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THE INDEPENDENCE NUMBER OF CONNECTED (claw, K_4)-FREE 4-REGULAR GRAPHS

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Abstract. An *independent set* of a graph G is a subset of the vertices of G such that no two vertices in the subset are joined by an edge in G. The *independence number* of G is the cardinality of a maximum independent set of G, and is denoted by $\alpha(G)$. In this paper we show that every 2-connected (claw, K_4)-free 4-regular graph G on n vertices has independence number exactly $\lfloor n/3 \rfloor$.

1. INTRODUCTION

All graphs considered here are finite, simple and nonempty. For standard terminology not given here we refer the reader to [2]. Let $G = (V, E)$ be a graph with *vertex set* V and *edge set* E. For a vertex $v \in V$, the *open neighborhood* $N(v)$ of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u \mid uv \in E\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of v is equal to $|N(v)|$, denoted by $d_G(v)$ or simply $d(v)$. The maximum and minimum degrees of G will be denoted by $\Delta(G)$ and $\delta(G)$, respectively. If $d_G(v) = k$ for all $v \in V$, then we call G *k-regular*. In particular, a 3-regular graph is also called a *cubic* graph. For a subset $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A *cut vertex* of G is a vertex v such that $c(G - v) > c(G)$. where $c(G)$ is the number of components of G. A *cut edge* can similarly defined. The *line graph* $L(G)$ of G is the graph on E in which $x, y \in E$ are adjacent as vertices if and only if they are adjacent as edges in G. As usual, K_n denotes the complete graph on n vertices, and P_n denotes the path on n vertices,. The graph $K_{1,3}$ is also called a *claw* and K_3 a *triangle*. For a given graph F, we say that a graph G is F-free if it does not contain F as an induced subgraph. In particular, $K_{1,3}$-free is called *claw-free*. For a family of graphs $(F_1, \ldots, F_k)$, we say that G is $(F_1, \ldots, F_k)$-free if it is F_i-free for all $i = 1, \ldots, k$. Two distinct edges in a

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graph G are independent if they are not adjacent in G. A set of pairwise independent edges in G is called a *matching* of G. The *matching number* of G, denoted by $\alpha'(G)$, is the largest cardinality among all matchings of G.

An *independent set* I of G is a subset of the vertices of G such that no two vertices of I are joined by an edge in G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G. The *independence ratio* of G, denoted by $i(G)$, is $\alpha(G)/n$, where G has n vertices. Independent sets in graphs is now well studied in graph theory.

For a connected graph G on n vertices with m edges, Harant and Schiermeyer [11] proved $\alpha(G) \geq [(2m+n+1) - \sqrt{(2m+n+1)^2 - 4n^2}]/2$ and discussed its algorithmic realization. Li and Virlouvet [16] showed that for every claw-free graph G on n vertices, $\alpha(G) \leq 2n/(\Delta(G)+2)$. In [5] this result on claw-free graphs was extended to $K_{1,r+1}$-free graphs. Ryjácek and Schiermeyer [20] used the degree sequence, order, size and vertex connectivity of a $K_{1,r+1}$-free graph or of an almost claw-free graph to obtain several upper bounds on its independence number.

Brooks [3] proved that every connected graph G which is neither a complete graph nor odd cycle must be $\Delta(G)$-colorable. Thus, such a graph must have $i(G) \geq 1/\Delta(G)$. Albertson, Bollobás and Tucker [1] proved that $i(G) \geq 1/k$ for a K_k-free graph G with $\Delta(G) = k = 3$ or $\Delta(G) = k \geq 6$. Fajtlowics [4] proved that $i(G) \geq 2/(\Delta(G)+k)$ for a K_k-free graph G. In 1979, Staton [21] proved that every triangle-free graph G with maximum degree k has $i(G) \geq 5/(5k-1)$. In particular, Fraughnaugh [6] and Heckman and Thomas [13] provided shorter proofs of this result for the case when G is a triangle-free graph with maximum degree three. Heckman [12] discussed the tightness of the $5/14$ independence ratio of the triangle-free graphs with maximum degree at most three. Harant et al. [10] proved that every K_4-free graph G on n vertices, size m and maximum degree at most three has $\alpha(G) \geq (4n - m - \lambda - tr)/7$, where λ counts the number of components of G whose blocks are each either isomorphic to one of four specific graphs or edges between two of these four specific graphs and tr is the maximum number of vertex-disjoint triangles in G. This result generalizes the bound due to Heckman and Thomas [13]. Fraughnaugh and Locke [8] proved that every connected triangle-free 3-regular graph G on n vertices has $\alpha(G) \geq 11n/30 - 2/15$; and Heckman and Thomas [14] proved that every triangle-free planar graph on n vertices with maximum degree three has $\alpha(G) \geq 3n/8$. Fraughnaugh [7] proved that for every triangle-free 4-regular graph G on n vertices, $\alpha(G) \geq 4n/13$. Kreher and Radziszowski [15] further extended this result to triangle-free graphs with average degree 4. Fraughnaugh and Locke [9] found a shorter proof of the result.

