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Research Article

New Global Synchronization Analysis for Complex Networks with Coupling Delay

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Global synchronization analysis for complex networks with coupling delay is investigated. Firstly the constant time delay is analyzed and then the case for time-varying delay is considered. Sufficient conditions for network synchronization are given based on Lyapunov functional, linear matrix inequality, and Kronecker product technique. The unknown variables in the sufficient conditions are fewer than those in the recent reference. Moreover, for the time-varying delay case, we find that the conditions are dependent on the bounds of both time delay and its derivative, and the derivative of the time-varying delay can be any value in the bounds. Finally, numerical examples are given to validate the effectiveness of the obtained results.

1. Introduction

Dynamical recurrent neural networks are used extensively in classification of patterns, associative memories, optimization [1–3], and so on. The networks are composed of a large number of highly interconnected dynamical units and exhibit very complicated dynamics. Therefore, theory analysis of complex networks has become a focal research field and attracted a great deal of attention.

Recently, it is found that synchronization is one of the most important dynamical properties of complex networks and has been extensively investigated in different ways [4–19]. In [5], Lü and Chen considered a dynamical network and gave the sufficient conditions for local synchronization. Because of the network traffic congestions as well as the finite speed of signal transmission over the links, time delays occur commonly in complex networks. Therefore, Zhou and Chen [7] and Gao et al. [8] improved the models with no delays, the case with constant coupling delays is considered, and they also analyzed the synchronization problem [9, 10]. However, in many real-world networks time delay is varying; therefore time varying coupling delays are considered in [11–13]. By using free-weighting matrices, some synchronization criteria for general complex dynamical networks with time-varying delays are proposed. However, the computation is huge, and there are a large amount of variables in the condition. Moreover, in order to derive the synchronization condition, in some papers [7–13], the time-varying delay $\tau(t)$ is usually confined to $0 \le \tau(t) \le \tau$ (lower bound of the delay is zero), and the derivative is restricted to less than 1. Therefore, how to improve the system performance by removing the redundant variables and reducing computation still remains unsolved.

Motivated by the mentioned work, we study the synchronization problems for general complex networks with time constant coupling delays and interval time-varying delays. Using different Lyapunov functions, the synchronization conditions derived turn out to be less conservative, and the addressed systems contain some models as their special cases; the more effective mathematical techniques are employed to reduce the conservatism.

The remainder of this paper is organized as follows. In Section 2, the investigated systems are formulated and some lemmas and notations are given. In Section 3, the conditions for synchronization are derived. In Section 4, two numerical examples are presented to demonstrate the effectiveness and the advantage of the proposed method. Finally, conclusions are drawn in Section 5.

2. Problem Formulations

In this paper, $R^{m\times n}$ denotes the set of $m \times n$ real matrixes, and $X \geq 0$ denotes that matrix X is positive semidefinite, while $X > 0$ denotes that matrix X is positive definite. We use diag$\{\cdots\}$ to denote a block-diagonal matrix; $\left[\begin{smallmatrix} X & Y \\ * & Z \end{smallmatrix}\right]$ stands for $\left[\begin{smallmatrix} X & Y \\ Y^T & Z \end{smallmatrix}\right]$. The notation $A \otimes B$ denotes the Kronecker product of matrices A and B; I_n represents the n-dimensional identity matrix. Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations.

We use $x_i(t)$ to denote the state of coupling nodes, $i \in \{1,\dots,N\}$, and then the dynamic neural networks (DNNs) of general form can be described by

$$\begin{aligned}
\dot{x}_i(t) = & -Cx_i(t) + Df(x_i(t)) + Bf(x_i(t-\tau)) + I(t) \\
& + c\sum_{j=1,j\neq i}^{N} G_{ij}A\left[x_j(t-\tau) - x_i(t-\tau)\right] \\
& + c\sum_{j=1,j\neq i}^{N} G_{ij}A_\tau\left[x_j(t-\tau) - x_i(t)\right], \\
\dot{x}_i(t) = & -Cx_i(t) + Df(x_i(t)) + Bf(x_i(t-\tau(t))) + I(t) \\
& + c\sum_{j=1,j\neq i}^{N} G_{ij}A\left[x_j(t-\tau(t)) - x_i(t-\tau(t))\right] \\
& + c\sum_{j=1,j\neq i}^{N} G_{ij}A_\tau\left[x_j(t-\tau(t)) - x_i(t)\right],
\end{aligned} \tag{1}$$

in which $x_i(t) = [x_{i1}(t), x_{i2}(t), \dots, x_{in}(t)]^T \in R^n$ is the state vector of the ith network at time t. The functions $f(x_i(t)) = [f_1(x_{i1}(t)), f_2(x_{i2}(t)), \dots, f_n(x_{in}(t))]^T$ are sufficiently smooth nonlinear vector fields, and $I(t) = [I_1(t), I_2(t), \dots, I_n(t)]^T \in R^n$ is the external input vector. $C = [c_{ij}]_{n\times n}$, $D = [d_{ij}]_{n\times n}$, and $B = [b_{ij}]_{n\times n}$ are coefficient matrixes, $A = [a_{ij}]_{n\times n}$ and $A_\tau = [a'_{ij}]_{n\times n}$ are inner-coupling matrixes, and $G = [G_{ij}]_{N\times N}$ represents the outer-coupling connections. The constant $c > 0$ represents the coupling strength; τ and $\tau(t)$ represent the time constant delay and time-varying delay, respectively.

For the networks (1), we have the following assumptions.

