semigroups of nonexpansive mappings, affine nonexpansive mappings and bounded linear operators in Banach spaces.

In this paper, we introduce the notion of weakly almost periodicity for vector-valued bounded functions defined on a semigroup which take values in a locally convex topological vector space and show mean ergodic theorems for vector-valued weakly almost periodic functions in the sense of Eberlein. Next, motivated by Fréchet [10], we study the relationship between almost periodicity of semigroups of mappings and their equicontinuity in order to prove a mean ergodic theorem for equicontinuous semigroups of mappings. We also show mean ergodic theorems for such semigroups in Banach spaces as special cases.

2. Preliminaries

Throughout this paper, we denote by S a semigroup with identity and by E a locally convex topological vector space (or l.c.s.). We also denote by $\mathbb{R}_+$ and $\mathbb{N}_+$ the set of non-negative real numbers and the set of non-negative integers, respectively. Let $\langle E, F\rangle$ be the duality between vector spaces E and F. For each $y \in F$, we define a linear functional f_y on E by $f_y(x) = \langle x, y\rangle$. We denote by $\sigma(E, F)$

the weak topology on E generated by $\{f_y : y \in F\}$. If X is a l.c.s., we denote by X' the topological dual of X. We also denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear form between E and E', that is, for $x \in E$ and $x' \in E'$, $\langle x, x' \rangle$ is the value of x' at x. Let A be a subset of E. We denote by A° the polar of A, that is, $A^\circ = \{x' \in E' : \langle x, x' \rangle \leq 1$ for each $x \in A\}$. We also denote by $\overline{A}$ the closure of A.

We denote by $l^\infty(S)$ the Banach space of bounded real-valued functions on S. For each $s \in S$, we define operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s)f)(t) = f(st) \quad \text{and} \quad (r(s)f)(t) = f(ts)$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. A subspace X of $l^\infty(S)$ is said to be *translation invariant* if $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$. Let X be a subspace of $l^\infty(S)$ which contains constants. A linear functional μ on X is said to be a *mean* on X if $\|\mu\| = \mu(e) = 1$, where $e(s) = 1$ for each $s \in S$. We often write $\mu_s f(s)$ instead of $\mu(f)$ for each $f \in X$. For $s \in S$, we define a *point evaluation* δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. A convex combination of point evaluations is called a *finite mean* on S. As is well known, μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each $f \in X$; see Day [6] and Takahashi [30] for more details. Let X be also translation invariant. Then, a mean μ on X is said to be *left (or right) invariant* if $\mu(l(s)f) = \mu(f)$ (or $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant. If there exists an invariant mean on X, then X is said to be *amenable*. We know from Day [6] that if S is commutative, then X is amenable. Let $\{\mu_\alpha\}$ be a net of means on X. Then $\{\mu_\alpha\}$ is said to be *(strongly) asymptotically invariant* if for each $s \in S$, both $l(s)'\mu_\alpha - \mu_\alpha$ and $r(s)'\mu_\alpha - \mu_\alpha$ converge to 0 in the weak topology $\sigma(X', X)$ (the norm topology), where $l(s)'$ and $r(s)'$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day [6].

We denote by $l^\infty(S, E)$ the vector space of vector-valued functions defined on S that take values in E such that for each $f \in l^\infty(S, E)$, $f(S) = \{f(s) : s \in S\}$ is bounded in E. Let $\mathfrak{U}$ be a neighborhood base of 0 in E and let $M(V) = \{f \in l^\infty(S, E) : f(S) \subset V\}$ for each $V \in \mathfrak{U}$. We denote by $\mathfrak{B}$ the filter base $\{M(V) : V \in \mathfrak{U}\}$. Then, $l^\infty(S, E)$ is a l.c.s. with the topology $\mathfrak{T}$ of uniform convergence on S that has a neighborhood base $\mathfrak{B}$ of 0. We also denote by $l_c^\infty(S, E)$ the subspace of $l^\infty(S, E)$ such that for each $f \in l_c^\infty(S, E)$, $f(S)$ is relatively weakly compact in E. Let X be a subspace of $l^\infty(S)$ containing constants such that for each $f \in l_c^\infty(S, E)$ and $x' \in E'$, the function $s \mapsto \langle f(s), x' \rangle$ is contained in X. Such an X is called *admissible*. Let $\mu \in X'$. Then, for each $f \in l_c^\infty(S, E)$, we define a linear functional $\tau(\mu)f$ on E' by

