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CYCLIC ODD $3K$-CYCLE SYSTEMS OF THE COMPLETE GRAPH

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Abstract. For any prime p and each admissible value n, a complete answer to the existence problem for cyclic $3p$-cycle systems of the complete graph K_n is given.

1. Introduction

Let K_n be the complete graph of order n and let $C = (c_0, c_1, \cdots, c_{m-1})$ denote an m-cycle or a closed m-trail. An *m-cycle system* of K_n is a pair $(V, \boldsymbol{C})$ where V is the vertex set of K_n and $\boldsymbol{C}$ is a collection of m-cycles whose edges partition the edges of K_n. The necessary conditions for the existence of an m-cycle system of K_n are

$$(*) \qquad n \equiv 1 \pmod 2, 3 \leq m \leq n, \text{ and } n(n-1) \equiv 0 \pmod{2m}.$$

Given an integer $m \geq 3$, an integer n satisfying the conditions in (*) is said to be *admissible*.

The study of m-cycle systems of the complete graph has been one of the most interesting problems in graph decompositions. A survey on cycle decompositions is given in [4]. Alspach and Gavlas [1] in the case of m odd and Sajna [15] in the even case proved the necessary conditions in (*) are also sufficient.

Let $\mathbb{Z}_n$ be the group of integers modulo n and $\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\}$. An m-cycle system $(V, \boldsymbol{C})$ of the complete graph K_n is said to be *cyclic* if $V = \mathbb{Z}_n$ and $C + 1 = (c_0+1, c_1+1, \cdots, c_{m-1}+1) \pmod n \in \boldsymbol{C}$ whenever $C \in \boldsymbol{C}$. The necessary conditions in (*) however are not sufficient for the existence of a cyclic m-cycle system of K_n. A cyclic n-cycle system of the complete graph K_n is called a cyclic *Hamiltonian* cycle system.

In 1938, Peltesohn [10] proved that for each admissible n ($\neq 9$), there exists a cyclic 3-cycle system of K_n. Since then, finding necessary and sufficient conditions for cyclic m-cycle systems of K_n has attracted much attention. Some partial solutions have been given by a number of authors [2, 3, 6-9, 11-13, 16, 17, 19].

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Theorem 1.1. ([3, 6, 7, 8, 16]).

(1) *Suppose $m \geq 3$ is a positive integer. Then there exists a cyclic m-cycle system of K_{2pm+1} for $p \geq 1$.*

(2) *Suppose $m \geq 3$ is an odd integer. Then there exists a cyclic m-cycle system of K_{2pm+m} for $p \geq 0$ except when $(m, p) \in \{(3, 1), (15, 0), (q^\alpha, 0)\}$ where q is a prime and $\alpha > 1$.*

Theorem 1.2. ([19]).

(1) *If $3 \leq m \leq 32$, then for each admissible value n, there exists a cyclic m-cycle system of K_n provided $(m, n) \neq (3, 9), (6, 9), (9, 9), (14, 21), (15, 15), (15, 21), (15, 25), (20, 25), (22, 33), (24, 33), (25, 25), (27, 27),$ and $(28, 49)$.*

(2) *If $m < n < 2m + 1$ and gcd(m, n) is an odd prime power, then there does not exist a cyclic m-cycle system of K_n.*

Theorem 1.3. ([17]). *For each even integer $m \geq 4$ and each admissible value n with $n > 2m$, there exists a cyclic m-cycle system of K_n.*

To construct a cyclic m-cycle system of K_n, it is crucial to further characterize the admissible values n. Assume $m = de$ to be any positive integer, where d is odd, $e \geq 1$, and gcd$(d, e) = 1$, and n to be an admissible value. If gcd$(m, n) = d = 1$, then it is easy to check from (*) that $n = 2pm+1$ for $p \geq 1$. Now, suppose gcd$(m, n) = d > 1$. Then it is obvious that $n = 2pm + ds$, where p is a nonnegative integer and s is odd with $1 \leq s < 2e$. Also, since $n(n - 1) \equiv 0 \pmod{2m}$ and gcd$(m, n) = d$, it follows that $n-1 \equiv 0 \pmod{2e}$, or equivalently, $n = 2pm + 2be + 1$, where p is a nonnegative integer and $1 \leq b < d$. In fact, since $n = 2pm + ds = 2pm + 2be + 1 \geq m$, we have $p \geq 1$, if $b < \frac{d-1}{2}$ or $s < e$, and $p \geq 0$, if $b \geq \frac{d-1}{2}$ or $s \geq e$. Moreover, we obtain $ds = 2be + 1$, where s is odd with $1 \leq s < 2e$ and $1 \leq b < d$.

Lemma 1.4. ([17]). *Let $m = de$ be any given integer (≥ 3) where d is odd, $e \geq 1$, and gcd$(d, e) = 1$, and let n be admissible with gcd$(m, n) = d$.*

(1) *If $d = 1$, then $n = 2pm + 1$ for $p \geq 1$.*

(2) *If $d > 1$ and $b < \frac{d-1}{2}$ or $s < e$, then $n = 2pm + 2be + 1 = 2pm + ds$ for $p \geq 1$ where $1 \leq b < d$ and s is odd with $1 \leq s < 2e$.*

(3) *If $d > 1$ and $b \geq \frac{d-1}{2}$ or $s \geq e$, then $n = 2pm + 2be + 1 = 2pm + ds$ for $p \geq 0$ where $1 \leq b < d$ and s is odd with $1 \leq s < 2e$. In particular, if m is odd and $e = 1$, then $n = 2pm + m$ for $p \geq 0$.*

In view of Lemma 1.4, if we take $m = p^k$ where p is an odd prime and $k \geq 1$, then $d = 1$ or p^k. It implies that $n = 2pm + 1$ for $p \geq 1$ or $n = 2pm + m$ for $p \geq 0$ and by utilizing Theorem 1.1, we obtain the following consequence.

Theorem 1.5. *Let $m = p^k$ where p is an odd prime and $k \geq 1$. Then for each admissible value n, there exists a cyclic m-cycle system of K_n except when $m = 3$ and $n = 9$ or $n = m$.*

In this paper, we focus our attention on the constructions of cyclic m-cycle systems of K_n where $m = 3k$ is an odd integer with gcd(3, k) = 1. Note that by Theorem 1.2(1), it is enough to consider the m-cycles where $m \geq 33$. The methods used here involve difference constructions and circulant graphs, and it should be mentioned that some basic techniques used in this paper also occurred in [18]. The main result is:

Theorem 1.6. *For any prime p and each admissible value n, there exist cyclic $3p$-cycle systems of the complete graph K_n.*

We remark that given an odd integer $m = 3k$ with gcd(3, k) = 1, it follows by Lemma 1.4 that $n \equiv 1$, 3, k, or $3k$ (mod $2m$), and using Theorem 1.1, it suffices to consider only the cases when $n \equiv 3$ or k (mod $2m$), that is, $\gcd(m, n) = 3$ or k. Moreover, in the light of Theorem 1.2(2), if k is a prime, then there is no cyclic $3k$-cycle system of K_n where $n < 6k$.

2. Definitions and Preliminaries

Let S be a subset of $\mathbb{Z}_n^*$ such that $S = -S$; that is, $s \in S$ implies that $-s \in S$. The *circulant graph* of order n, $X(n, S)$, is defined as the graph whose vertices are the elements of $\mathbb{Z}_n$, with an edge between vertices u and v if and only if $v = u + s$ for some $s \in S$. The set S is called the *connection set* of $X(n, S)$. Since for each edge $\{u, v\}$ in $X(n, S)$, there is an element s in S such that $\{u, v\} = \{u, u + s\} = \{v + n - s, v\}$ (mod n), we will write $-s$ for $n - s$ when n is understood, and the elements $\pm s$ in S are said to be the *differences* of the edge $\{u, v\}$ in $X(n, S)$, and we denote it by $d(u, v) = \pm s$. In what follows, we will use $\|D(H)\|$ to denote the number of distinct differences of edges in H where H is the subgraph of $X(n, S)$.

Given an m-cycle $C = (c_0, c_1, \cdots, c_{m-1})$ in $X(n, S)$ where $m = de$ is an odd integer, the cycle C is of *type* d if its stabilizer under the natural action of $\mathbb{Z}_n$ has order d. In other words, d is the common divisor of n and m such that $C = C + n/d$ (mod n). Following [5], the *list of partial differences* of C of type d is the multiset

$$\partial C = \{\pm(c_{i+1} - c_i) : 0 \leq i \leq m/d - 1\}.$$

An m-cycle C of type d on $X(n, S)$ is called *full* if $d = 1$, otherwise *short*. The *cycle orbit* $\mathcal{O}$ of C is the set of m-cycles in the collection $\{C + i : 0 \leq i < n/d\}$. The *length* of a cycle orbit is its cardinality. A *base cycle* of a cycle orbit $\mathcal{O}$ is a cycle $C \in \mathcal{O}$ that is chosen arbitrarily. Any cyclic m-cycle system of a graph of order n is generated from base cycles, and each full m-cycle corresponds to a cycle orbit with length n.