Assumption (H1). $\tau(t)$ is the interval time-varying delay satisfying

$$\begin{aligned}
& 0 \leq h_1 \leq \tau(t) \leq h_2, \\
& \dot{\tau}(t) \leq \mu < +\infty, \\
& h_{21} = h_2 - h_1.
\end{aligned} \tag{2}$$

Assumption (H2). The outer-coupling configuration matrices of the network satisfy

$$\begin{aligned}
& G_{ij} \geq 0, \quad i \neq j, \\
& G_{ii} = -\sum_{j=1,j\neq i}^{N} G_{ij}, \quad i,j = 1,2,\dots,N.
\end{aligned} \tag{3}$$

Assumption (H3). There exist constants σ_i^-, σ_i^+, and the functions satisfy

$$\sigma_i^- \leq \frac{f_i(x_1) - f_i(x_2)}{x_1 - x_2} \leq \sigma_i^+, \quad i = 1,2,\dots,n. \tag{4}$$

We denote

$$\begin{aligned}
& \Lambda_1 = \operatorname{diag}\left(\sigma_1^+\sigma_1^-, \sigma_2^+\sigma_2^-, \dots, \sigma_n^+\sigma_n^-\right), \\
& \Lambda_2 = \operatorname{diag}\left(\frac{\sigma_1^+ + \sigma_1^-}{2}, \frac{\sigma_2^+ + \sigma_2^-}{2}, \dots, \frac{\sigma_n^+ + \sigma_n^-}{2}\right).
\end{aligned} \tag{5}$$

Based on Assumption (H2), system (1) can be rewritten as the following form:

$$\begin{aligned}
\dot{x}_i(t) = & -Cx_i(t) + Df(x_i(t)) + Bf(x_i(t-\tau)) + I(t) \\
& + c\sum_{j=1}^{N} G_{ij}Ax_j(t-\tau) \\
& + c\sum_{j=1}^{N} G_{ij}A_\tau x_j(t-\tau) - cG_{ii}A_\tau\left[x_i(t-\tau) - x_i(t)\right],
\end{aligned} \tag{6}$$

$$\begin{aligned}
\dot{x}_i(t) = & -Cx_i(t) + Df(x_i(t)) + Bf(x_i(t-\tau(t))) + I(t) \\
& + c\sum_{j=1}^{N} G_{ij}Ax_j(t-\tau(t)) \\
& + c\sum_{j=1}^{N} G_{ij}A_\tau x_j(t-\tau(t)) \\
& - cG_{ii}A_\tau\left[x_i(t-\tau(t)) - x_i(t)\right].
\end{aligned} \tag{7}$$

The initial conditions of (6) are given by $x_i(s) = \phi_i(s)$. $s \in [t_0 - \tau, t_0]$, $i = 1,2,\dots,N$, where $\phi_i(\cdot) = [\phi_{i1}(\cdot), \phi_{i2}(\cdot), \dots, \phi_{in}(\cdot)]^T \in C([t_0 - \tau, t_0], R^n)$. The initial conditions of (7) are given by $x_i(s) = \phi_i(s)$. $s \in [t_0 - h_2, t_0]$, $i = 1,2,\dots,N$, where $\phi_i(\cdot) = [\phi_{i1}(\cdot), \phi_{i2}(\cdot), \dots, \phi_{in}(\cdot)]^T \in C([t_0 - h_2, t_0], R^n)$, and in order to simplify we set $G_{ii} = -l$.

Remark 1. The constants σ_i^-, σ_i^+ in Assumption (H3) are allowed to be any value. Then, most of the previous results in similar networks are just special cases of this assumption, which means that the activation functions are more general than those of other works.

Considering the sign of Kronecker product, models (6) and (7) can be rewritten as

$$\begin{aligned}\dot{x}(t) = &-(I_N \otimes C)\,x(t) + (I_N \otimes D)\,F(x(t)) + (I_N \otimes B) \\ &\times F(x(t-\tau)) + I'(t) + c\,(G \otimes (A + A_\tau))\,x(t-\tau) \\ &+ cl\,(I_N \otimes A_\tau)\,[x(t-\tau) - x(t)],\end{aligned} \tag{8}$$

$$\begin{aligned}\dot{x}(t) = &-(I_N \otimes C)\,x(t) + (I_N \otimes D)\,F(x(t)) \\ &+ (I_N \otimes B)\,F(x(t-\tau(t))) \\ &+ I'(t) + c\,(G \otimes (A + A_\tau))\,x(t-\tau(t)) \\ &+ cl\,(I_N \otimes A_\tau)\,[x(t-\tau(t)) - x(t)],\end{aligned} \tag{9}$$

with $x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T$, $F(x(t)) = [f^T(x_1(t)), f^T(x_2(t)), \ldots, f^T(x_N(t))]^T$, and $I'(t) = [\underbrace{I^T(t), I^T(t), \ldots, I^T(t)}_{N}]^T$.

Definition 2 (see [14]). Dynamical networks (6) are said to be global asymptotic synchronization, if for any ε, any $\varphi_i(s), \varphi_j(s) \in C([t_0 - \tau, t_0], R^n)$ $(i, j = 1, 2, \ldots, N)$. There exists $T_1 > t_0$ such that $\|x_i(t) - x_j(t)\| \le \varepsilon$, where $t > T_1$, $i, j = 1, 2, \ldots, N$, and $\|\cdot\|$ denotes the Euclidean norm.