$$\tau(\mu)f : x' \mapsto \mu\langle f(\cdot), x' \rangle.$$

It follows from the bipolar theorem that $\tau(\mu)f$ is contained in E. We know that $\tau(\mu)$ is a continuous linear mapping of $l_c^\infty(S,E)$ into E. If μ is a mean on X, then $\tau(\mu)f$ is contained in the closure of convex hull of $f(S)$ for each $f \in l_c^\infty(S,E)$. Such a $\tau(\mu)$ is called a *vector-valued mean* (generated by a mean μ on X). If E is a Banach space, then $\tau(\mu)$ is also a mean on $l_c^\infty(S,E)$ in the sense of Goldberg and Irwin [11]. See also Takahashi [27] and Kada and Takahashi [13]. For each $s \in S$, we define the operators $R(s)$ and $L(s)$ on $l^\infty(S,E)$ by

$$(R(s)f)(t) = f(ts) \quad \text{and} \quad (L(s)f)(t) = f(st)$$

for each $t \in S$ and $f \in l^\infty(S,E)$, respectively. Note that if μ is a left (or right) invariant mean on X, then $\tau(\mu)(L(s)f)$ (or $\tau(\mu)(R(s)f)) = \tau(\mu)f$.

Let C be a closed convex subset of a l.c.s. E and let $\mathfrak{F}$ be the semigroup of continuous self-mappings of C under operator multiplication. If T is a semigroup homomorphism of S into $\mathfrak{F}$, then T is said to be a *representation* of S as continuous self-mappings of C. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous self-mappings of C such that for each $x \in C$, the orbit $\mathcal{O}(x) = \{T(s)x : s \in S\}$ of x is relatively weakly compact in C and let X be a subspace of $l^\infty(S)$ containing constants such that for each $x \in C$ and $x' \in E'$, the function $s \mapsto \langle T(s)x, x'\rangle$ is contained in X. Such an X is called *admissible* with respect to $\mathcal{S}$. If no confusion will occur, then X is simply called admissible. Let $\mu \in X'$. Then, there exists a unique point x_0 of E such that $\mu\langle T(\cdot)x, x'\rangle = \langle x_0, x'\rangle$ for each $x' \in E'$. We denote such a point x_0 by $T(\mu)x$. Note that if μ is a mean on X, then for each $x \in C$, $T(\mu)x$ is contained in the closure of convex hull of the orbit $\mathcal{O}(x)$ of x.

Let $f \in l^\infty(S,E)$. We denote by $\mathcal{RO}(f)$ the right orbit of f, that is, the set $\{R(s)f \in l^\infty(S,E) : s \in S\}$ of right translates of f. Similarly, we also denote by $\mathcal{LO}(f)$ the left orbit of f, that is, the set $\{L(s)f \in l^\infty(S,E) : s \in S\}$ of left translates of f. A function $f \in l^\infty(S,E)$ is said to be *almost periodic* if $\mathcal{RO}(f)$ is relatively compact in $(l^\infty(S,E), \mathfrak{T})$; the notion of almost periodicity for real-valued functions on an abstract group is due to von Neumann [19]. We denote by $AP(S,E)$ the set of almost periodic functions defined on S which take values in E. See also Bochner and von Neumann [4]. A function $f \in l^\infty(S,E)$ is said to be *right (or left) weakly almost periodic* (in the sense of Eberlein) if $\mathcal{RO}(f)$ (or $\mathcal{LO}(f)$) is relatively weakly compact in $(l^\infty(S,E), \mathfrak{T})$; the notion of weakly almost periodicity was introduced by Eberlein [8]. If $f \in l^\infty(S,E)$ is both left and right weakly almost periodic, then f is said to be *weakly almost periodic* in the sense of Eberlein. We denote by $WR(S,E)$ (or $WL(S,E)$) the set of right (or left) weakly almost periodic functions defined on S which take values in E. See also de Leeuw and Glicksberg [7], Goldberg and Irwin [11] and Miyake and Takahashi [18]. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous mappings of a bounded, closed and convex subset C of E into itself and define a mapping $\phi_{\mathcal{S}}$ of C into $l^\infty(S,E)$ by $(\phi_{\mathcal{S}}(x))(s) = T(s)x$ for each $s \in S$. Then, $\mathcal{S}$ is also said to be *(weakly) almost periodic* if, for each $x \in C$, $\phi_{\mathcal{S}}(x) \in AP(S, E)$ ($WR(S, E)$). Note that every right (or left) weakly almost periodic function $f \in l^\infty(S, E)$ is contained in $l_c^\infty(S, E)$.