In 1997, Locke and Lou [17] gave a lower bound on the independence number of a connected K_4-free 4-regular graph.

Theorem 1. ([17]). *If G is a connected K_4-free 4-regular graph on n vertices, then $\alpha(G) \geq (7n-4)/26$.*

In this paper we continue to investigate the independence number in K_4-free 4-regular graphs. We shall show that every 2-connected (claw, K_4)-free 4-regular graph has independence number exactly $\lfloor n/3 \rfloor$, where G has n vertices.

2. Main Results

Let us introduce some more notation and terminology. If the graphs G and G' are disjoint, we denote by $G * G'$ the graph obtained from $G \cup G'$ by joining all the vertices of G to all the vertices of G'. The graph $C_n * K_1$ is called an *n-wheel* and the graph $C_n * \overline{K_2}$ $(n \geq 4)$ a *double wheel*, where $\overline{K_2}$ is the *complement* of K_2.

The well-known Petersen Theorem will be useful.

Lemma 1. ([19]). *Every cubic graph without cut edges has a perfect matching.*

Let $\mathcal{G}$ denote the class of 2-connected (claw, K_4)-free 4-regular graphs. To obtain our main result, we first give a lower bound on the independence number for graphs in $\mathcal{G}$.

Theorem 2. *For $G \in \mathcal{G}$ and $|V(G)| = n$, $\alpha(G) \geq (n-2)/3$.*

Proof. We may assume that G is 2-connected. Since G is a K_4-free 4-regular graph, we have $n \geq 6$. We prove by induction on n. For $n = 6$, it is easy to see that G is the double wheel $C_4 * \overline{K}_2$. Clearly $\alpha(G) = 2 \geq (n-2)/3$, and the assertion holds. Now let G be given with $n > 6$, and assume the assertion holds for graphs with fewer vertices.

For each $v \in V(G)$, by the claw-freeness and K_4-freeness of G, we see that the induced subgraph $G[N(v)]$ is triangle-free and has $\alpha(G[N(v)]) = 2$. Hence $G[N(v)]$ is isomorphic to one of the three graphs $K_2 \cup K_2$, P_4 and C_4. We distinguish the following three cases.

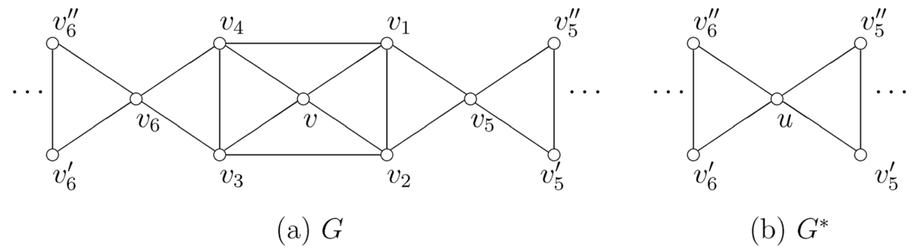


Fig. 1. Case 1.1.

Case 1. There exists a vertex $v \in V(G)$ such that $G[N(v)]$ is isomorphic to C_4.

In this case, clearly $G[N[v]]$ is a 4-wheel. Let $C_4 = v_1v_2v_3v_4v_1$ be the cycle induced by $N(v)$ in G. We consider the fourth neighbor, say v_5, of v_1. Note that

$G \neq C_4 * \overline{K}_2$ as $n > 6$. This implies that v_5 is adjacent to exactly one of v_2 and v_4 by the claw-freeness of G. Without loss of generality, assume $v_5v_2 \in E(G)$. Then $v_5v_4 \notin E(G)$. Now let v_6 be the fourth neighbor of v_4. Similarly, we have $v_6v_3 \in E(G)$. Further, let $v_5', v_5'' \in N(v_5) \setminus \{v_1, v_2\}$ and $v_6', v_6'' \in N(v_6) \setminus \{v_3, v_4\}$. Then $v_5'v_5'' \in E(G)$ and $v_6'v_6'' \in E(G)$ by the claw-freeness of G.