Definition 3 (see [15]). Dynamical networks (7) are said to be global exponential synchronization, if for any initial conditions $\varphi_i(s), \varphi_j(s) \in C([t_0 - h_2, t_0], R^n)$ $(i, j = 1, 2, \ldots, N)$, there exist $T_1 > t_0$ and $\varepsilon > 0$ such that $\|x_i(t) - x_j(t)\| \le Me^{-\varepsilon t}$, in which $t > T_1$ and $\|\cdot\|$ denotes the Euclidean norm.

Lemma 4. *Let $\otimes$ denote the notation of Kronecker product; then one has the following conclusions:*

(1) $(\alpha A) \otimes B = A \otimes (\alpha B)$

(2) $(A + B) \otimes C = A \otimes C + B \otimes C$

(3) $(A \otimes B)(C \otimes D) = (AC) \otimes BD$

Lemma 5 (see [16]). *Let $e = (1, 1, \ldots, 1)^T$, $E_N = ee^T$, $U = NI_N - E_N$, $K \in R^{n\times n}$, $x = [x_1^T, x_2^T, \ldots, x_N^T]^T$, $y = [y_1^T, y_2^T, \ldots, y_N^T]^T$ with $x_k, y_k \in R^n$ $(k = 1, 2, \ldots, N)$, and then $x^T(U \otimes K)y = \sum_{1\le i<j\le N}^{N}(x_i - x_j)^T K(y_i - y_j)$.*

Lemma 6 (see [17]). *For any constant matrix $X \in R^{n\times n}$, $X = X^T \ge 0$, a scalar functional $h := h(t) \ge 0$, and a vector function $\dot{x} : [-h, 0] \to R^n$, the following inequality holds $-h\int_{t-h}^{t} \dot{x}(s)^T \times \dot{x}(s)ds \le -[x(t) - x(t-h)]^T \times [x(t) - x(t-h)]$.*

3. Main Results

3.1. Synchronization Condition for Constant Time Delay. In this section, we state and investigate the global asymptotic synchronization for system (6).

Theorem 7. *Suppose Assumptions (H1)–(H3) hold, if there exist three symmetric positive definite matrices P, Q, W and three positive diagonal matrices H, K, V, such that the LMIs hold for each i, j $(i < j)$ in $(1, 2, \ldots, N)$. Consider*

$$\Pi_{ij} = \begin{bmatrix} \alpha & -cNG_{ij}P\,(A + A_\tau) + clPA_\tau + W & -C^TK^T - clA_\tau^TK^T & PD + H\Lambda_2 & PB \\ * & -Q - W - V\Lambda_1 & \beta & 0 & V\Lambda_2 \\ * & * & \tau^2W - K - K^T & KD & KB \\ * & * & * & -H & 0 \\ * & * & * & * & -V \end{bmatrix} < 0, \tag{10}$$

where $\alpha = -PC - C^TP + Q - clPA_\tau - clA_\tau^TP - W - H\Lambda_1$, $\beta = -cNG_{ij}(A + A_\tau)^TK^T + clA_\tau^TK^T$.

Then system (8) is global asymptotic synchronization.

Proof. Choose a Lyapunov-Krasovskii functional as

$$\begin{aligned}V(x(t)) = &\,x^T(t)\,(U \otimes P)\,x(t) + \int_{t-\tau}^{t} x^T(s)\,(U \otimes Q)\,x(s)\,ds \\ &+ \int_{-\tau}^{0}\int_{t+\theta}^{t} \tau\dot{x}(s)^T\,(U \otimes W)\,\dot{x}(s)\,ds\,d\theta.\end{aligned} \tag{11}$$

Now by directly computing $\dot{V}(x(t))$ along the trajectory of system (8), we have

$$\begin{aligned}\dot{V}(x(t)) = &\,2x^T(t)\,(U \otimes P)\,\dot{x}(t) + x^T(t)\,(U \otimes Q)\,x(t) \\ &- x^T(t-\tau)\,(U \otimes Q)\,x(t-\tau) \\ &+ \tau^2\dot{x}^T(t)\,(U \otimes W) \\ &\times \dot{x}(t) - \int_{t-\tau}^{t} \tau\dot{x}(s)^T\,(U \otimes W)\,\dot{x}(s)\,ds.\end{aligned} \tag{12}$$

In view of Lemma 6, we have

$$\begin{aligned}\dot{V}(x(t)) \le &\,2x^T(t)\,(U \otimes P)\,\dot{x}(t) + x^T(t)\,(U \otimes Q)\,x(t) \\ &- x^T(t-\tau)\,(U \otimes Q)\,x(t-\tau) \\ &+ \tau^2\dot{x}^T(t)\,(U \otimes W)\,\dot{x}(t) \\ &- [x(t) - x(t-\tau)]^T\,(U \otimes W)\,[x(t) - x(t-\tau)].\end{aligned} \tag{13}$$