3. Mean Ergodic Theorems for Weakly Almost Periodic Functions

In 1934, von Neumann first proved the existence of the mean values for real-valued almost periodic functions defined on an abstract group. Later, Bochner and von Neumann [4] extended von Neumann's result to vector-valued almost periodic functions defined on a group which take values in a complete locally convex space.

Theorem 1. (von Neumann [19]). *Let G be a group, let $AP(G)$ be the Banach space of real-valued almost periodic functions defined on G and let $f \in AP(G)$. Then, there exists the unique constant function c_f in the closure of convex hull of $\mathcal{RO}(f)$. In this case, putting $\mu(f) = c_f$, μ is an invariant mean on $AP(G)$ such that $\mu_x(f(x^{-1})) = \mu_x f(x)$ for each $f \in AP(G)$.*

In 1949, Eberlein [8] introduced the notion of weakly almost periodicity for real-valued bounded functions defined on a locally compact abelian group. Goldberg and Irwin [11] studied weakly almost periodicities for vector-valued functions defined on a semigroup whose ranges are relatively compact in a Banach space.

The following lemma is crucial for proving main results of this paper, which can be obtained as in the proof of Lemma 3.3 in [18].

Lemma 1. *Let S be a semigroup with identity, let E be a l.c.s., let $f \in WR(S, E)$, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let μ be a mean on X. Then, the function $s \mapsto \tau(l(s)'\mu)f = \tau(\mu)(L(s)f)$ is contained in the closure of convex hull of $\mathcal{RO}(f)$.*

Remark 1. Let μ be a mean on X and let $f \in WR(S, E)$. Motivated by this lemma, we call such a $\tau(l(\cdot)'\mu)f$ an "*ergodic mean*" of f. In particular, if λ is a finite mean on S, then $\tau(l(\cdot)'\lambda)f$ is a convex combination of right translates of f.

Using Lemma 1, we can prove the existence of the mean values for vector-valued weakly almost periodic functions in the sense of Eberlein as in the proof of Lemma 3.5 in [18].

Theorem 2. *Let S be a semigroup with identity, let E be a l.c.s., let $f \in WR(S, E)$ and let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants. If X has an invariant mean μ, then there exists the unique constant function c_f in the closure $\mathcal{K}$ of convex hull of $\mathcal{RO}(f)$. In this case, $\tau(\mu)f = c_f$.*

Proof. Let μ is an invariant mean on X. It is clear from invariance of μ that $\tau(l(\cdot)'\mu)f = \tau(\mu)f$ is a constant function in $\mathcal{K}$.

Let $g = \sum_{i=1}^{n} \lambda_i R(s_i)f$ with $\lambda_i \geq 0$ ($i = 1, \ldots, n$) and $\sum_{i=1}^{n} \lambda_i = 1$. Then, as in the proof of Lemma 3.5 in [18], we have

$$\tau(\mu)g = \tau\left(\sum_{i=1}^{n} \lambda_i r(s_i)'\mu\right) f = \tau(\mu)f.$$

So, since $\tau(\mu)$ is a continuous linear mapping of $l_c^{\infty}(S, E)$ into E, it follows that $\tau(\mu)h = \tau(\mu)f$ for each $h \in \mathcal{K}$. If c is a constant function in $\mathcal{K}$, then $c = \tau(\mu)c = \tau(\mu)f$. This completes the proof. ■

As in the proofs of Theorem 3.7 and Theorem 3.8 in [18], we can also prove mean ergodic theorems for vector-valued weakly almost periodic functions in the sense of Eberlein by using Lemma 1 and Theorem 2.