Since n is odd, the connection set S can be partitioned into subsets A, $-A$ such that for every element s in A, $s = i$ or $-i$ for $1 \le i \le \frac{n-1}{2}$, so we may assume $S = \pm A$. It is evident that the complete graph K_n is isomorphic to the circulant graph $X(n, S)$ with $S = \mathbb{Z}_n^* = \pm\{1, 2, \cdots, \frac{n-1}{2}\}$, so $\|D(K_n)\| = n - 1$.

By $[a, b]$ we mean the set of consecutive integers $a, a+1, \cdots, b$ where $1 \le a < b \le \frac{n-1}{2}$. Given an odd integer m, the connection set $S = \{d_i, d_i + j_i : \ j_i = 1$ or $2,\ 1 \le i \le k\}$ is called *proper* if all elements in it are pairwise distinct, $1 \le d_1 < d_2 < \cdots < d_k < \frac{n-1}{2}$, and $d_i + j_i < d_{i+1}$ for $1 \le i \le k-1$. Note that $|S| = 2k$. If $j_1 = \cdots = j_k = 1$ (resp. $j_1 = j_k = 2$, $j_2 = \cdots = j_{k-1} = 1$), we say the proper set S is of *type* 1 (resp. *type* 2); if $j_1 = 2$ and $j_2 = \cdots = j_k = 1$ (resp. $j_1 = \cdots = j_{k-1} = 1$ and $j_k = 2$), the proper set S is said to be of *type* 3 (resp. *type* 4). By S_i we mean the proper set S of type i for $1 \le i \le 4$.

A *Skolem sequence* of order p is a collection of ordered pairs $\{(s_i, t_i) : t_i - s_i = i, 1 \le i \le p\}$ with $\bigcup_{i=1}\{s_i, t_i\} = \{1, 2, \cdots, 2p\}$ or $\{1, 2, \cdots, 2p-1, 2p+1\}$. In the second case one usually speaks of a *hooked* Skolem sequence.

Theorem 2.1. ([14]).

(1) *A Skolem sequence of order* p *exists if and only if* $p \equiv 0$ *or* $1 \ (mod\ 4)$.

(2) *A hooked Skolem sequence of order* p *exists if and only if* $p \equiv 2$ *or* $3 \ (mod\ 4)$.

A set $\{r, s_r + r, t_r + r\}$ where r is a positive integer with $1 \le r \le p$ is called a r*-Skolem set*, denoted T_r, if (s_r, t_r) is an ordered pair in a Skolem sequence of order p.

Corollary 2.2.

(1) *If* $p \equiv 0$ *or* $1 \ (mod\ 4)$, *then* $[1, 3p]$ *can be partitioned into the union of* r*-Skolem subsets for* $1 \le r \le p$.

(2) *If* $p \equiv 2$ *or* $3 \ (mod\ 4)$, *then* $[1, 3p+1] \setminus \{3p\}$ *can be partitioned into the union of* r*-Skolem subsets for* $1 \le r \le p$.

Given a r-Skolem set T_r and a proper set of type i S_i where $1 \le r \le p$ and $1 \le i \le 4$, the connection set $S = T_r \bigcup S_i$ is said to be i*-proper* if $T_r \bigcap S_i = \emptyset$.

The following two consequences will be used as the main tools to construct the full base cycles on circulant graphs. In what follows, we shall assume $C = (c_0 = 0, c_1, \cdots, c_{m-1})$ to be a closed m-trail and $T_r = \{r, s_r + r, t_r + r\}$ to be a r-Skolem set.

Proposition 2.3. *Suppose the connection set* S *is* 1*-proper or* 2*-proper. Then for* $m = 4k + 3$ *with* $k \ge 1$, *there exists a cyclic* m*-cycle system of* $X(n, \pm S)$.

Proof. Suppose $S = T_r \bigcup S_1$ is 1-proper where $S_1 = \{e_i, e_i + 1 : \ 1 \le i \le 2k\}$ is a proper set of type 1. Let us define the vertices c_i in C for $1 \le i \le m-1$ as

$$c_i = \begin{cases} e_{k+1-j} + j, & \text{if } i = 2j - 1 \text{ for } 1 \leq j \leq k; \\ j, & \text{if } i = 2j \text{ for } 1 \leq j \leq k; \\ r + k, & \text{if } i = 2k + 1; \\ s_r + 2r + k, & \text{if } i = 2k + 2; \\ -e_{2k} + t_r + r + k - 1, & \text{if } i = 2k + 3; \\ -e_{2k} + k - j, & \text{if } i = 2k + 2 + 2j \text{ for } 1 \leq j \leq k; \text{ and} \\ -e_{2k} + e_{k+j} + k - j, & \text{if } i = 2k + 3 + 2j \text{ for } 1 \leq j \leq k - 1. \end{cases}$$

Let $<C> = \langle c_0 = 0, c_2, c_4, \cdots, c_{2k}, c_{2k+1}, c_{2k+2}, c_{2k-1}, c_{2k-3}, \cdots, c_1, c_{4k+2}, c_{4k}, \cdots, c_{2k+4}, c_{2k+3}, c_{2k+5}, \cdots, c_{4k+1}\rangle$ be a sequence obtained from the vertices c_i in C where $c_{4k+1} = n - e_2 + t_r + r$ if $k = 1$ and $c_{4k+1} = n - e_{2k} + e_{2k-1} + 1$ if $k \geq 2$. Since $<C>$ is increasing, it means that C is an m-cycle, and since $d(c_{2i}, c_{2i+1}) = \pm(e_{k-i} + 1)$ and $d(c_{2i+1}, c_{2i+2}) = \pm e_{k-i}$ for $0 \leq i \leq k - 1$, $d(c_{2k}, c_{2k+1}) = \pm r$, $d(c_{2k+1}, c_{2k+2}) = \pm(s_r + r)$, $d(c_{2k+2}, c_{2k+3}) = \pm(e_{2k} + 1)$, $d(c_{2k+3}, c_{2k+4}) = \pm(t_r + r)$, $d(c_{2k+2+2i}, c_{2k+3+2i}) = \pm e_{k+i}$ for $1 \leq i \leq k - 1$, $d(c_{2k+3+2i}, c_{2k+4+2i}) = \pm(e_{k+i} + 1)$ for $1 \leq i \leq k - 1$, and $d(c_0, c_{4k+2}) = \pm e_{2k}$, we have that C is indeed an m-cycle with $\partial C = \pm S$.

The similar proof can be used for the case when $S = T_r \bigcup S_2$ is 2-proper, i.e., replacing c_i in C with $c_i + 1$ for $2k - 1 \leq i \leq 2k + 2$. We leave it to the reader. ■

Proposition 2.4. *Suppose the connection set S is 3-proper or 4-proper. Then for $m = 4k + 5$ with $k \geq 1$, there exists a cyclic m-cycle system of $X(n, \pm S)$.*

Proof. The proof is divided into two cases according as whether S is 3-proper or 4-proper.

Suppose $S = T_r \bigcup S_3$ is 3-proper where $S_3 = \{e_1, e_1 + 2\} \bigcup \{e_i, e_i + 1 : 2 \leq i \leq 2k + 1\}$. The vertices c_i in C for $1 \leq i \leq m - 1$ are given by

$$c_i = \begin{cases} e_{k+1-j} + j, & \text{if } i = 2j - 1 \text{ for } 1 \leq j \leq k - 1; \\ j, & \text{if } i = 2j \text{ for } 1 \leq j \leq k - 1; \\ e_1 + k + 1, & \text{if } i = 2k - 1; \\ k + 1, & \text{if } i = 2k; \\ r + k + 1, & \text{if } i = 2k + 1; \\ s_r + 2r + k + 1, & \text{if } i = 2k + 2; \\ -e_{2k+1} + t_r + r + k, & \text{if } i = 2k + 3; \\ -e_{2k+1} + k + 1 - j, & \text{if } i = 2k + 2 + 2j \text{ for } 1 \leq j \leq k + 1; \text{ and} \\ -e_{2k+1} + e_{k+j} + k + 1 - j, & \text{if } i = 2k + 3 + 2j \text{ for } 1 \leq j \leq k. \end{cases}$$

Suppose $S = T_r \bigcup S_4$ is 4-proper where $S_4 = \{e_i, e_i + 1 : \ 1 \le i \le 2k\} \bigcup \{e_{2k+1}, e_{2k+1} + 2\}$. For $1 \le i \le m-1$, the vertices c_i in C are defined as

$$c_i = \begin{cases} e_{k+2-j} + j, & \text{if } i = 2j-1 \text{ for } 1 \le j \le k+1; \\ j, & \text{if } i = 2j \text{ for } 1 \le j \le k+1; \\ r + k + 1, & \text{if } i = 2k+3; \\ s_r + 2r + k + 1, & \text{if } i = 2k+4; \\ -e_{2k+1} + t_r + r + k - 1, & \text{if } i = 2k+5; \\ -e_{2k+1} + k - j, & \text{if } i = 2k+4+2j \text{ for } 1 \le j \le k; \text{ and} \\ -e_{2k+1} + e_{k+1+j} + k - j, & \text{if } i = 2k+5+2j \text{ for } 1 \le j \le k-1. \end{cases}$$

The rest of the proof is analogous to those in Proposition 2.3, and we omit the details. ■

Establishing a cyclic m-cycle system of K_n, the vital key is to construct short base m-cycles in it. Lemma 2.5 provides a useful method for constructing short m-cycles on circulant graphs. For the convenience of notation, by $[c_0, c_1, \cdots, c_{e-1}]_{k \cdot n/d}$ we mean an m-cycle (or a closed m-trail) of the form $(c_0, c_1, \cdots, c_{m-1})$ (mod n) where $c_{i+j \cdot e} = c_i + j \cdot k \cdot n/d$ for $0 \le i \le e-1$ and $0 \le j \le d-1$.