Noting the facts that $(U \otimes P)I(t) = 0$ and $UG = NG$, using Lemmas 4 and 5, we have

$$\begin{aligned}
&\dot{V}(x(t)) \\
&\le 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(x_i(t)-x_j(t)\right)^T\left[\left(-PC - clPA_\tau + \frac{1}{2}Q\right)\right. \\
&\qquad \times\left(x_i(t)-x_j(t)\right) \\
&\qquad + PD\left(f\left(x_i(t)\right)-f\left(x_j(t)\right)\right) \\
&\qquad + PB\left(f\left(x_i(t-\tau)\right)\right. \\
&\qquad\qquad \left.-f\left(x_j(t-\tau)\right)\right) \\
&\qquad - cNG_{ij}P\left(A + A_\tau\right) \\
&\qquad \times\left(x_i(t-\tau)-x_j(t-\tau)\right) \\
&\qquad + clPA_\tau \\
&\qquad \left.\times\left(x_i(t-\tau)-x_j(t-\tau)\right)\right] \\
&\quad -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(x_i(t-\tau)-x_j(t-\tau)\right)^T \\
&\qquad \times Q\left(x_i(t-\tau)-x_j(t-\tau)\right) \\
&\quad +\tau^2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\dot{x}_i(t)-\dot{x}_j(t)\right)^T W \\
&\qquad \times\left(\dot{x}_i(t)-\dot{x}_j(t)\right) \\
&\quad -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[\left(x_i(t)-x_j(t)\right)^T W\left(x_i(t)-x_j(t)\right)\right. \\
&\qquad -2\left(x_i(t)-x_j(t)\right)^T W \\
&\qquad \times\left(x_i(t-\tau)-x_j(t-\tau)\right) \\
&\qquad +\left(x_i(t-\tau)-x_j(t-\tau)\right)^T W \\
&\qquad \left.\times\left(x_i(t-\tau)-x_j(t-\tau)\right)\right].
\end{aligned} \tag{14}$$

On the other hand, it is easy to see from the formulation of (8) that the following equation also holds for any matrices $K \in R^{n\times n}$:

$$\begin{aligned}
2\dot{x}^T(t)(U\otimes K)\big(&-\dot{x}(t)-\left(I_N\otimes C\right)x(t) \\
&+\left(I_N\otimes D\right)F(x(t)) \\
&+\left(I_N\otimes B\right)F(x(t-\tau))+I'(t) \\
&+c\left(G\otimes\left(A+A_\tau\right)\right)x(t-\tau)\big) \\
+cl&\left(I_N\otimes A_\tau\right)(x(t-\tau)-x(t))=0.
\end{aligned} \tag{15}$$

For convenience, let $x_{ij}(t) = x_i(t) - x_j(t)$, $\dot{x}_{ij}(t) = \dot{x}_i(t) - \dot{x}_j(t)$, $f(x_{ij}(t)) = f(x_i(t)) - f(x_j(t))$, and $f(x_{ij}(t-\tau)) = f(x_i(t-\tau)) - f(x_j(t-\tau))$.

For any $n \times n$ diagonal matrices $H > 0$, $V > 0$, and Λ_1, Λ_2 from Assumption (H3), we have

$$\begin{aligned}
\sum_{1\le i<j\le N}\Big\{&-\Big[x_{ij}^T(t)H\Lambda_1x_{ij}(t)-2x_{ij}^T(t)H\Lambda_2f\left(x_{ij}(t)\right) \\
&\qquad+f^T\left(x_{ij}(t)\right)Hf\left(x_{ij}(t)\right)\Big] \\
&-\Big[x_{ij}^T(t-\tau)V\Lambda_1x_{ij}(t-\tau) \\
&\qquad-2x_{ij}^T(t-\tau)V\Lambda_2f\left(x_{ij}(t-\tau)\right) \\
&\qquad+f^T\left(x_{ij}(t-\tau)\right)Vf\left(x_{ij}(t-\tau)\right)\Big]\Big\}\ge 0.
\end{aligned} \tag{16}$$

Adding up (14)–(16) from both sides, we have

$$\begin{aligned}
&\dot{V}(t) \\
&\le\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Big\{2x_{ij}(t)^T\left(-PC+\frac{1}{2}Q-clPA_\tau\right)x_{ij}(t) \\
&\qquad+2x_{ij}(t)^TPDf\left(x_{ij}(t)\right) \\
&\qquad+2x_{ij}(t)^TPBf\left(x_{ij}(t-\tau)\right) \\
&\qquad+2x_{ij}(t)^T\left(-cNG_{ij}P\left(A+A_\tau\right)\right) \\
&\qquad\times x_{ij}(t-\tau)+2x_{ij}(t)^T\left(clPA_\tau\right)x_{ij}(t-\tau)\Big\} \\
&\quad-\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}x_{ij}(t-\tau)^TQx_{ij}(t-\tau) \\
&\quad+\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\tau^2\dot{x}_{ij}(t)^TW\dot{x}_{ij}(t) \\
&\quad-\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Big\{x_{ij}(t)^TWx_{ij}(t)-2x_{ij}(t)^TWx_{ij}(t-\tau) \\
&\qquad+x_{ij}^T(t-\tau)Wx_{ij}(t-\tau)\Big\} \\
&\quad-\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Big\{2\dot{x}_{ij}^T(t)K\dot{x}_{ij}(t)+2\dot{x}_{ij}^T(t)\left(KC+clKA_\tau\right) \\
&\qquad\times x_{ij}(t)-2\dot{x}_{ij}^T(t)KDf\left(x_{ij}(t)\right) \\
&\qquad-2\dot{x}_{ij}^T(t)KBf\left(x_{ij}(t-\tau)\right) \\
&\qquad+2\dot{x}_{ij}^T(t)cNG_{ij}K\left(A+A_\tau\right)x_{ij}(t-\tau) \\
&\qquad-2\dot{x}_{ij}^T(t)\left(clKA_\tau\right)x_{ij}(t-\tau)\Big\}
\end{aligned}$$