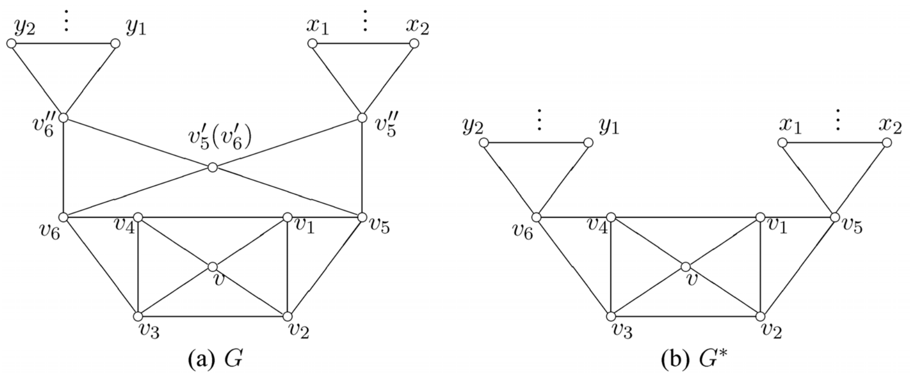


Fig. 2. Case 1.2.

Case 1.1. $N(v_5) \cap N(v_6) = \emptyset$ (see Fig. 1 (a)).

Let G^* be the graph obtained from G by deleting the vertices $v, v_1, v_2, v_3, v_4, v_5, v_6$ and adding one new vertex u and new edges $uv_5',\ uv_5'',\ uv_6',\ uv_6''$ (see Fig. 1 (b)). Since G is 2-connected, both v_5 and v_6 are not cut-vertices of G, so u is not a cut-vertex of G^*. Hence $G^* \in \mathcal{G}$. Let $|V(G^*)| = n^*$. Then $n^* = n - 6$. By applying the induction hypothesis to G^*, we have $\alpha(G^*) \geq (n^* - 2)/3$. Let I^* be a maximum independent set of G^*. If $u \notin I^*$, then let $I = I^* \cup \{v_1,\ v_3\}$ or $I^* \cup \{v_2,\ v_4\}$. Otherwise, let $I = (I^* - \{u\}) \cup \{v,\ v_5,\ v_6\}$. It is easy to see that I is an independent set of G. So

$$\alpha(G) \geq \alpha(G^*) + 2 \geq \frac{n^* - 2}{3} + 2 = \frac{n - 2}{3},$$

and the desired result follows.

Case 1.2. $N(v_5) \cap N(v_6) \neq \emptyset$ (see Fig. 2 (a)).

Let $v_5' = v_6' \in N(v_5) \cap N(v_6)$. We claim that $v_5'' \neq v_6''$. Otherwise, it would produce a claw centered at v_5' or v_5''. Furthermore, suppose $v_5''v_6'' \in E(G)$. Then the fourth neighbor, say v_7, of v_5'' must be adjacent to v_6''. This implies that v_7 is a cut-vertex of G, which contradicts that G is 2-connected. So $v_5''v_6'' \notin E(G)$. Let $x_1,\ x_2$ and $y_2,\ y_2$ be the other two neighbors of v_5'' and v_6'', respectively. By the claw-freeness of G, we have $x_1x_2, y_1y_2 \in E(G)$ and $|N(v_5'') \cap N(v_6'')| \leq 2$. Hence $|\{x_1, x_2\} \cap \{y_1, y_2\}| \leq 1$.

Now let G^* be the graph obtained from G by deleting the vertices $v_5'(v_6'), v_5'', v_6''$ and adding the edges v_5x_1, v_5x_2, v_6y_1 and v_6y_2 (see Fig. 2 (b)). Clearly, $G^* \in \mathcal{G}$. Let

$|V(G^*)| = n^*$. Then $n^* = n - 3$. By the induction hypothesis, we have $\alpha(G^*) \geq (n^* - 2)/3$.

Let I^* be a maximum independent set of G^* and let $B = \{v, v_1, v_2, v_3, v_4, v_5, v_6\}$. We construct an independent set of G as follows.