Theorem 3. *Let S be a semigroup with identity, let E be a l.c.s., let $f \in WR(S, E)$, let X be a closed, translation invariant and admissible subspace of $l^{\infty}(S)$ containing constants and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, $\{\tau(l(\cdot)'\mu_\alpha)f\}$ converges weakly to the constant function c_f in the closure $\mathcal{K}$ of convex hull of $\mathcal{RO}(f)$. In this case, $\tau(\mu)f = c_f$ for each invariant mean μ on X.*

Proof. For each α, we define a vector-valued function $\mu_\alpha.f \in l^{\infty}(S, E)$ by $(\mu_\alpha.f)(s) = \tau(l(s)'\mu_\alpha)f$ for each $s \in S$. Then, by Lemma 1, a net $\{\mu_\alpha.f\}$ is contained in $\mathcal{K}$. Suppose that a subnet $\{\mu_{\alpha_\beta}.f\}$ of $\{\mu_\alpha.f\}$ converges weakly to g in $\mathcal{K}$. Since $\{\mu_\alpha\}$ is asymptotically invariant, $\{\mu_{\alpha_\beta}\}$ has a cluster point μ in X' in the topology $\sigma(X', X)$. We can assume that $\{\mu_{\alpha_\beta}\}$ converges to μ without loss of generality. Then, as in the proof of Theorem 3.7 in [18] (see also [30]), μ is an invariant mean on X and hence

$$g = \lim_{\beta} \mu_{\alpha_\beta}.f = \lim_{\beta} \tau(l(\cdot)'\mu_{\alpha_\beta})f = \tau(l(\cdot)'\mu)f = \tau(\mu)f.$$

It follows from Theorem 2 that g is the unique constant function c_f in $\mathcal{K}$. In this case, $c_f(\cdot) = \tau(\mu)f$ where μ is an invariant mean on X. Hence, $\{\tau(l(\cdot)'\mu_\alpha)f\}$ converges weakly to the unique constant function c_f in $\mathcal{K}$. This completes the proof. ■

Theorem 4. *Let S be a semigroup with identity, let E be a l.c.s., let $f \in WR(S, E)$, let X be a closed, translation invariant and admissible subspace of $l^{\infty}(S)$ containing constants and let $\{\mu_\alpha\}$ be a strongly asymptotically invariant net* *of means on X. Then, $\{\tau(l(\cdot)'\mu_\alpha)f\}$ converges to the constant function c_f in the closure $\mathcal{K}$ of convex hull of $\mathcal{RO}(f)$. In this case, $\tau(\mu)f = c_f$ for each invariant mean μ on X.*

Proof. Let U be a neighborhood of 0 and let $g \in \mathcal{K}$. Choose a closed, convex and circled neighborhood V of 0 such that $V + V \subset U$. Then, there exists a convex combination $h = \sum_{i=1}^n \lambda_i R(s_i)f$ of right translates of f such that $g - h \in M(V)$, that is, $g(s) - h(s) \in V$ for each $s \in S$, where $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ and $s_i \in S$ ($i = 1, \ldots, n$). So, we have, for each α and $t \in S$,

$$\begin{aligned}\tau(l(t)'\mu_\alpha)g - \tau(l(t)'\mu_\alpha)h &= \tau(\mu_\alpha)(L(t)g) - \tau(\mu_\alpha)(L(t)h)\\ &= \tau(\mu_\alpha)(L(t)(g-h)) \in V.\end{aligned}$$

By boundedness of f, there exists a $\beta > 0$ such that $f(s) \in \beta V$ for each $s \in S$. Since, for each $s, t \in S$ and $x' \in V^\circ$,

$$\begin{aligned}&\langle \tau(l(t)'\mu_\alpha)(R(s)f) - \tau(l(t)'\mu_\alpha)f, x'\rangle\\ &= \langle \tau(\mu_\alpha)(L(t)R(s)f) - \tau(\mu_\alpha)(L(t)f), x'\rangle\\ &= \langle \tau(\mu_\alpha)(R(s)L(t)f) - \tau(\mu_\alpha)(L(t)f), x'\rangle\\ &= \langle \tau(r(s)'\mu_\alpha)(L(t)f) - \tau(\mu_\alpha)(L(t)f), x'\rangle\\ &= (r(s)'\mu_\alpha - \mu_\alpha)\langle L(t)f(\cdot), x'\rangle\\ &\leq \|r(s)'\mu_\alpha - \mu_\alpha\| \sup_{w\in S}\langle f(tw), x'\rangle\\ &\leq \beta\|r(s)'\mu_\alpha - \mu_\alpha\|,\end{aligned}$$

it follows that for each $s \in S$, there exists an α_s such that for each $\alpha \geq \alpha_s$ and $t \in S$,

$$\langle \tau(l(t)'\mu_\alpha)(R(s)f) - \tau(l(t)'\mu_\alpha)f, x'\rangle \leq 1$$

and hence $\tau(l(t)'\mu_\alpha)(R(s)f) - \tau(l(t)'\mu_\alpha)f \in V$.