$$-\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left\{x_{ij}^{T}(t)\,H\Lambda_{1}x_{ij}(t)\right.$$
$$-2x_{ij}^{T}(t)\,H\Lambda_{2}f\left(x_{ij}(t)\right)$$
$$+f^{T}\left(x_{ij}(t)\right)Hf\left(x_{ij}(t)\right)$$
$$+x_{ij}^{T}(t-\tau)\,V\Lambda_{1}x_{ij}(t-\tau)$$
$$-2x_{ij}^{T}(t-\tau)\,V\Lambda_{2}f\left(x_{ij}(t-\tau)\right)$$
$$\left.+f\left(x_{ij}^{T}(t-\tau)\right)Vf\left(x_{ij}(t-\tau)\right)\right\}$$
$$=\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\eta_{ij}^{T}(t)\,\Pi_{ij}\eta_{ij}(t), \tag{17}$$

where Π_{ij} is from (10), and

$$\eta_{ij}^{T}(t)=\left[x_{ij}^{T}(t)\;\; x_{ij}^{T}(t-\tau)\;\; \dot{x}_{ij}^{T}(t)\;\; f\left(x_{ij}^{T}(t)\right)\;\; f\left(x_{ij}^{T}(t-\tau)\right)\right]. \tag{18}$$

Following Theorem 7, we have $V(t)\le V(0)$, and then we have $x_i(t)-x_j(t)\to 0$ for all $1\le i<j\le N$. Therefore, system (8) is global asymptotic synchronization. Then end the proof. □

Corollary 8. *Suppose Assumptions (H1)–(H3) hold and $B=0$ in the system (8), if there exist three symmetric positive definite matrices P, Q, W and three positive diagonal matrices H, K, V, such that the LMIs holds for each i, j $(i<j)$ in $(1,2,\ldots,N)$. Consider*

$$\Pi_{ij}=\begin{bmatrix}\alpha & -cNG_{ij}P\left(A+A_{\tau}\right)+clPA_{\tau}+W & -C^{T}K^{T}-clA_{\tau}^{T}K^{T} & PD+H\Lambda_{2} & 0\\ * & -Q-W-V\Lambda_{1} & \beta & 0 & V\Lambda_{2}\\ * & * & \tau^{2}W-K-K^{T} & KD & 0\\ * & * & * & -H & 0\\ * & * & * & * & -V\end{bmatrix}<0, \tag{19}$$

where $\alpha=-PC-C^{T}P+Q-clPA_{\tau}-clA_{\tau}^{T}P-W-H\Lambda_{1}$, $\beta=-cNG_{ij}(A+A_{\tau})^{T}K^{T}+clA_{\tau}^{T}K^{T}$.

Then system (8) is global asymptotic synchronization.

The proof is obvious from Theorem 7 and we omit the details.

3.2. Time-Varying Delay Synchronization Condition. In this section, by utilizing the improved techniques used in [18], we obtain the following global exponential synchronization criterion for system (7).

Theorem 9. *Suppose Assumptions (H1)–(H3) hold, if there exist five symmetric positive definite matrices P, Q_i, R_i, Z_1, Z_2 for each i in $(1,2,3)$ and four positive diagonal matrices H, K, W, V, such that the LMIs holds for each i, j $(i<j)$ in $(1,2,\ldots,N)$. Consider*

$$\Phi_{ij}^{1}=\Phi_{ij}-\left[0_{n,3n}\;\; -I_{n}\;\; I_{n}\;\; 0_{n,3n}\right]^{T}\times Z_{2}\left[0_{n,3n}\;\; -I_{n}\;\; I_{n}\;\; 0_{n,3n}\right]<0, \tag{20}$$

$$\Phi_{ij}^{2}=\Phi_{ij}-\left[0_{n,3n}\;\; I_{n}\;\; 0_{n}\;\; -I_{n}\;\; 0_{n,2n}\right]^{T}\times Z_{2}\left[0_{n,3n}\;\; I_{n}\;\; 0_{n}\;\; -I_{n}\;\; 0_{n,2n}\right]<0, \tag{21}$$

where

$$\Phi_{ij}=\begin{bmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & Z_{1} & 0 & 0 & 0\\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & 0 & 0 & 0 & 0\\ * & * & \Omega_{33} & \Omega_{34} & 0 & 0 & 0 & 0\\ * & * & * & \Omega_{44} & Z_{2} & Z_{2} & 0 & 0\\ * & * & * & * & \Omega_{55} & 0 & W\Lambda_{2} & 0\\ * & * & * & * & * & -Q_{2}-Z_{2}-V\Lambda_{1} & 0 & V\Lambda_{2}\\ * & * & * & * & * & * & -R_{1}-W & 0\\ * & * & * & * & * & * & * & -R_{2}-V\end{bmatrix},$$

$$\Omega_{11}=-PC-C^{T}P+\sum_{i=1}^{3}Q_{i}-clPA_{\tau}-clA_{\tau}^{T}P-Z_{1}$$
$$+C^{T}\left(h_{1}^{2}Z_{1}+h_{12}^{2}Z_{2}\right)\left(C+clA_{\tau}\right)+c^{2}l^{2}A_{\tau}^{T}\left(h_{1}^{2}Z_{1}+h_{12}^{2}Z_{2}\right)A_{\tau}$$
$$+clC^{T}\left(h_{1}^{2}Z_{1}+h_{12}^{2}Z_{2}\right)A_{\tau}-H\Lambda_{1},$$