(1) If $v_5, v_6 \in I^*$, then $v \in I^*$ and $|I^* \cap B| = 3$. Let $I = \{v_5'', v_6'', v_1, v_3\} \cup (I^* - (I^* \cap B))$.

(2) If $v_5 \in I^*$, $v_6 \notin I^*$, then $|I^* \cap B| = 2$. Let $I = \{v_5'', v_1, v_3\} \cup (I^* - (I^* \cap B))$.

(3) If $v_5 \notin I^*$, $v_6 \in I^*$, then $|I^* \cap B| = 2$. Let $I = \{v_6'', v_1, v_3\} \cup (I^* - (I^* \cap B))$.

(4) If $v_5, v_6 \notin I^*$, then $|I^* \cap B| = 2$. Let $I = \{v, v_5, v_6\} \cup (I^* - (I^* \cap B))$.

In all cases, it is easy to check that I is an independent set of G. So

$$\alpha(G) \geq \alpha(G^*) + 1 \geq \frac{n^* - 2}{3} + 1 = \frac{n - 2}{3},$$

and the assertion holds.

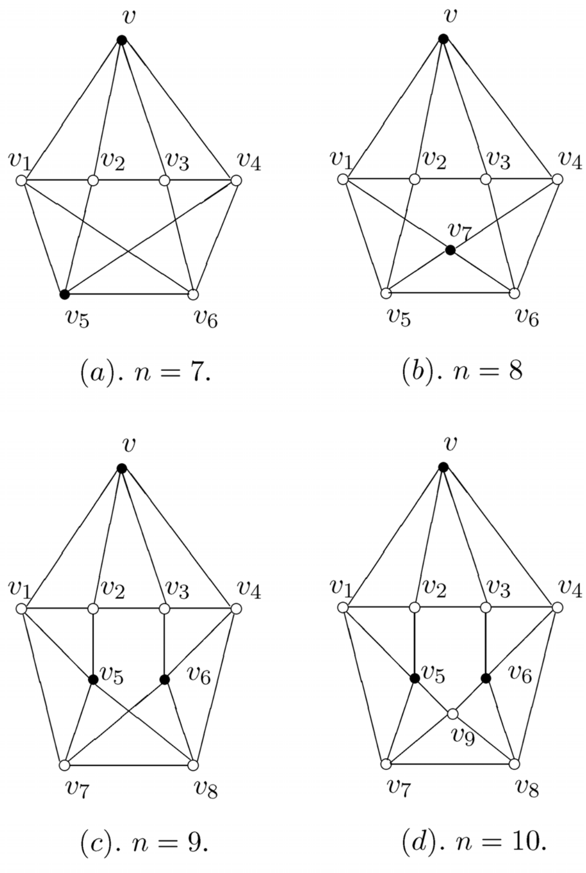


Fig. 3. $n = 7, 8, 9, 10$.

In what follows we may assume that

$(*1)$ there is no vertex $v \in V(G)$ such that $G[N(v)]$ is isomorphic to C_4, i.e., $G[N[v]]$ is not a 4-wheel.

Case 2. There exists a vertex $v \in V(G)$ such that $G[N(v)]$ is isomorphic to P_4.

Let $N(v) = \{v_1,\ v_2,\ v_3,\ v_4\}$ and let $P_4 = v_1v_2v_3v_4$ be the path induced by $N(v)$. We consider the fourth neighbor, say v_5, of v_2. Then, by the claw-freeness of G and $(*1)$, v_5 is adjacent to exactly one of v_1 and v_3. We consider the following two subcases depending on $v_1v_5 \in E(G)$ or $v_3v_5 \in E(G)$.

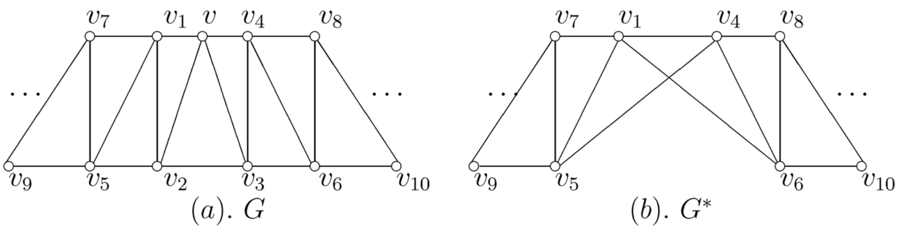


Fig. 4. Case 2.1.