Lemma 2.5. *Let $m = de$ be an odd integer where $d \ge 3$, $e \ge 1$, and $gcd(d, e) = 1$, and let n be admissible with $gcd(m, n) = d$. If there exists an m-cycle $C = [c_0, c_1, \cdots, c_{e-1}]_{k \cdot n/d}$ with $gcd(k, d) = 1$ on a circulant graph $X(n, \pm S)$ satisfying*
(1) for $0 \le i \ne j \le e-1$, $c_i \not\equiv c_j$ (mod n/d) and
(2) the differences $d(c_{i-1}, c_i) = \pm d_i$ for $1 \le i \le e$ are all distinct,
then there exists a cyclic m-cycle system of $X(n, \partial C)$ where $\partial C = \pm\{d_1, d_2, \cdots, d_e\}$.

Note that the set $\{C + i : \ 0 \le i < n/d\}$ forms a cycle orbit of C with length n/d, and the cycle C can be regarded as a short base cycle of this cycle orbit. For convenience, the cycle $C = [c_0, c_1, \cdots, c_{e-1}]_{k \cdot n/d}$ in Lemma 2.5 is said to be an m-cycle of *index* $k \cdot n/d$. The m-cycle C itself, of course, is of type d on $X(n, \partial C)$.

The circulant graphs will also play a crucial role for constructing a cyclic m-cycle system of K_n.

Theorem 2.6. *There exists a cyclic m-cycle system of K_n if and only if there are cyclic m-cycle systems of the circulant graphs $X(n, \partial C_i) (1 \le i \le t)$ such that $\bigcup_{i=1}^{t} \partial C_i = \mathbb{Z}_n^*$ and $\partial C_i \bigcap \partial C_j = \emptyset$ for $i \ne j$.*

By virtue of Lemma 1.4, for each specified integer $m = de$, we have $n = 2pm + 2be + 1 = 2pm + ds = d(2pe + s)$ and so $n/d = 2pe + s$. To construct a cyclic m-cycle system of K_n, it is natural that we will try to set up p full base m-cycles and b short base m-cycles C of index $k \cdot n/d$ for some positive integer k with $\gcd(k, d) = 1$ and $\|D(C)\| = 2e$ each since $\|D(K_n)\| = n - 1 = 2(pm + be)$.

3. $\mathrm{Gcd}(m, n) = 3$

In this section, we shall assume that $d = 3$, i.e., $m = 3e$ with $\gcd(3, e) = 1$, and let n be admissible with $\gcd(m, n) = 3$. Recall that it suffices to consider $m = 3e \geq 33$, that is, $e \geq 11$. Since $\gcd(3, e) = 1$, it follows that $e = 12a + 11$, $12a + 13$, $12a + 17$, or $12a + 19$ for $a \geq 0$. By virtue of Lemma 1.4, we have:

if $e = 12a + 11$, then $b = 2$, $s = 16a + 15$, and $n = 6pe + 48a + 45$ for $p \geq 0$;
if $e = 12a + 13$, then $b = 1$, $s = 8a + 9$, and $n = 6pe + 24a + 27$ for $p \geq 1$;
if $e = 12a + 17$, then $b = 2$, $s = 16a + 23$, and $n = 6pe + 48a + 69$ for $p \geq 0$; and
if $e = 12a + 19$, then $b = 1$, $s = 8a + 13$, and $n = 6pe + 24a + 39$ for $p \geq 1$.

That is, if $e = 12a + 13$ or $12a + 19$ (resp. $12a + 11$ or $12a + 17$), then we will construct p full base m-cycles and a short base m-cycle (resp. two short base m-cycles).

Next, consider an e-set $W = \{w_1, w_2, \ldots, w_e\}$ where $w_i \in \mathbb{Z}_n^*$. The set W is called *strong* if $1 \leq w_1 < w_2 < \ldots < w_e < n/3$ and $\sum_{i=1}^{\frac{e-1}{2}} (w_{2i} - w_{2i-1}) + w_e = n/3$. The strong e-set will be used to establish the short base m-cycles of index $n/3$.

Lemma 3.1. *If $W = \{w_1, w_2, \ldots, w_e\}$ is a strong e-set, then there exists a cyclic m-cycle system of $X(n, \pm W)$.*

Proof. Let $C = [c_0 = 0, c_1, \ldots, c_{e-1}]_{n/3}$ be a closed m-trail defined as

$$\begin{aligned} c_{2i-1} &= w_{e-2i+1} + \sum_{j=1}^{i-1} (w_{e-2j+1} - w_{e-2j}) \text{ and} \\ c_{2i} &= \sum_{j=1}^{i} (w_{e-2j+1} - w_{e-2j}) \text{ for } 1 \leq i \leq \frac{e-1}{2}. \end{aligned}$$

Consider the sequence $<C> = \langle c_0 = 0, c_2, c_4, \cdots, c_{e-1}, c_{e-2}, c_{e-4}, \cdots, c_1 = w_{e-1} \rangle$ from the vertices c_i ($0 \leq i \leq e-1$) in C. Since the sequence $<C>$ is increasing and $c_i \not\equiv c_j \pmod{n/3}$ for $0 \leq i < j \leq e - 1$, we have that C is an m-cycle of index $n/3$, and since $d(c_i, c_{i+1}) = \pm w_{e-1-i}$ for $0 \leq i \leq e - 2$ and $d(c_{e-1}, c_e) = \pm w_e$, it follows that C is an m-cycle with $\partial C = \pm W$.

The thesis follows by Lemma 2.5. ■

By $[a, b] = \biguplus_{i=1}^{t} A_i$ we mean that the set $[a, b]$ can be partitioned into the union of disjoint subsets A_i for $1 \leq i \leq t$. A set U is *even* if $|U| \equiv 0 \pmod 2$. Throughout we will use $T_r \biguplus S_i$, $T_r \biguplus S_{i,r}$ as i-proper connection sets where $1 \leq r \leq p$ and $1 \leq i \leq 4$.

Proposition 3.2. *Suppose $m = 3e$ where $e = 12a + 13$ or $12a + 19$ for $a \geq 0$ and let n be admissible with $gcd(m, n) = 3$. Then there exists a cyclic m-cycle system of K_n.*

Proof. It is clear that $m \equiv 3 \pmod 4$ if $e = 12a + 13$ and $m \equiv 1 \pmod 4$ if $e = 12a + 19$. Recall that $[1, 3p] = \biguplus_{i=1}^{p} T_i$ if $p \equiv 0$ or $1 \pmod 4$ and $[1, 3p+1] \setminus \{3p\} = \biguplus_{i=1}^{p} T_i$ if $p \equiv 2$ or $3 \pmod 4$ by Corollary 2.2. The proof is split into the following 4 cases.

Case 1. $e = 12a + 13$ and $p \equiv 1 \pmod 4$ or $e = 12a + 19$ and $p \equiv 0 \pmod 4$.

$[1, \frac{n-1}{2}] = [1, 3p] \uplus U \uplus W$ where $U = \{3p + 1, 3p + 3\} \uplus [3p + e + 2, \frac{n}{3} - \frac{e+3}{2}] \uplus [\frac{n}{3} - \frac{e-1}{2}, \frac{n-1}{2}]$ and $W = \{3p + 2, 3p + 4, 3p + 5, \cdots, 3p + e + 1, \frac{n}{3} - \frac{e+1}{2}\}$.

If $e = 12a + 13$, then partition the set $[1, 3p] \bigcup U$ into a 2-proper subset $T_p \uplus S_2$ and $p - 1$ 1-proper subsets $T_i \uplus S_{1,i}$ for $1 \leq i \leq p - 1$, i.e., $[1, 3p] \bigcup U = (\biguplus_{i=1}^{p-1} T_i \uplus S_{1,i}) \uplus (T_p \uplus S_2)$.