$$\Omega_{12} = -C^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) D + PD - clA_\tau^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) D + H\Lambda_2,$$
$$\Omega_{13} = -C^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) B - clA_\tau^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) B + PB,$$
$$\begin{aligned}\Omega_{14} = {} & C^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right)\left(cNG_{ij}\left(A + A_\tau\right) - clA_\tau\right) \\ & - cNG_{ij}P\left(A + A_\tau\right) + clPA_\tau - c^2 lNG_{ij}A_\tau^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right) \\ & \times \left(A_\tau + A\right) - c^2 l^2 A_\tau^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) A_\tau,\end{aligned}$$
$$\Omega_{22} = D^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) D + \sum_{i=1}^{3} R_i - H,$$
$$\Omega_{23} = D^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) B,$$
$$\begin{aligned}\Omega_{24} = {} & D^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right)\left(-cNG_{ij}\left(A + A_\tau\right) + clA_\tau\right) \\ & + clD^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) A_\tau,\end{aligned}$$
$$\Omega_{33} = B^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) B - (1-\mu) R_3 - K,$$
$$\begin{aligned}\Omega_{34} = {} & -cNG_{ij}B^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right)\left(A + A_\tau\right) + K\Lambda_2 \\ & + clB^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) A_\tau,\end{aligned}$$
$$\begin{aligned}\Omega_{44} = {} & c^2 NG_{ij}^2 \left(A + A_\tau\right)^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right)\left(A + A_\tau\right) - (1-\mu) Q_3 - 2Z_2 - K\Lambda_1 \\ & - c^2 lNG_{ij}\left(A + A_\tau\right)^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) A_\tau + c^2 l^2 A_\tau^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) A_\tau,\end{aligned}$$
$$\Omega_{55} = -Z_1 - Z_2 - Q_1 - W\Lambda_1. \tag{22}$$

Then system (9) is global exponential synchronization.

Proof. We construct the Lyapunov-Krasovskii functional as follows:

$$\begin{aligned} V(x(t)) = {} & x^T(t)(U \otimes P)x(t) \\ & + \int_{t-\tau(t)}^{t} x^T(s)\left(U \otimes Q_3\right)x(s)\,ds \\ & + \sum_{i=1}^{2}\int_{t-h_i}^{t} x^T(s)\left(U \otimes Q_i\right)x(s)\,ds \\ & + \int_{t-\tau(t)}^{t} f^T(x(s))\left(U \otimes R_3\right)f(x(s))\,ds \\ & + \sum_{i=1}^{2}\int_{t-h_i}^{t} f^T(x(s))\left(U \otimes R_i\right)f(x(s))\,ds \\ & + \int_{-h_1}^{0}\int_{t+\theta}^{t} h_1 \dot{x}^T(s)\left(U \otimes Z_1\right)\dot{x}(s)\,ds\,d\theta \\ & + \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} h_{12}\dot{x}^T(s)\left(U \otimes Z_2\right)\dot{x}(s)\,ds\,d\theta. \end{aligned} \tag{23}$$

Calculating the time derivative of $V(x(t))$ along the trajectories of system (9), from Lemmas 4 and 5, we have

$$\begin{aligned} \dot{V}(x(t)) = {} & 2x^T(t)(U \otimes P)\dot{x}(t) + \sum_{i=1}^{3} x^T(t)\left(U \otimes Q_i\right)x(t) \\ & - \sum_{i=1}^{2} x^T\left(t-h_i\right)\left(U \otimes Q_i\right)x\left(t-h_i\right) \\ & - (1-\mu)x^T(t-\tau(t))\left(U \otimes Q_3\right) \\ & \times x(t-\tau(t)) + \sum_{i=1}^{3} f^T(x(t))\left(U \otimes R_i\right)f(x(t)) \\ & - \sum_{i=1}^{2} f^T\left(x\left(t-h_i\right)\right)\left(U \otimes R_i\right)f\left(x\left(t-h_i\right)\right) \\ & - (1-\mu)f^T(x(t-\tau(t)))\left(U \otimes R_3\right) \\ & \times f(x(t-\tau(t))) \\ & + \dot{x}^T(t)\left[h_1^2\left(U \otimes Z_1\right) + h_{12}^2\left(U \otimes Z_2\right)\right]\dot{x}(t) \end{aligned}$$

$$
\begin{aligned}
&-\int_{t-h_1}^{t} h_1 \dot{x}^T(s)\left(U\otimes Z_1\right)\dot{x}(s)\,ds\\
&-\int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds.
\end{aligned}
\tag{24}
$$

Using Lemma 4, we have

$$
\begin{aligned}
&-\int_{t-h_1}^{t} h_1\dot{x}^T(s)\left(U\otimes Z_1\right)\dot{x}(s)\,ds\\
&\quad\leq -\left(x(t)-x\left(t-h_1\right)\right)^T\left(U\otimes Z_1\right)\left(x(t)-x\left(t-h_1\right)\right).
\end{aligned}
\tag{25}
$$

On the other hand, based on the approach in [18], and using $h_{12}=[h_2-\tau(t)]+[\tau(t)-h_1]$, then we have

$$
\begin{aligned}
&-\int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&\quad= -\int_{t-h_2}^{t-\tau(t)} h_{12}\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&-\int_{t-\tau(t)}^{t-h_1} h_{12}\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&\quad= -\int_{t-h_2}^{t-\tau(t)}\left(h_2-\tau(t)\right)\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&-\int_{t-h_2}^{t-\tau(t)}\left(\tau(t)-h_1\right)\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&-\int_{t-\tau(t)}^{t-h_1}\left(\tau(t)-h_1\right)\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&-\int_{t-\tau(t)}^{t-h_1}\left(h_2-\tau(t)\right)\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds.
\end{aligned}
\tag{26}
$$