Case 2.1. $v_1v_5 \in E(G)$.

Then $v_3v_5 \notin E(G)$. By the claw-freeness, the fourth neighbor, say v_6, of v_3 must be adjacent to v_4, and the fourth neighbor, say v_7, of v_1 must be adjacent to v_5. Suppose $v_7 = v_6$. Then $v_4v_5 \in E(G)$ for otherwise a claw would occur centered at v_4. This means that G is the graph of order 7 shown in Fig. 3 (a) that satisfies the conditions of theorem. It is easy to check that $\alpha(G) = 2 \geq (n-2)/3$. So we may assume $v_7 \neq v_6$. Similarly, the fourth neighbor, say v_8, of v_4 must be adjacent to v_6. Suppose $v_8 = v_7$. Then $v_5v_6 \in E(G)$ for otherwise a claw would occur centered at v_5. This means that G is the graph of order 8 shown in Fig. 3. (b) that satisfies the conditions of theorem. It is not difficult to check that $\alpha(G) = 2 \geq (n-2)/3$. So we may assume $v_8 \neq v_7$. Note that the fourth neighbor, say v_9, of v_5 is adjacent to v_7, for otherwise it would create a claw centered at v_5. Suppose $v_9 = v_8$. To avoid a claw centered at v_6 or v_7, it must be the case that $v_7v_6 \in E(G)$. So G is the graph of order 9 shown in Fig. 3 (c). It is easy to check $\alpha(G) = 3 \geq (n-2)/3$. So we may assume $v_9 \neq v_8$. Note that the fourth neighbor, say v_{10}, of v_6 must be adjacent to v_8. Suppose $v_{10} = v_9$. Then $v_7v_8 \in E(G)$. So G is the graph of order 10 shown in Fig. 3. (d). It is easy to check that $\alpha(G) = 3 \geq (n-2)/3$. So we may assume $v_{10} \neq v_9$ (see Fig. 4 (a)).

Now let G^* be the graph obtained from G by deleting $v,\ v_2,\ v_3$ and adding edges $v_1v_4,\ v_1v_6,\ v_4v_5$ (see Fig. 4 (b)). Clearly, $G^* \in \mathcal{G}$ and $|V(G^*)| = n^* = n-3$. By the induction hypothesis, we have $\alpha(G^*) \geq (n^*-2)/3$. Let I^* be a maximum independent set of G^*. Note that $|I^* \cap \{v_1,\ v_4,\ v_5\}| \leq 1$; we construct an independent set of G as follows.

(1) If $v_1 \in I^*$, then $v_4,\ v_6 \notin I^*$ and let $I = I^* \cup \{v_3\}$.

(2) If $v_4 \in I^*$, then then $v_1,\ v_5 \notin I^*$ and let $I = I^* \cup \{v_2\}$.
(3) If $v_1,\ v_4 \notin I^*$, then let $I = I^* \cup \{v\}$.
In all cases, clearly I is an independent set of G. So

$$\alpha(G) \geq \alpha(G^*) + 1 \geq \frac{n^* - 2}{3} + 1 = \frac{n-2}{3},$$

and the assertion follows.

Case 2.2. $v_3v_5 \in E(G)$.
Then $v_1v_5 \notin E(G)$. By $(*1)$, we have $G[N(v_3)]$ is not isomorphic to C_4, so $v_4v_5 \notin E(G)$.

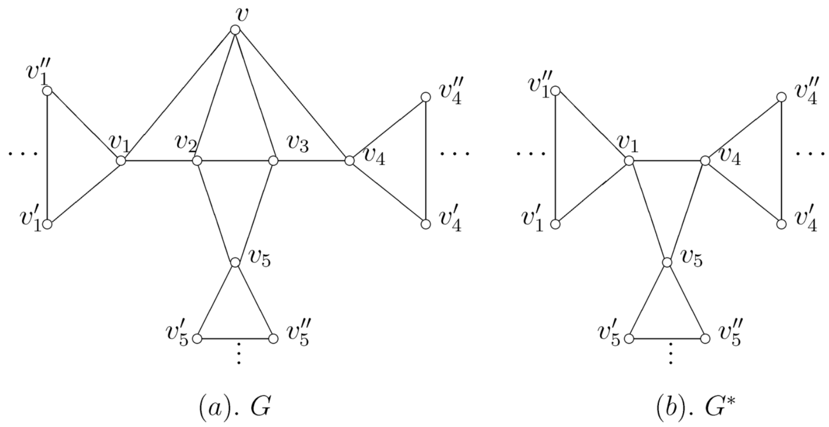


Fig. 5. Case 2.2.1.