If $e = 12a + 19$, then $[1, 3p] \bigcup U = (\biguplus_{i=1}^{p-1} T_i \uplus S_{3,i}) \uplus (T_p \uplus S_4)$.

Note that the elements $\frac{n-3}{2}$, $\frac{n+1}{2}$ are included in S_2, S_4, respectively.

Case 2. $e = 12a + 13$ and $p \equiv 0 \pmod 4$ or $e = 12a + 19$ and $p \equiv 1 \pmod 4$.

$[1, \frac{n-1}{2}] = [1, 3p] \uplus U \uplus W$ where $U = [3p + e, \frac{n}{3} - \frac{e+1}{2}] \uplus [\frac{n}{3} - \frac{e-3}{2}, \frac{n-1}{2}]$ and $W = \{3p + 1, 3p + 2, \cdots, 3p + e - 1, \frac{n}{3} - \frac{e-1}{2}\}$.

If $e = 12a + 13$, then $[1, 3p] \bigcup U = \biguplus_{i=1}^{p} T_i \uplus S_{1,i}$.

If $e = 12a + 19$, then $[1, 3p] \bigcup U = (\biguplus_{i=1}^{p-1} T_i \uplus S_{3,i}) \uplus (T_p \uplus S_4)$.

Case 3. $e = 12a + 13$ and $p \equiv 2 \pmod 4$ or $e = 12a + 19$ and $p \equiv 3 \pmod 4$.

$[1, \frac{n-1}{2}] = ([1, 3p + 1] \setminus \{3p\}) \uplus U \uplus W$ where $U = \{3p, 3p + 2\} \uplus [3p + e + 2, \frac{n}{3} - \frac{e+1}{2}] \uplus [\frac{n}{3} - \frac{e-3}{2}, \frac{n-1}{2}]$ and $W = \{3p + 3, 3p + 4, \cdots, 3p + e + 1, \frac{n}{3} - \frac{e-1}{2}\}$.

If $e = 12a + 13$, then $([1, 3p + 1] \setminus \{3p\}) \bigcup U = (\biguplus_{i=1}^{p-1} T_i \uplus S_{1,i}) \uplus (T_p \uplus S_2)$.

If $e = 12a + 19$, then $([1, 3p + 1] \setminus \{3p\}) \bigcup U = (\biguplus_{i=1}^{p} T_i \uplus S_{3,i})$.

Case 4. $e = 12a + 13$ and $p \equiv 3 \pmod 4$ or $e = 12a + 19$ and $p \equiv 2 \pmod 4$.

$[1, \frac{n-1}{2}] = ([1, 3p + 1] \setminus \{3p\}) \uplus U \uplus W$ where $U = [3p + e, \frac{n}{3} - \frac{e+3}{2}] \uplus [\frac{n}{3} - \frac{e-1}{2}, \frac{n-1}{2}]$ and $W = \{3p, 3p + 2, 3p + 3, \cdots, 3p + e - 1, \frac{n}{3} - \frac{e+1}{2}\}$.

If $e = 12a + 13$, then $([1, 3p + 1] \setminus \{3p\}) \bigcup U = (\biguplus_{i=1}^{p} T_i \uplus S_{1,i})$.

If $e = 12a + 19$, then $([1, 3p + 1] \setminus \{3p\}) \bigcup U = (\biguplus_{i=1}^{p} T_i \uplus S_{3,i})$.

Note that in each case, U is an even $p(m-3)$-set and W is a strong e-set. By virtue of Lemma 3.1, there is a cyclic m-cycle system of $X(n, \pm W)$. Moreover, if $e = 12a + 13$ (resp. $e = 12a + 19$), by Proposition 2.3 (resp. Proposition 2.4), there exist cyclic m-cycle systems of $X(n, \pm([1, 3p] \bigcup U))$ and $X(n, \pm(([1, 3p + 1] \setminus \{3p\}) \bigcup U))$.

Since for each case, $\mathbb{Z}_n^* = \pm([1, 3p] \uplus U \uplus W)$ or $\pm(([1, 3p+1] \setminus \{3p\}) \uplus U \uplus W)$, by Theorem 2.6, there is a cyclic m-cycle system of K_n. ∎

Proposition 3.3. *Suppose $m = 3e$ where $e = 12a + 11$ or $12a + 17$ for $a \geq 0$ and let n be admissible with $gcd(m, n) = 3$ and $n > 2m$. Then there exists a cyclic m-cycle system of K_n.*

Proof. Obviously, $m \equiv 1 \pmod 4$ if $e = 12a + 11$ and $m \equiv 3 \pmod 4$ if $e = 12a + 17$. We divide the proof into 4 cases as follows.

Case 1. $e = 12a + 11$ and $p \equiv 1 \pmod 4$ or $e = 12a + 17$ and $p \equiv 0 \pmod 4$.

$[1, \frac{n-1}{2}] = [1, 3p] \biguplus U \biguplus W_1 \biguplus W_2$ where $U = \{3p + 4, 3p + 6\} \biguplus [3p + 2e + 1, \frac{n}{3} - \frac{e+5}{2}] \biguplus [\frac{n}{3} - \frac{e-1}{2}, \frac{n-1}{2}]$, $W_1 = \{3p+2, 3p+5, 3p+7, \cdots, 3p+e+3, \frac{n}{3} - \frac{e+3}{2}\}$, and $W_2 = \{3p + 1, 3p + 3, 3p + e + 4, \cdots, 3p + 2e, \frac{n}{3} - \frac{e+1}{2}\}$.

If $e = 12a + 11$, then $[1, 3p] \bigcup U = \biguplus_{i=1}^{p} T_i \biguplus S_{3,i}$.

If $e = 12a + 17$, then $[1, 3p] \bigcup U = (\biguplus_{i=1}^{p-1} T_i \biguplus S_{1,i}) \biguplus (T_p \biguplus S_2)$.

Case 2. $e = 12a + 11$ and $p \equiv 0 \pmod 4$ or $e = 12a + 17$ and $p \equiv 1 \pmod 4$.

$[1, \frac{n-1}{2}] = [1, 3p] \biguplus U \biguplus W_1 \biguplus W_2$ where $U = \{3p + 1, 3p + 3\} \biguplus [3p + 2e + 1, \frac{n}{3} - \frac{e+3}{2}] \biguplus [\frac{n}{3} - \frac{e-3}{2}, \frac{n-1}{2}]$, $W_1 = \{3p+2, 3p+4, 3p+5, \cdots, 3p+e+1, \frac{n}{3} - \frac{e+1}{2}\}$, and $W_2 = \{3p + e + 2, 3p + e + 3, \cdots, 3p + 2e, \frac{n}{3} - \frac{e-1}{2}\}$.

If $e = 12a + 11$, then $[1, 3p] \bigcup U = (\biguplus_{i=1}^{p-1} T_i \biguplus S_{3,i}) \biguplus (T_p \biguplus S_4)$.

If $e = 12a + 17$, then $[1, 3p] \bigcup U = (\biguplus_{i=1}^{p-1} T_i \biguplus S_{1,i}) \biguplus (T_p \biguplus S_2)$.

Case 3. $e = 12a + 11$ and $p \equiv 2 \pmod 4$ or $e = 12a + 17$ and $p \equiv 3 \pmod 4$.

$[1, \frac{n-1}{2}] = ([1, 3p + 1] \setminus \{3p\}) \biguplus U \biguplus W_1 \biguplus W_2$ where $U = [3p + 2e - 1, \frac{n}{3} - \frac{e+3}{2}] \biguplus [\frac{n}{3} - \frac{e-3}{2}, \frac{n-1}{2}]$, $W_1 = \{3p, 3p + 2, 3p + 3, \cdots, 3p + e - 1, \frac{n}{3} - \frac{e+1}{2}\}$, and $W_2 = \{3p + e, 3p + e + 1, \cdots, 3p + 2e - 2, \frac{n}{3} - \frac{e-1}{2}\}$.

If $e = 12a + 11$, then $([1, 3p + 1] \setminus \{3p\}) \bigcup U = \biguplus_{i=1}^{p} T_i \biguplus S_{3,i}$.

If $e = 12a + 17$, then $([1, 3p + 1] \setminus \{3p\}) \bigcup U = (\biguplus_{i=1}^{p} T_i \biguplus S_{1,i})$.

Case 4. $e = 12a + 11$ and $p \equiv 3 \pmod 4$ or $e = 12a + 17$ and $p \equiv 2 \pmod 4$.

$[1, \frac{n-1}{2}] = ([1, 3p + 1] \setminus \{3p\}) \biguplus U \biguplus W_1 \biguplus W_2$ where $U = [3p + 2e - 1, \frac{n}{3} - \frac{e+5}{2}] \biguplus [\frac{n}{3} - \frac{e-1}{2}, \frac{n-1}{2}]$, $W_1 = \{3p + 2, 3p + 4, 3p + 5, \cdots, 3p + e + 1, \frac{n}{3} - \frac{e+1}{2}\}$, and $W_2 = \{3p, 3p + 3, 3p + e + 2, 3p + e + 3, \cdots, 3p + 2e - 2, \frac{n}{3} - \frac{e+3}{2}\}$.