Choose an α_0 with $\alpha_0 \geq \alpha_{s_i}$ ($i = 1, \ldots, n$). Then, we have, for each $t \in S$ and $\alpha \geq \alpha_0$,

$$\begin{aligned}&\tau(l(t)'\mu_\alpha)g - \tau(l(t)'\mu_\alpha)f\\ &= \tau(l(t)'\mu_\alpha)g - \tau(l(t)'\mu_\alpha)h\\ &\quad + \tau(l(t)'\mu_\alpha)h - \tau(l(t)'\mu_\alpha)f\\ &\in V + \sum_{i=1}^n \lambda_i\{\tau(l(t)'\mu_\alpha)(R(s_i)f) - \tau(l(t)'\mu_\alpha)f\}\\ &\subset V + V \subset U.\end{aligned}$$

Hence, for each $g \in \mathcal{K}$, $\{\tau(l(\cdot)'\mu_\alpha)g - \tau(l(\cdot)'\mu_\alpha)f\}$ converges to 0 in $l^\infty(S, E)$ with the topology $\mathfrak{T}$ of uniform convergence on S. We know from Theorem 2 that there exists the unique constant function $p(\cdot) = \tau(\mu)f$, for each invariant mean μ on X, which is contained in $\mathcal{K}$. So, it follows that $\{\tau(l(\cdot)'\mu_\alpha)f\}$ converges to $p = \tau(l(\cdot)'\mu_\alpha)p$ in $(l^\infty(S, E), \mathfrak{T})$. This completes the proof. ■

Corollary 1. *Let E be a Banach space and let $f \in WR(\mathbb{R}_+, E)$. Then,*

$$\frac{1}{t}\int_0^t f(r+h)\ dr$$

converges uniformly in $h \in \mathbb{R}_+$ as $t \to +\infty$.

Corollary 2. *Let E be a Banach space and let $f \in WR(\mathbb{R}_+, E)$. Then, the Abel means*

$$r\int_0^\infty \exp(-rt)f(t+h)\ dt$$

converge uniformly in $h \in \mathbb{R}_+$ as $r \to +\infty$.

4. Mean Ergodic Theorems for Almost Periodic Semigroups

We can apply mean ergodic theorems for vector-valued weakly almost periodic functions in the sense of Eberlein in order to obtain new and well-known mean ergodic theorems for semigroups of linear and non-linear operators in a locally convex space E. See also Ruess and Summers [22, 23] and Miyake and Takahashi [18]. For example, the following theorem follows from Theorem 4.

Theorem 5. *Let S be a semigroup with identity, let E be a Banach space, let $\mathcal{S} = \{T(s) : s \in S\}$ be a weakly almost periodic representation of S as bounded linear operators on E, that is, $\mathcal{S}$ be a representation of S as bounded linear operators on E such that for each $x \in E$, the orbit $\mathcal{O}(x)$ of x is relatively weakly compact, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be a strongly asymptotically invariant net of means on X. Then, for each $x \in E$, $\{T(l(h)'\mu_\alpha)x\}$ converges to a fixed point p for $\mathcal{S}$ uniformly in $h \in S$. In this case, $p = T(\mu)x$ for each invariant mean μ on X.*

Proof. For each $x \in E$, we define a vector-valued function $\phi(x) \in l^\infty(S, E)$ by $(\phi(x))(s) = T(s)x$ for each $s \in S$. Since $\mathcal{S}$ is a weakly almost periodic representation of S, $\phi(x)$ is right weakly almost periodic for each $x \in E$.

It follows from Theorem 4 that $\{\tau(l(\cdot)'\mu_\alpha)\phi(x)\}$ converges to the unique constant function p in the closure of convex hull of $\mathcal{RO}(\phi(x))$. In this case, $p(\cdot) =$