Case 2.2.1. Suppose that v_1, v_4, v_5 have no common neighbors other than v, v_2, v_3 (see Fig. 5(a)).

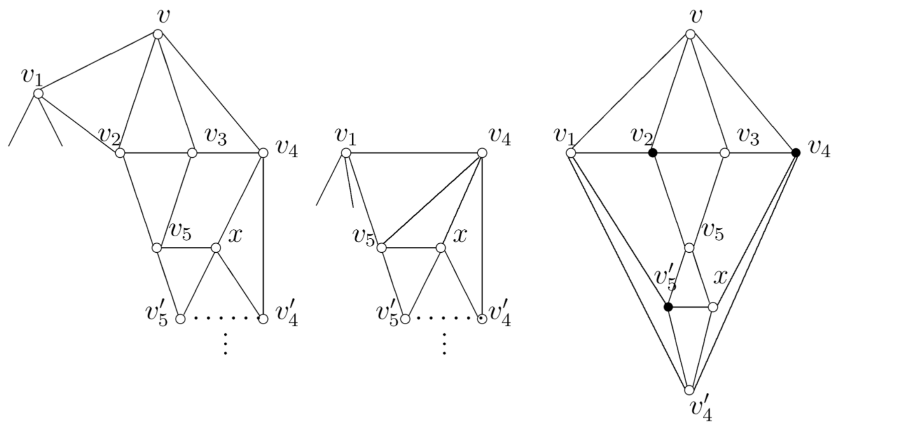
Let $v_i',\ v_i''$ be the other two neighbors of v_i. Clearly, v_i' and v_i'' must be adjacent by claw-freeness, for $i = 1, 4, 5$. To complete our inductive proof, let G^* be the graph obtained from G by deleting the vertices v, v_2, v_3 and adding edges $v_1v_4,\ v_1v_5,\ v_4v_5$ (see Fig. 5(b)). Clearly, $G^* \in \mathcal{G}$ and $|V(G^*)| = n^* = n - 3$. Applying the induction hypothesis to G^*, we have $\alpha(G^*) \geq (n^* - 2)/3$. Let I^* be a maximum independent set of G^*. Note that $|I^* \cap \{v_1,\ v_4,\ v_5\}| \leq 1$. We construct an independent set of G as follows.

(1) If $v_1 \in I^*$, then $v_4,\ v_5 \notin I^*$ and let $I = I^* \cup \{v_3\}$.
(2) If $v_4 \in I^*$, then $v_1,\ v_5 \notin I^*$ and let $I = I^* \cup \{v_2\}$.
(3) If $v_5 \in I^*$, then $v_1,\ v_4 \notin I^*$ and let $I = I^* \cup \{v\}$.
(4) If $v_1,\ v_4,\ v_5 \notin I^*$, then let $I = I^* \cup \{v\}$.
Clearly I is an independent set of G. So

$$\alpha(G) \geq \alpha(G^*) + 1 \geq \frac{n^* - 2}{3} + 1 = \frac{n-2}{3},$$

and the assertion follows.

Case 2.2.2. By symmetry, we may assume that $N(v_4) \cap N(v_5) \setminus \{v_3\} \neq \emptyset$.



(a) G $\quad$ (b) G^* $\quad$ (c) $v_4'v_5' \in E(G)$ and $v_1v_4' \in E(G)$

Fig. 6. Case 2.2.2.

Let $x \in N(v_4) \cap N(v_5) \setminus \{v_3\}$. By the claw-freeness of G, $v_1x \notin E(G)$. We claim that $N(v_4) \cap N(v_5) = \{v_3, x\}$. Indeed, if there exists $y \in N(v_4) \cap N(v_5) \setminus \{v_3, x\}$, then $xy \in E(G)$ by the claw-freeness. Let $z \in N(x)$ be the fourth neighbor of x except for v_4, v_5 and y. Recall that $v_4v_5 \notin E(G)$. Hence $G[\{v_4, v_5, x, z\}]$ is a claw centered at x, a contradiction. The fourth neighbor of v_4, v_5 is denoted by v_4', v_5', respectively. Then $v_4'x$, $v_5'x \in E(G)$ by the claw-freeness.