If $e = 12a + 11$, then $([1, 3p + 1] \setminus \{3p\}) \bigcup U = (\biguplus_{i=1}^{p-1} T_i \biguplus S_{3,i}) \biguplus (T_p \biguplus S_4)$.

If $e = 12a + 17$, then $([1, 3p + 1] \setminus \{3p\}) \bigcup U = \biguplus_{i=1}^{p} T_i \biguplus S_{1,i}$.

It can be checked in each case that U is an even $p(m - 3)$-set and both W_1 and W_2 are strong e-subsets.

Similarly to Proposition 3.2, the proof follows by virtue of Lemma 3.1, Propositions 2.3, 2.4, and Theorem 2.6. ■

Together with Propositions 3.2 and 3.3, we obtain the first main consequence.

Theorem 3.4. *Suppose $m = 3e$ is an odd integer with $gcd(3, e) = 1$, and let n be admissible with $gcd(m, n) = 3$ and $n > 2m$. Then there exists a cyclic m-cycle system of K_n.*

Example 1. A cyclic 69-cycle system of K_{507} is presented. Given $d = 3$, $e = 23$, and $p = 3$, we have $m = 69$, b = 2, $s = 31$, and $n = 507$ where gcd(m, n) = 3 and so $\frac{n-1}{2} = 253$ and n/d = 169.

Taking $U = [54, 155] \uplus [158, 253]$, $W_1 = \{11, 13, \ldots, 33, 157\}$, and $W_2 = \{9, 12, 34, \ldots, 53, 156\}$, it follows that $[1, \frac{n-1}{2}]$ = $([1, 10] \setminus \{9\}) \uplus U \uplus W_1 \uplus W_2$. Note that both W_1 and W_2 are strong 23-sets.

Let $T_1 \uplus S_{3,1}$, $T_2 \uplus S_{3,2}$, $T_3 \uplus S_4$ be respectively connection sets defined as

$$T_1 \uplus S_{3,1} = \{1, 4, 5\} \uplus \{54, 56\} \uplus [58, 121],$$

$$T_2 \uplus S_{3,2} = \{2, 6, 8\} \uplus \{55, 57\} \uplus [122, 155] \uplus [158, 187], \text{ and}$$

$$T_3 \uplus S_4 = \{3, 7, 10\} \uplus [188, 251] \uplus \{252, 254\}.$$

It is clear that both $T_1 \uplus S_{3,1}$ and $T_2 \uplus S_{3,2}$ are 3-proper and $T_3 \uplus S_4$ is 4-proper.

By Proposition 2.4, there are cyclic 69-cycle systems of $X(507, \pm(T_i \uplus S_{3,i}))$ $(1 \leq i \leq 2)$ and $X(507, \pm(T_3 \uplus S_4))$, and by Lemma 3.1, there exist cyclic 69-cycle systems of $X(507, \pm W_i)$ $(1 \leq i \leq 2)$.

Now, by virtue of Theorem 2.6, we obtain a cyclic 69-cycle system of K_{507}.

4. Gcd(m, n) = d

Finally, assume gcd$(m, n) = d$, that is, $e = 3$ and $m = 3d$ where gcd(3, d) = 1. Note that we just consider $d \geq 11$ because $m \geq 33$. Since d is odd with gcd(d, 3) = 1, we have d = $6a$ + 5 or $6a$ + 7 for $a \geq 1$. If $d = 6a + 5$, by Lemma 1.4.(3), s = 5, $b = 5a + 4$, and $n = 2pm + 30a + 25$ for $p \geq 0$; in this case, $m \equiv 3$ (resp. 1) (mod 4) if $a \equiv 0$ (resp. 1) (mod 2). Analogously, by Lemma 1.4(2), if $d = 6a + 7$, then s = 1, $b = a + 1$, and n = $2pm + 6a + 7$ for $p \geq 1$, and it follows that $m \equiv 1$ (resp. 3) (mod 4) if $a \equiv 0$ (resp. 1) (mod 2).

Lemma 4.1. *Let* $m = 3d$ *where* $d = 6a + 5$ *or* $6a + 7$ *for* $a \geq 1$ *and* n *admissible with gcd*$(m, n) = d$.

(1) *If* $d = 6a + 5$, *then* $s = 5$, $b = 5a + 4$, $n = 2pm + 30a + 25$ *for* $p \geq 0$, *and* $m \equiv 3$ (*resp.* 1) (*mod* 4) *if* $a \equiv 0$ (*resp.* 1) (*mod* 2).

(2) *If* $d = 6a + 7$, *then* $s = 1$, $b = a + 1$, $n = 2pm + 6a + 7$ *for* $p \geq 1$, *and* $m \equiv 1$ (*resp.* 3) (*mod* 4) *if* $a \equiv 0$ (*resp.* 1) (*mod* 2).

Hence, besides p full base cycles, $5a + 4$ (resp. $a + 1$) short base cycles C with $\|D(C)\| = 2e$ will be constructed if $d = 6a + 5$ (resp. $d = 6a + 7$). Recall that n = $2pm + 2be + 1 = 2pm + ds = d(2pe + s)$. Assume $b = 4q + r$ where $q \geq 0$ and $0 \leq r \leq 3$ to be the Euclidean division of b by 4. Let Q, A, B, D, and F be subsets of $[1, \frac{n-1}{2}]$ defined by

$$Q = \begin{cases} [1,3p] \bigcup [3p+1, n/d-2], & \text{if } p \equiv 1 \pmod 4, \\ ([1,3p+1] \setminus \{3p\}) \bigcup \{3p, 3p+2\} \bigcup [3p+3, n/d-3], & \text{if } p \equiv 2 \pmod 4, \\ ([1,3p+1] \setminus \{3p\}) \bigcup \{3p, 3p+2\} \bigcup [3p+3, n/d-2], & \text{if } p \equiv 3 \pmod 4, \\ [1,3p] \bigcup [3p+1, n/d-3], & \text{if } p \equiv 0 \pmod 4, \end{cases}$$

$A = \bigcup_{i=0}^{q-1} A_i$, where

$$A_i = \begin{cases} \{(2i+1)\cdot n/d-1, (2i+1)\cdot n/d \\ \quad +2, (2i+2)\cdot n/d-2, (2i+2)\cdot n/d+1\}, & \text{if } p \equiv 1 \text{ or } 3 \pmod 4, \\ \{(2i+1)\cdot n/d-2, (2i+1)\cdot n/d \\ +1, (2i+2)\cdot n/d-1, (2i+2)\cdot n/d+2\}, & \text{if } p \equiv 0 \text{ or } 2 \pmod 4, \end{cases}$$

$B = \bigcup_{i=0}^{q-1} B_i$, where

$$B_i = \begin{cases} \{(2i+1)\cdot n/d, (2i+1)\cdot n/d \\ \quad +1, (2i+2)\cdot n/d-1, (2i+2)\cdot n/d\}, & \text{if } p \equiv 1 \text{ or } 3 \pmod 4, \\ \{(2i+1)\cdot n/d-1, (2i+1)\cdot n/d, \\ \quad (2i+2)\cdot n/d, (2i+2)\cdot n/d+1\}, & \text{if } p \equiv 0 \text{ or } 2 \pmod 4, \end{cases}$$

$D = \bigcup_{i=0}^{q-1} D_i$, where

$$D_i = \begin{cases} [(2i+1)\cdot n/d+3, (2i+2)\cdot n/d-3] \bigcup [(2i+2)\cdot n/d \\ \quad +2, (2i+3)\cdot n/d-2], & \text{if } p \equiv 1 \text{ or } 3 \pmod 4, \\ [(2i+1)\cdot n/d+2, (2i+2)\cdot n/d-2] \bigcup [(2i+2)\cdot n/d \\ \quad +3, (2i+3)\cdot n/d-3], & \text{if } p \equiv 0 \text{ or } 2 \pmod 4, \end{cases}$$

$$F = \begin{cases} [(2q+1)\cdot n/d-1, \frac{n-1}{2}], & \text{if } p \equiv 1 \text{ or } 3 \pmod 4, \text{ and} \\ [(2q+1)\cdot n/d-2, \frac{n-1}{2}], & \text{if } p \equiv 0 \text{ or } 2 \pmod 4. \end{cases}$$

It is easy to see that if $p \equiv 1$ or $3 \pmod 4$, then $A \bigcup B \bigcup D = [n/d-1, (2q+1)n/d-2]$, and if $p \equiv 0$ or $2 \pmod 4$, then $A \bigcup B \bigcup D = [n/d-2, (2q+1)n/d-3]$. Moreover, F is not empty. An easy verification shows that the union of subsets Q, A, B, D, and F forms a partition of $[1, \frac{n-1}{2}]$.