$\tau(\mu)\phi(x) = T(\mu)x$ for each invariant mean μ on X. Hence, for each $x \in E$, $\{T(l(h)'\mu_\alpha)x\}$ converges to a point $T(\mu)x$ in C uniformly in $h \in S$ where μ is an invariant mean on X. Since, for each $s \in S$ and $x' \in E'$,

$$\begin{aligned}\langle T(s)T(\mu)x, x'\rangle &= \langle T(\mu)x, T(s)'x'\rangle = \mu\langle T(\cdot)x, T(s)'x'\rangle\\ &= \mu\langle T(s)T(\cdot)x, x'\rangle = \mu\langle T(s\cdot)x, x'\rangle\\ &= l(s)'\mu\langle T(\cdot), x'\rangle = \mu\langle T(\cdot), x'\rangle\\ &= \langle T(\mu)x, x'\rangle\end{aligned}$$

where $T(s)'$ is the adjoint operator of $T(s)$, we have $T(s)T(\mu)x = T(\mu)x$ for each $s \in S$. This completes the proof. ∎

Remark 2. By the uniform boundedness theorem, every weakly almost periodic representation $\mathcal{S} = \{T(s) : s \in S\}$ of S as bounded linear operators on E is uniformly bounded, that is, there exists a $K > 0$ such that $\|T(s)\| \leq K$ for each $s \in S$.

In 1941, Fréchet proved a mean ergodic theorem for one-parameter equicontinuous semigroups of mappings in an Euclidian space.

Theorem 6. (Fréchet [10]). *Let C be a bounded, closed and convex subset of an Euclidian space $\mathbb{R}^m$ and let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a one-parameter semigroup of continuous mappings of C into itself such that $\mathcal{S}$ is equicontinuous. Then, for each $x \in C$,*

$$\frac{1}{t}\int_0^t T(r)x\, dr$$

converges to a point of C as $t \to +\infty$.

Motivated by Fréchet, we study the relationship between almost periodicity of semigroups of mappings in a locally convex space and their equicontinuity.

Lemma 2. *Let S be a semigroup with identity, let C be a bounded, closed and convex subset of a l.c.s. E and let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous mappings of C into itself. Then, the following are equivalent:*

(*i*) *$\mathcal{S}$ is almost periodic;*

(*ii*) *for each $x \in C$, $\mathcal{O}(x)$ is relatively compact and $\mathcal{S}$ is equicontinuous on the closure of $\mathcal{O}(x)$.*

Proof. (i) $\Rightarrow$ (ii): By (i), $\mathcal{RO}(T(\cdot)x)$ is relatively compact. For each $s \in S$, let $p_s : f \mapsto f(s)$ be a projection of $l^\infty(S, E)$ into E. Then, each p_s is continuous on $l^\infty(S, E)$ with the topology $\mathfrak{T}$ of uniform convergence on S. From $p_e(\mathcal{RO}(T(\cdot)x)) = \mathcal{O}(x)$, $\mathcal{O}(x)$ is relatively compact, where e is the identity of S.

Let $x_0 \in \overline{\mathcal{O}(x)}$ and let U be a neighborhood of 0. Choose a closed neighborhood V of 0 such that $V+V+V \subset U$. Since $\mathcal{LO}(T(\cdot)x)$ is totally bounded, there exist $s_1, \ldots, s_n$ in S such that

$$\mathcal{LO}(T(\cdot)x) \subset \cup_{i=1}^{n}(L(s_i)T(\cdot)x) + M(V),$$

where $M(V) = \{f \in l^\infty(S,E) : f(S) \subset V\}$. Then, there exists a neighborhood W of 0 such that $T(s_i)x_0 - T(s_i)y \in V$ for each $y \in x_0 + W$ $(i = 1, \ldots, n)$. Let $t \in S$. Choose a s_k such that for each $s \in S$, $T(t)T(s)x - T(s_k)T(s)x \in V$. Hence, $T(t)y - T(s_k)y \in V$ for each $y \in \overline{\mathcal{O}(x)}$ from continuity of $T(t)$ and $T(s_k)$. So, we have, for each $y \in (x_0 + W) \cap \overline{\mathcal{O}(x)}$,

$$\begin{aligned} T(t)x_0 - T(t)y &= (T(t)x_0 - T(s_k)x_0) + (T(s_k)x_0 - T(s_k)y) \\ &\quad + (T(s_k)y - T(t)y) \in V + V + V \subset U. \end{aligned}$$

Since $t \in S$ is arbitrary, $\mathcal{S}$ is equicontinuous on $\overline{\mathcal{O}(x)}$.