Suppose $v_4'v_5' \notin E(G)$. Then v_1 is adjacent to at most one of v_4', v_5' by the claw-freeness. In fact, regardless of whether v_1 is adjacent to v_4' or v_5', let G^* be the graph obtained from G by deleting the vertices v, v_2, v_3 and adding edges $v_1v_4,\ v_1v_5,\ v_4v_5$ (see, Fig. 6 (b)). Clearly, $G^* \in \mathcal{G}$ and $|V(G^*)| = n^* = n - 3$. The remaining proof is the same as that of Case 2.2.1.

On the other hand, suppose $v_4'v_5' \in E(G)$. If $v_1v_4' \in E(G)$, then, since G is claw-free, we have $v_1v_5' \in E(G)$. Similarly, if $v_1v_5' \in E(G)$, we have $v_1v_4' \in E(G)$. Thus G is the graph of order 9 shown in Fig. 6 (c). It is easy to check that $\alpha(G) = 3 \geq (n-2)/3$. Hence, we may suppose $v_1v_4', v_1v_5' \notin E(G)$. Now we construct the graph G^* described as in Case 2.2.1, the remaining proof is the same as that of Case 2.2.1.

In the following, we therefore may assume that

$(*2)$ there is no vertex $v \in V(G)$ such that $G[N(v)]$ is isomorphic to P_4. By $(*1)$ and $(*\ 2)$, we consider the following final case.

Case 3. For any $v \in V(G)$, $G[N(v)]$ is isomorphic to $K_2 \cup K_2$.

Then, for every vertex $v \in V(G)$, $G[N[v]]$ consists of two edge-disjoint triangles with only v in common. This implies that every edge of G exactly lies in one triangle. Let H be the graph whose vertices are the triangles of G, such that two vertices of H are adjacent if and only if the corresponding triangles of G intersect (at a vertex). Clearly, H is a 3-regular graph. For the graph H, we have

Claim 1. H is 2-connected.

Suppose not, then there exists a vertex x which is a cut-vertex of H. For x, the corresponding triangle of G is denoted by A_x. Thus G is disconnected by deleting A_x in G. This implies that there exists a vertex v in A_x such that v is a cut-vertex of G, which contradicts that G is 2-connected. ∎

By Claim 1 and Lemma 1, H has a perfect matching. Let M be a perfect matching of H. Then $|M| = |V(H)|/2$. Note that $|V(H)| = 2n/3$. Hence $|M| = n/3$. Let $I = \{x \in V(G) \mid x$ is the only common vertex of two triangles in G corresponding to u and v of H, for all $uv \in M\}$. Clearly, I is a independent set of vertices of G. So $\alpha(G) \geq |I| = |M| = n/3 \geq (n-2)/3$.

This completes the proof of Theorem 2. ∎

Li and Virlouvet [16] proved the following result involving the independence number of a claw-free graph.

Lemma 2. ([16]). *For any claw-free graph G on n vertices, $\Delta(G) \leq 2(n - \alpha(G))/\alpha(G)$.*

By Lemma 2, we know that $\alpha(G) \leq n/3$ for a claw-free 4-regular graph G on n vertices. By Theorem 2, we immediately obtain our main result.

Theorem 3. *If $G \in \mathcal{G}$ and $|V(G)| = n$, then $\alpha(G) = \lfloor n/3 \rfloor$.*

3. Concluding Remarks

In this paper we determine the exact value of the independence number $\alpha(G)$ for (claw, K_4)-free 4-regular graphs without cut vertices. For (claw, K_4)-free 4-regular graphs with cut vertices, we propose the following conjecture.

Conjecture 1. *If G is a connected (claw, K_4)-free 4-regular graph on n vertices, then $\alpha(G) \geq (8n-3)/27$.*

By using the following known result, it is easy to show that the conjecture is true for the line graph of a cubic graph.

Lemma 3. ([18]) *If G is a connected cubic graph on n vertices, then $\alpha'(G) \geq (4n-1)/9$, and this is sharp infinitely often.*

Theorem 4. *If G is a connected cubic graph on n vertices, then $\alpha(L(G)) \geq (8|E(G)|-3)/27$, and this is sharp infinitely often.*

By Theorem 4, if the Conjecture 1 is true, then the lower bound is sharp. This also means that the condition “without cut vertices” in Theorem 2 and Theorem 3 is necessary.

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