And in order to combine some of the terms in (26), letting $\omega=(\tau(t)-h_1)/h_2$, we have

$$
\begin{aligned}
&-\int_{t-h_2}^{t-\tau(t)}\left(\tau(t)-h_1\right)\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&\quad= -\omega\int_{t-h_2}^{t-\tau(t)} h_{12}\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&\quad\leq -\omega\int_{t-h_2}^{t-\tau(t)}\left(h_2-\tau(t)\right)\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&\qquad -\int_{t-\tau(t)}^{t-h_1} h_{12}\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&\quad= -(1-\omega)\int_{t-\tau(t)}^{t-h_1} h_{12}\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds\\
&\quad\leq -(1-\omega)\int_{t-\tau(t)}^{t-h_1}\left(\tau(t)-h_1\right)\dot{x}^T(s)\left(U\otimes Z_2\right)\dot{x}(s)\,ds.
\end{aligned}
\tag{27}
$$

For any $n\times n$, diagonal matrices $H>0$, $K>0$, $W>0$, and $V>0$, and Λ_1,Λ_2 from Assumption (H3), the following inequality holds:

$$
\begin{aligned}
\sum_{1\leq i<j\leq N}\Big\{&-\left[x_{ij}^T(t)H\Lambda_1x_{ij}(t)-2x_{ij}^T(t)H\Lambda_2f\left(x_{ij}(t)\right)\right.\\
&\quad\left.+f^T\left(x_{ij}(t)\right)Hf\left(x_{ij}(t)\right)\right]\\
&-\left[x_{ij}^T(t-\tau(t))K\Lambda_1x_{ij}(t-\tau(t))\right.\\
&\quad-2x_{ij}^T(t-\tau(t))K\Lambda_2f\left(x_{ij}(t-\tau(t))\right)\\
&\quad\left.+f^T\left(x_{ij}(t-\tau(t))\right)Kf\left(x_{ij}(t-\tau(t))\right)\right]\\
&-\left[x_{ij}^T(t-h_1)W\Lambda_1x_{ij}(t-h_1)\right.\\
&\quad-2x_{ij}^T(t-h_1)W\Lambda_2f\left(x_{ij}(t-h_1)\right)\\
&\quad\left.+f^T\left(x_{ij}(t-h_1)\right)Wf\left(x_{ij}(t-h_1)\right)\right]\\
&-\left[x_{ij}^T(t-h_2)V\Lambda_1x_{ij}(t-h_2)\right.\\
&\quad-2x_{ij}^T(t-h_2)V\Lambda_2f\left(x_{ij}(t-h_2)\right)\\
&\quad\left.+f^T\left(x_{ij}(t-h_2)\right)Vf\left(x_{ij}(t-h_2)\right)\right]\Big\}\geq 0.
\end{aligned}
\tag{28}
$$

Adding up (24)–(28) from both sides, we have

$$
\begin{aligned}
&\dot{V}(x(t))\\
&\leq \sum_{1\leq i<j\leq N}\xi_{ij}^T(t)\Phi_{ij}\xi_{ij}(t)\\
&\quad-\omega[x(t-\tau(t))-x(t-h_2)]^T\\
&\quad\times\left(U\otimes Z_2\right)\left[x(t-\tau(t))-x(t-h_2)\right]\\
&\quad-(1-\omega)\left[x(t-h_1)-x(t-\tau(t))\right]^T\\
&\quad\times\left(U\otimes Z_2\right)\left[x(t-h_1)-x(t-\tau(t))\right]\\
&=\sum_{1<i<j\leq N}\xi_{ij}^T(t)\left[\Phi_{ij}-\omega I_1^TZ_2I_1\right.\\
&\qquad\qquad\left.-(1-\omega)I_2^TZ_2I_2\right]\xi_{ij}(t)\\
&=\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\xi_{ij}(t)^T\left[(1-\omega)\Phi_{ij}^1+\omega\Phi_{ij}^2\right]\xi_{ij}(t),
\end{aligned}
\tag{29}
$$

where ϕ_{ij} ϕ_{ij}^1 ϕ_{ij}^2 are given in (20), (21), and (9), respectively, and $I_1=[0_{n,3n}\ \ -I_n\ \ 0_n\ \ -I_n\ \ 0_{n,2n}]$ and $I_2=[0_{n,3n}\ \ -I_n\ \ I_n\ \ 0_{n,3n}]$. Consider

$$
\begin{aligned}
&\xi_{ij}^T(t)\\
&=\left[x_{ij}^T(t)\ \ f^T\left(x_{ij}(t)\right)\ \ f^T\left(x_{ij}(t-\tau(t))\right)\ \ x_{ij}^T(t-\tau(t))\right.\\
&\qquad\left.x_{ij}^T(t-h_1)\ \ x_{ij}^T(t-h_2)\ \ f^T\left(x_{ij}(t-h_1)\right)\ \ f^T\left(x_{ij}(t-h_2)\right)\right].
\end{aligned}
\tag{30}
$$

Following Theorem 9, we have $(1-\omega)\Phi_{ij}^1 + \omega\Phi_{ij}^2 < 0$, and there exists a positive constant $M > 0$ that satisfied $(1-\omega)\Phi_{ij}^1 + \omega\Phi_{ij}^2 \le -MI < 0$, such that

$$\begin{aligned} \dot{V}(x(t)) &\le \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \xi_{ij}(t)^T \left[(1-\omega)\Phi_{ij}^1 + \omega\Phi_{ij}^2\right]\xi_{ij}(t) \\ &\le -M\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[\left\|x_{ij}(t)\right\|^2 + \left\|x_{ij}(t-\tau(t))\right\|^2\right]. \end{aligned} \tag{31}$$