Lemma 4.2. *The interval* $[1, \frac{n-1}{2}]$ *can be partitioned into the union of subsets* $Q, A, B, D,$ *and* F.

In view of the subsets D_i $(0 \le i \le q-1)$ in D, we can partition it into the union of subsets $D_{i,1}$, $D_{i,2}$, and $D_{i,3}$ and set $D_i^* = \bigcup_{i=0}^{q-1} D_{i,3}$ as follows.

If $p \equiv 1$ or $3 \pmod 4$, then

$$\begin{cases} D_{i,1} = [(2i+1)\cdot n/d+3, (2i+1)\cdot n/d+6]; \\ D_{i,2} = [(2i+2)\cdot n/d+2, (2i+2)\cdot n/d+5]; \text{ and} \\ D_{i,3} = [(2i+1)\cdot n/d+7, (2i+2)\cdot n/d-3] \bigcup [(2i+2)\cdot n/d+6, (2i+3)\cdot n/d-2]. \end{cases}$$

If $p \equiv 0$ or 2 (mod 4), then

$$\begin{cases} D_{i,1}=[(2i+1)\cdot n/d+2,(2i+1)\cdot n/d+5]; \\ D_{i,2}=[(2i+2)\cdot n/d+3,(2i+2)\cdot n/d+6]; \text{ and} \\ D_{i,3}=[(2i+1)\cdot n/d+6,(2i+2)\cdot n/d-2]\bigcup[(2i+2)\cdot n/d+7,(2i+3)\cdot n/d-3]. \end{cases}$$

To prove the second main result, we need some auxiliary lemmas. Throughout we will assume d to be an odd prime (≥ 11).

Lemma 4.3. *For each i with $1 \leq i \leq 3$, there exists a cyclic m-cycle system of $X(n, \pm W_i)$ where $W_1 = \{(2q+1)\cdot n/d-1, (2q+1)\cdot n/d+2, (2q+1)\cdot n/d+3\}$, $W_2 = \{3p+1, 3p+3, (2q+1)\cdot n/d-2\}$, and $W_3 = \{3p, 3p+2, (2q+1)\cdot n/d-2\}$.*

Proof. Let C_i $(1 \leq i \leq 3)$ be closed m-trails defined as

$C_1 = [0, (2q+1)\cdot n/d-1, (4q+2)\cdot n/d+2]_{(2q+1)\cdot n/d}$,
$C_2 = [0, (2q+1)\cdot n/d-2, (2q+1)\cdot n/d+3p+1]_{(2q+1)\cdot n/d}$, and
$C_3 = [0, (2q+1)\cdot n/d-2, (2q+1)\cdot n/d+3p]_{(2q+1)\cdot n/d}$.

It can be checked that each C_i $(1 \leq i \leq 3)$ is an m-cycle of index $(2q+1)\cdot n/d$ with $\partial C_i = \pm W_i$. The thesis then follows from Lemma 2.5. ∎

Lemma 4.4. *For each i with $1 \leq i \leq 4$, there exists a cyclic m-cycle system of $X(n, \pm W_i)$ where $W_1 = \{3p+1, 3p+3, (2q+1)\cdot n/d-1, (2q+1)\cdot n/d+2, (2q+1)\cdot n/d+3, (2q+1)\cdot n/d+4\}$, $W_2 = \{3p+1, 3p+3, (2q+1)\cdot n/d-2, (2q+1)\cdot n/d+1, (2q+1)\cdot n/d+2, (2q+1)\cdot n/d+3\}$, $W_3 = \{3p, 3p+2, (2q+1)\cdot n/d-1, (2q+1)\cdot n/d+2, (2q+1)\cdot n/d+3, (2q+1)\cdot n/d+4\}$, and $W_4 = \{3p, 3p+2, (2q+1)\cdot n/d-2, (2q+1)\cdot n/d+1, (2q+1)\cdot n/d+2, (2q+1)\cdot n/d+3\}$.*

Proof. For $1 \leq i \leq 4$, let C_i be the union of closed m-trails $C_{i,1}$, $C_{i,2}$ given by

$C_{1,1} = C_{3,1} = [0, (2q+1)\cdot n/d-1, (4q+2)\cdot n/d+3]_{(2q+1)\cdot n/d}$,
$C_{1,2} = [0, (2q+1)\cdot n/d+2, (2q+1)\cdot n/d+3p+3]_{(2q+1)\cdot n/d}$,
$C_{2,1} = C_{4,1} = [0, (2q+1)\cdot n/d+1, (4q+2)\cdot n/d+3]_{(2q+1)\cdot n/d}$,
$C_{2,2} = [0, (2q+1)\cdot n/d-2, (2q+1)\cdot n/d+3p+1]_{(2q+1)\cdot n/d}$,
$C_{3,2} = [0, (2q+1)\cdot n/d+2, (2q+1)\cdot n/d+3p+2]_{(2q+1)\cdot n/d}$, and
$C_{4,2} = [0, (2q+1)\cdot n/d-2, (2q+1)\cdot n/d+3p]_{(2q+1)\cdot n/d}$.

Similarly, we have the thesis by Lemma 2.5 since $C_{i,1}$, $C_{i,2}$ $(1 \leq i \leq 4)$ are m-cycles of index $(2q+1)\cdot n/d$ and $\partial C_i = \partial(C_{i,1} \bigcup C_{i,2}) = \pm W_i$ for $1 \leq i \leq 4$. ∎

Lemma 4.5. *For each i with $1 \leq i \leq 3$, there exists a cyclic m-cycle system of $X(n, \pm W_i)$ where $W_1 = \{3p+1, 3p+3, (2q+1)\cdot n/d-1, (2q+1)\cdot n/d+2, \cdots, (2q+1)\cdot n/d+7\}$, $W_2 = \{3p, 3p+2, (2q+1)\cdot n/d-1, (2q+1)\cdot n/d+2, \cdots, (2q+1)\cdot n/d+7\}$, and $W_3 = \{(2q+1)\cdot n/d-2, (2q+1)\cdot n/d+1, \cdots, (2q+1)\cdot n/d+8\}$.*

Proof. The thesis follows from Lemma 2.5 by taking $C_i = \bigcup_{j=1}^{3} C_{i,j}$ where each $C_{i,j}$ $(1 \leq i, j \leq 3)$ defined as follows is an m-cycle of index $(2q+1) \cdot n/d$ and $\partial C_i = \pm W_i$ for $1 \leq i \leq 3$.

$C_{1,1} = C_{2,1} = [0, (2q+1) \cdot n/d - 1, (4q+2) \cdot n/d + 5]_{(2q+1) \cdot n/d}$,
$C_{1,2} = [0, (2q+1) \cdot n/d + 2, (2q+1) \cdot n/d + 3p + 3]_{(2q+1) \cdot n/d}$,
$C_{2,2} = [0, (2q+1) \cdot n/d + 2, (2q+1) \cdot n/d + 3p + 2]_{(2q+1) \cdot n/d}$,
$C_{1,3} = C_{2,3} = [0, (2q+1) \cdot n/d + 3, (4q+2) \cdot n/d + 7]_{(2q+1) \cdot n/d}$,
$C_{3,1} = [0, (2q+1) \cdot n/d + 1, (4q+2) \cdot n/d + 4]_{(2q+1) \cdot n/d}$,
$C_{3,2} = [0, (2q+1) \cdot n/d + 2, (4q+2) \cdot n/d + 7]_{(2q+1) \cdot n/d}$, and
$C_{3,3} = [0, (2q+1) \cdot n/d - 2, (4q+2) \cdot n/d + 6]_{(2q+1) \cdot n/d}$. ■

Throughout assume $W = \bigcup_{i=0}^{q-1} (A_i \bigcup D_{i,1} \bigcup D_{i,2})$ and $\epsilon = 0$ or 1 according to whether $p \equiv 1, 3$ or $0, 2 \pmod 4$.

Lemma 4.6. *There exists a cyclic m-cycle system of $X(n, \pm W)$.*

Proof. For $0 \leq i \leq q-1$ and $1 \leq j \leq 4$, let $C_{i,j}$ be an m-cycle of index $(2i+1) \cdot n/d$ or $(2i+2) \cdot n/d$ defined as follows:

If $p \equiv 1$ or $3 \pmod 4$, then set

$C_{i,1} = [0, (2i+1) \cdot n/d - 1, (4i+2) \cdot n/d + 3]_{(2i+1) \cdot n/d}$,
$C_{i,2} = [0, (2i+1) \cdot n/d + 2, (4i+3) \cdot n/d + 4]_{(2i+1) \cdot n/d}$,
$C_{i,3} = [0, (2i+2) \cdot n/d - 2, (4i+4) \cdot n/d + 3]_{(2i+2) \cdot n/d}$, and
$C_{i,4} = [0, (2i+2) \cdot n/d + 1, (4i+3) \cdot n/d + 6]_{(2i+2) \cdot n/d}$.