(ii) $\Rightarrow$ (i): Define a mapping $\phi_{\mathcal{S}} : x \mapsto T(\cdot)x$ of C into $l^\infty(S,E)$. Then, from equicontinuity of $\mathcal{S}$, $\phi_{\mathcal{S}}$ is continuous on $\overline{\mathcal{O}(x)}$. Since, for each $s \in S$,

$$R(s)T(\cdot)x = T(\cdot s)x = T(\cdot)T(s)x = \phi_{\mathcal{S}}(T(s)x),$$

we have $\mathcal{RO}(T(\cdot)x) = \phi_{\mathcal{S}}(\mathcal{O}(x))$. It follows from continuity of $\phi_{\mathcal{S}}$ that $\mathcal{RO}(T(\cdot)x)$ is relatively compact. This completes the proof. ∎

As in the proof of Theorem 4.1 in [18], we can prove a mean ergodic theorem for equicontinuous semigroups of mappings of a compact convex subset of a locally convex space E into itself by using Lemma 2 and Theorem 3.

Theorem 7. *Let S be a semigroup with identity, let C be a compact convex subset of a l.c.s. E, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous mappings of C into itself such that $\mathcal{S}$ is equicontinuous, let X be a closed, translation invariant and admissible subspace of $l^{\infty}(S)$ containing constants and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, for each $x \in C$, $\{T(l(h)'\mu_\alpha)x\}$ converges to a point p of C uniformly in $h \in S$. In this case, $p = T(\mu)x$ for each invariant mean μ on X.*

Proof. For each $x \in E$, we define a vector-valued function $\phi(x) \in l^\infty(S,E)$ by $(\phi(x))(s) = T(s)x$ for each $s \in S$. Then, by Lemma 2, $\phi(x)$ is almost periodic. It follows from Theorem 3 that $\{\tau(l(\cdot)'\mu_\alpha)\phi(x)\}$ converges to the unique constant function p in the closure of convex hull of $\mathcal{RO}(\phi(x))$. In this case, $p(\cdot) = \tau(\mu)\phi(x) = T(\mu)x$ for each invariant mean μ on X. Hence, for each $x \in C$, $\{T(l(h)'\mu_\alpha)x\}$ converges to a point $T(\mu)x$ in C uniformly in $h \in S$ where μ is an invariant mean on X. This completes the proof. ∎

Remark 3. Note that the limit point $T(\mu)x$ is *not* always a common fixed point for $\mathcal{S}$. In fact, we know that there exists a nonexpansive mapping T of C into itself such that for some $x \in C$, its Cesàro means $\{1/n \sum_{k=0}^{n-1} T^k x\}$ converge, but its limit point is not a fixed point of T; see also Suzuki and Takahashi [26] and Suzuki [25].

The following corollaries are the case when E is a Banach space with the norm topology.

Corollary 3. *Let S be a semigroup with identity, let C be a compact convex subset of a Banach space E, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous mappings of C into itself such that $\mathcal{S}$ is equicontinuous, let X be a closed, translation invariant and admissible subspace of $l^{\infty}(S)$ containing constants and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, for each $x \in C$, $\{T(l(h)'\mu_\alpha)x\}$ converges strongly to a point p of C uniformly in $h \in S$. In this case, $p = T(\mu)x$ for each invariant mean μ on X.*

Corollary 4. *Let C be a compact convex subset of a Banach space E, let U and W be continuous mappings of C into itself such that $UW = WU$ and the families $\{U^n\}$ and $\{W^n\}$ are equicontinuous. Then, for each $x \in C$, the Cesàro means*

$$\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} U^{i+h} W^{j+h} x$$

converge to a point p of C uniformly in $h \in \mathbb{N}_+$.

Corollary 5. *Let C be a compact convex subset of a Banach space and let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a one-parameter semigroup of continuous mappings of C into itself such that $\mathcal{S}$ is equicontinuous. Then, for each $x \in C$, the Abel means*

$$r \int_0^\infty \exp(-rt) T(t+h)x \, dt$$

converge to a point p of C uniformly in $h \in \mathbb{R}_+$ as $r \to +\infty$.

The following corollary is the case when E is a Banach space with the weak topology.

Corollary 6. *Let S be a semigroup with identity, let C be a weakly compact convex subset of a Banach space E, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as weakly continuous mappings of C into itself such that $\mathcal{S}$ is weak-to-weak equicontinuous, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, for each $x \in C$, $\{T(l(h)'\mu_\alpha)x\}$ converges weakly to a point p of C uniformly in $h \in S$. In this case, $p = T(\mu)x$ for each invariant mean μ on X.*

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