If $p \equiv 0$ or $2 \pmod 4$, then set

$C_{i,1} = [0, (2i+1) \cdot n/d - 2, (4i+3) \cdot n/d + 3]_{(2i+1) \cdot n/d}$,
$C_{i,2} = [0, (2i+1) \cdot n/d + 1, (4i+2) \cdot n/d + 3]_{(2i+1) \cdot n/d}$,
$C_{i,3} = [0, (2i+2) \cdot n/d - 1, (4i+3) \cdot n/d + 4]_{(2i+2) \cdot n/d}$, and
$C_{i,4} = [0, (2i+2) \cdot n/d + 2, (4i+4) \cdot n/d + 6]_{(2i+2) \cdot n/d}$.

Let $C = \bigcup_{i=0}^{q-1} \bigcup_{j=1}^{4} C_{i,j}$ be the union of m-cycles $C_{i,j}$ $(0 \leq i \leq q-1$ and $1 \leq j \leq 4)$, we then obtain the thesis since in each case $\partial C = \pm W$. ■

Proposition 4.7. *Suppose $m = 3d$ where $d = 6a+5$ for $a \geq 1$ and let n be admissible with $gcd(m, n) = d$. Then there exists a cyclic m-cycle system of K_n.*

Proof. Recall that $m \equiv 3$ (resp. 1) (mod 4) if $a \equiv 0$ (resp. 1) (mod 2). The proof is split into 4 cases according to whether $a \equiv 0, 1, 2$, or $3 \pmod 4$.

Case 1. $a \equiv 0 \pmod 4$.

If $p \equiv 0$ or $1 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \uplus U \uplus W$ where $U = [3p+1, n/d - 2 - \epsilon] \uplus B \uplus D_i^* \uplus F$, and $[1, 3p] \bigcup U = \biguplus_{i=1}^{p} (T_i \uplus S_{1,i})$.

If $p \equiv 2$ or $3 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \uplus U \uplus W$ where $U = \{3p, 3p+2\} \uplus [3p+3, n/d - 2 - \epsilon] \uplus B \uplus D_i^* \uplus F$, and $([1, 3p+1] \setminus \{3p\}) \bigcup U = \biguplus_{i=1}^{p-1} (T_i \uplus S_{1,i}) \uplus (T_p \uplus S_2)$.

By Proposition 2.3, Lemma 4.6, and Theorem 2.6, for each subcase there is a cyclic m-cycle system of K_n.

Case 2. $a \equiv 1 \pmod 4$.

If $p \equiv 1 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \biguplus U \biguplus W \biguplus W^*$ where $W^* = \{(2q+1)\cdot n/d-1, (2q+1)\cdot n/d+2, (2q+1)\cdot n/d+3\}$ and $U = [3p+1, n/d-2] \biguplus B \biguplus D_i^* \biguplus (F \setminus W^*)$; $[1, 3p] \bigcup U = \biguplus_{i=1}^{p-1}(T_i \biguplus S_{3,i}) \biguplus (T_p \biguplus S_4)$.

If $p \equiv 2 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \biguplus U \biguplus W \biguplus W^*$ where $W^* = \{3p, 3p+2, (2q+1)\cdot n/d-2\}$ and $U = [3p+3, n/d-3] \biguplus B \biguplus D_i^* \biguplus (F \setminus W^*)$; $([1, 3p+1] \setminus \{3p\}) \bigcup U = \biguplus_{i=1}^{p}(T_i \biguplus S_{3,i})$.

If $p \equiv 3 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \biguplus U \biguplus W \biguplus W^*$ where $W^* = \{(2q+1)\cdot n/d-1, (2q+1)\cdot n/d+2, (2q+1)\cdot n/d+3\}$ and $U = \{3p, 3p+2\} \biguplus [3p+3, n/d-2] \biguplus B \biguplus D_i^* \biguplus (F \setminus W^*)$; $([1, 3p+1] \setminus \{3p\}) \bigcup U = \biguplus_{i=1}^{p}(T_i \biguplus S_{3,i})$.

If $p \equiv 0 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \biguplus U \biguplus W \biguplus W^*$ where $W^* = \{3p+1, 3p+3, (2q+1)\cdot n/d-2\}$ and $U = \{3p+2, 3p+4\} \biguplus [3p+5, n/d-3] \biguplus B \biguplus D_i^* \biguplus (F \setminus W^*)$; $[1, 3p] \bigcup U = \biguplus_{i=1}^{p-1}(T_i \biguplus S_{3,i}) \biguplus (T_p \biguplus S_4)$.

By utilizing Proposition 2.4, Lemmas 4.3, 4.6, and Theorem 2.6, a cyclic m-cycle system of K_n exists.

Case 3. $a \equiv 2 \pmod 4$.

If $p \equiv 0$ or $1 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \biguplus U \biguplus W \biguplus W^*$ where $W^* = \{3p+1, 3p+3, (2q+1)\cdot n/d-1-\epsilon, (2q+1)\cdot n/d+2-\epsilon, (2q+1)\cdot n/d+3-\epsilon, (2q+1)\cdot n/d+4-\epsilon\}$ and $U = \{3p+2, 3p+4\} \biguplus [3p+5, n/d-2-\epsilon] \biguplus B \biguplus D_i^* \biguplus (F \setminus W^*)$; $[1, 3p] \bigcup U = \biguplus_{i=1}^{p-1}(T_i \biguplus S_{1,i}) \biguplus (T_p \biguplus S_2)$.

If $p \equiv 2$ or $3 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \biguplus U \biguplus W \biguplus W^*$ where $W^* = \{3p, 3p+2, (2q+1)\cdot n/d-1-\epsilon, (2q+1)\cdot n/d+2-\epsilon, (2q+1)\cdot n/d+3-\epsilon, (2q+1)\cdot n/d+4-\epsilon\}$ and $U = [3p+3, n/d-2-\epsilon] \biguplus B \biguplus D_i^* \biguplus (F \setminus W^*)$; $([1, 3p+1] \setminus \{3p\}) \bigcup U = \biguplus_{i=1}^{p}(T_i \biguplus S_{1,i})$.

By virtue of Proposition 2.3, Lemmas 4.4, 4.6, and Theorem 2.6, there is a cyclic m-cycle system of K_n.

Case 4. $a \equiv 3 \pmod 4$.

If $p \equiv 1 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \biguplus U \biguplus W \biguplus W^*$ where $W^* = \{3p+1, 3p+3\} \biguplus \{(2q+1)\cdot n/d-1, (2q+1)\cdot n/d+2, \cdots, (2q+1)\cdot n/d+7\}$ and $U = \{3p+2, 3p+4\} \biguplus [3p+5, n/d-2] \biguplus B \biguplus D_i^* \biguplus (F \setminus W^*)$.

If $p \equiv 0 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \biguplus U \biguplus W \biguplus W^*$ where $W^* = \{(2q+1)\cdot n/d-2, (2q+1)\cdot n/d+1, \cdots, (2q+1)\cdot n/d+8\}$ and $U = [3p+1, n/d-3] \biguplus B \biguplus D_i^* \biguplus (F \setminus W^*)$.

Then for each subcase, $[1, 3p] \bigcup U = \biguplus_{i=1}^{p}(T_i \biguplus S_{3,i})$.

If $p \equiv 2 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \biguplus U \biguplus W \biguplus W^*$ where $W^* = \{(2q+1)\cdot n/d-2, (2q+1)\cdot n/d+1, \cdots, (2q+1)\cdot n/d+8\}$ and $U = \{3p, 3p+2\} \biguplus [3p+3, n/d-3] \biguplus B \biguplus D_i^* \biguplus (F \setminus W^*)$.

If $p \equiv 3 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \biguplus U \biguplus W \biguplus W^*$ where

$W^* = \{3p, 3p+2\} \uplus \{(2q+1)\cdot n/d - 1, (2q+1)\cdot n/d + 2, \cdots, (2q+1)\cdot n/d + 7\}$ and $U = [3p+3, n/d-2] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

Also, for each subcase, $([1, 3p+1] \setminus \{3p\}) \bigcup U = \biguplus_{i=1}^{p-1}(T_i \uplus S_{3,i}) \uplus (T_p \uplus S_4)$.

According to Proposition 2.4, Lemmas 4.5, 4.6, and Theorem 2.6, it follows that for each subcase, there is a cyclic m-cycle system of K_n. ■

Lemma 4.8. *Suppose* $m = 3d$ *where* $d = 6a + 7$ *and* $n = 42a + 49, a \geq 1$. *Then there exists a cyclic* m*-cycle system of* K_n.

Proof. Note that if $a \equiv 1$ (resp. 0) (mod 2), then $m \equiv 3$ (resp. 1) (mod 4) and $b = a + 1$. Let $C_{1,i}, C_{2,i}, C_3$ be closed m-trails defined as

$$C_{1,i} = [0, 17 + 14i, 39 + 28i]_{14+14i},$$
$$C_{2,i} = [0, 19 + 14i, 37 + 28i]_{21+14i}, \text{ and}$$
$$C_3 = [0, \tfrac{n-5}{2}, \tfrac{n+3}{2}]_{\frac{n-7}{2}}.$$

It can be checked that both $C_{1,i}$ and $C_{2,i}$ $(0 \leq i \leq \lfloor \frac{b}{2} \rfloor - 1)$ are m-cycles of index $14 + 14i$ or $21 + 14i$, respectively, and C_3 is an m-cycle of index $\frac{n-7}{2}$. Moreover, $\partial C_{1,i} = \pm W_{1,i}$ where $W_{1,i} = \{17 + 14i, 22 + 14i, 25 + 14i\}$, $\partial C_{2,i} = \pm W_{2,i}$ where $W_{2,i} = \{16 + 14i, 18 + 14i, 19 + 14i\}$ and $\partial C_3 = \pm W_3$ where $W_3 = \{4, 5, \frac{n-5}{2}\}$.

Now, set $U = [4, \frac{n-1}{2}] \setminus Y$ where $Y = \bigcup_{i=0}^{\lfloor \frac{b}{2} \rfloor - 1}(W_{1,i} \uplus W_{2,i})$ if $a \equiv 1$ (mod 2) and $Y = \bigcup_{i=0}^{\lfloor \frac{b}{2} \rfloor - 1}(W_{1,i} \uplus W_{2,i}) \uplus W_3$ if $a \equiv 0$ (mod 2). A routine verification shows that $[1, 3] \bigcup U = T_1 \uplus S_1$ if $a \equiv 1$ (mod 2), and $[1, 3] \bigcup U = T_1 \uplus S_4$ if $a \equiv 0$ (mod 2).

The thesis follows by Propositions 2.3, 2.4, Lemma 2.5, and Theorem 2.6. ■

Proposition 4.9. *Suppose* $m = 3d$ *where* $d = 6a + 7$ *for* $a \geq 1$ *and let* n *be admissible with* $gcd(m, n) = d$ *and* $n > 2m$. *Then there exists a cyclic* m*-cycle system of* K_n.

Proof. Recall that $n = 2pm + ds$, so, by the hypothesis on d, we have $n = (6a + 7)(6p + 1)$. If $p = 1$, i.e., $n = 42a + 49$, the proof is done by Lemma 4.8, so it is enough to consider the cases where $p > 1$. The proof is divided into 2 cases according to whether $a \equiv 0$ or 1 (mod 2). The proof here is similar to those in Proposition 4.7, and to simplify, we just provide the construction methods and leave the details to the reader.

Case 1. $a \equiv 0$ (mod 2).

Then $b = 4q + 1$ or $4q + 3$.

Subcase 1.1 $b = 4q + 1$.

If $p \equiv 1$ (mod 4), then $[1, \frac{n-1}{2}] = [1, 3p] \uplus U \uplus W \uplus W^*$ where $W^* = \{(2q + 1)\cdot n/d - 1, (2q+1)\cdot n/d + 2, (2q+1)\cdot n/d + 3\}$ and $U = [3p+1, n/d-2] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

If $p \equiv 0 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \uplus U \uplus W \uplus W^*$ where $W^* = \{3p+1, 3p+3, (2q+1) \cdot n/d - 2\}$ and $U = \{3p+2, 3p+4\} \uplus [3p+5, n/d-3] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

If $p \equiv 3 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \uplus U \uplus W \uplus W^*$ where $W^* = \{(2q+1) \cdot n/d - 1, (2q+1) \cdot n/d + 2, (2q+1) \cdot n/d + 3\}$ and $U = \{3p, 3p+2\} \uplus [3p+3, n/d-2] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

If $p \equiv 2 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \uplus U \uplus W \uplus W^*$ where $W^* = \{3p, 3p+2, (2q+1) \cdot n/d - 2\}$ and $U = [3p+3, n/d-3] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

Subcase 1.2 $b = 4q + 3$.

If $p \equiv 1 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \uplus U \uplus W \uplus W^*$ where $W^* = \{3p+1, 3p+3, (2q+1) \cdot n/d - 1, (2q+1) \cdot n/d + 2, \cdots, (2q+1) \cdot n/d + 7\}$ and $U = \{3p+2, 3p+4\} \uplus [3p+5, n/d-2] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

If $p \equiv 0 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \uplus U \uplus W \uplus W^*$ where $W^* = \{(2q+1) \cdot n/d - 2, (2q+1) \cdot n/d + 1, \cdots, (2q+1) \cdot n/d + 8\}$ and $U = [3p+1, n/d-3] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

If $p \equiv 3 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \uplus U \uplus W \uplus W^*$ where $W^* = \{3p, 3p+2, (2q+1) \cdot n/d - 1, (2q+1) \cdot n/d + 2, \cdots, (2q+1) \cdot n/d + 7\}$ and $U = [3p+3, n/d-2] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

If $p \equiv 2 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \uplus U \uplus W \uplus W^*$ where $W^* = \{(2q+1) \cdot n/d - 2, (2q+1) \cdot n/d + 1, \cdots, (2q+1) \cdot n/d + 8\}$ and $U = \{3p, 3p+2\} \uplus [3p+3, n/d-3] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

Case 2. $a \equiv 1 \pmod 2$.

Then $b = 4q$ or $4q + 2$.

Subcase 2.1 $b = 4q$.

If $p \equiv 0$ or $1 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \uplus U \uplus W$ where $U = [3p+1, n/d - 2 - \epsilon] \uplus B \uplus D_i^* \uplus F$.

If $p \equiv 2$ or $3 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \uplus U \uplus W$ where $U = \{3p, 3p+2\} \uplus [3p+3, n/d-2-\epsilon] \uplus B \uplus D_i^* \uplus F$.

Subcase 2.2 $b = 4q + 2$.

If $p \equiv 0$ or $1 \pmod 4$, then $[1, \frac{n-1}{2}] = [1, 3p] \uplus U \uplus W \uplus W^*$ where $W^* = \{3p+1, 3p+3, (2q+1) \cdot n/d - 1 - \epsilon, (2q+1) \cdot n/d + 2 - \epsilon, (2q+1) \cdot n/d + 3 - \epsilon, (2q+1) \cdot n/d + 4 - \epsilon\}$ and $U = \{3p+2, 3p+4\} \uplus [3p+5, n/d-2-\epsilon] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$.

If $p \equiv 2$ or $3 \pmod 4$, then $[1, \frac{n-1}{2}] = ([1, 3p+1] \setminus \{3p\}) \uplus U \uplus W \uplus W^*$ where $W^* = \{3p, 3p+2, (2q+1) \cdot n/d - 1 - \epsilon, (2q+1) \cdot n/d + 2 - \epsilon, (2q+1) \cdot n/d + 3 - \epsilon, (2q+1) \cdot n/d + 4 - \epsilon\}$ and $U = [3p+3, n/d-2-\epsilon] \uplus B \uplus D_i^* \uplus (F \setminus W^*)$. ∎

Combining Propositions 4.7 and 4.9, we obtain the second main result.

Theorem 4.10. *Suppose $m = 3d$ with d a prime and let n be admissible with $gcd(m, n) = d$ and $n > 2m$. Then there exists a cyclic m-cycle system of K_n.*

Example 2. There is a cyclic 111-cycle system of K_{925}. Taking m = 111 with d = 37 and e = 3, by Lemma 4.1, we have that s = 1, b = 6, and $n = 222p + 37$, and letting p = 4, it follows that n = 925, n/d = 25, and $\frac{n-1}{2}$ = 462. Note that in this situation, $q = \epsilon$ = 1.

Then $[1, 462] = [1, 12] \biguplus U \biguplus W \biguplus W^*$ where $W = A_0 \biguplus D_{0,1} \biguplus D_{0,2} = \{23, 26, 49, 52\} \biguplus [27, 30] \biguplus [53, 56], W^* = \{13, 15, 73, 76, 77, 78\}$, and $U = \{14, 16\} \biguplus [17, 22] \biguplus B \biguplus D_0^* \biguplus (F \setminus W^*)$ where $B = \{24, 25, 50, 51\}$, $D_0^* = [31, 48] \biguplus [57, 72]$, and $F \setminus W^* = [74, 75] \biguplus [79, 462]$.

Since $[1, 12] \bigcup U = \bigcup_{i=1}^{3}(T_i \biguplus S_{1,i}) \biguplus (T_4 \biguplus S_2)$, by Proposition 2.3, a cyclic 111-cycle system of $X(925, \pm([1, 12] \biguplus U))$ exists, and by virtue of Lemmas 4.4 and 4.6, we obtain cyclic 111-cycle systems of $X(925, \pm W^*)$ and $X(925, \pm W)$.

According to Theorem 2.6, a cyclic 111-cycle system of K_{925} does exist.

Now, by utilizing Theorems 3.4 and 4.10, the thesis of Theorem 1.6 follows.

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