Furthermore, based on the proof in [12], there exist two positive scalars $\beta > 0$ and $k > 0$, such that

$$\left\|x_{ij}(t)\right\| \le \beta\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sup_{-2h_2\le s\le 0}\left\|\varphi_i(s) - \varphi_j(s)\right\|\cdot e^{-kt} \tag{32}$$

for $t \ge T_1$. By Definition 3, therefore, system (9) is global exponential synchronization. Then end the proof. □

Remark 10. In [10–13], the authors studied the synchronization of an array of linearly coupled networks with constant coupling delay or time-varying coupling delay, and the derivative of the time-varying delay is confined to be less than 1. We remove this restrictiveness and the derivative of the time-varying delay can be any value.

Remark 11. During the estimation, $-\int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(s)(U\otimes Z_2)\dot{x}(s)ds$ is separated into two parts as follows: $-\int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(s)(U\otimes Z_2)\dot{x}(s)ds = -\int_{t-h_2}^{t-\tau(t)} h_{12}\dot{x}^T(s)(U\otimes Z_2)\dot{x}(s)ds - \int_{t-\tau(t)}^{t-h_1} h_{12}\dot{x}^T(s)(U\otimes Z_2)\dot{x}(s)ds$, and we estimate each part, respectively, ignoring the direct estimate for $-\int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(s)(U\otimes Z_2)\dot{x}(s)ds$, and then we have more accurate estimate than those from the reference [12, 13].

When $h_1 = 0$ in Assumption (H1), we have the following corollary from Theorem 9.

Corollary 12. *Suppose Assumptions (H1)–(H3) hold, if there exist four symmetric positive definite matrices P, Q_i, R_i, Z_2 for each i in $(1,2)$ and four positive diagonal matrices H, K, V, such that the LMIs holds for each i, j $(i<j)$ in $(1,2,\ldots,N)$. Consider*

$$\begin{aligned} \overline{\Phi}_{ij}^1 &= \overline{\Phi}_{ij} - \begin{bmatrix}0_{n,3n} & -I_n & 0_{n,2n}\end{bmatrix}^T \\ &\quad\times Z_2\begin{bmatrix}0_{n,3n} & -I_n & 0_{n,2n}\end{bmatrix} < 0 \\ \overline{\Phi}_{ij}^2 &= \overline{\Phi}_{ij} - \begin{bmatrix}0_{n,3n} & I_n & -I_n & 0_n\end{bmatrix}^T \\ &\quad\times Z_2\begin{bmatrix}0_{n,3n} & I_n & -I_n & 0_n\end{bmatrix} < 0 \\ \overline{\Phi}_{ij} &= \begin{bmatrix} \overline{\Omega}_{11} & \overline{\Omega}_{12} & \overline{\Omega}_{13} & \overline{\Omega}_{14} & 0 & 0 \\ * & \overline{\Omega}_{22} & \overline{\Omega}_{23} & \overline{\Omega}_{24} & 0 & 0 \\ * & * & \overline{\Omega}_{33} & \overline{\Omega}_{34} & 0 & 0 \\ * & * & * & \overline{\Omega}_{44} & Z_2 & 0 \\ * & * & * & * & \overline{\Omega}_{55} & V\Lambda_2 \\ * & * & * & * & * & -R_2 - V \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \overline{\Omega}_{11} &= -PC - C^TP + \sum_{i=1}^{3}Q_i - clPA_\tau - clA_\tau^TP - Z_1 \\ &\quad + h_{12}^2C^TZ_2(C + clA_\tau) + c^2l^2h_{12}^2A_\tau^TZ_2A_\tau \\ &\quad + clh_{12}^2C^TZ_2A_\tau - H\Lambda_1 \\ \overline{\Omega}_{12} &= -h_{12}^2C^TZ_2D + PD - clh_{12}^2A_\tau^TZ_2D + H\Lambda_2 \\ \overline{\Omega}_{13} &= -h_{12}^2C^TZ_2B - clh_{12}^2A_\tau^TZ_2B + PB \\ \overline{\Omega}_{14} &= h_{12}^2C^TZ_2\left(cNG_{ij}(A + A_\tau) - clA_\tau\right) \\ &\quad - cNG_{ij}P(A + A_\tau) + clPA_\tau \\ &\quad - c^2lNh_{12}^2G_{ij}A_\tau^TZ_2(A_\tau + A) \\ &\quad - c^2l^2h_{12}^2A_\tau^TZ_2A \\ \overline{\Omega}_{22} &= h_{12}^2D^TZ_2D + \sum_{i=1}^{3}R_i - H \\ \overline{\Omega}_{23} &= h_{12}^2D^TZ_2B \\ \overline{\Omega}_{24} &= h_{12}^2D^TZ_2\left(-cNG_{ij}(A + A_\tau) + clA_\tau\right) \\ &\quad + clh_{12}^2D^TZ_2A_\tau \\ \overline{\Omega}_{33} &= h_{12}^2B^TZ_2B - (1-\mu)R_3 - K \\ \overline{\Omega}_{34} &= -cNh_{12}^2G_{ij}B^TZ_2(A + A_\tau) + K\Lambda_2 + clh_{12}^2B^TZ_2A_\tau \\ \overline{\Omega}_{44} &= c^2Nh_{12}^2G_{ij}^2(A + A_\tau)^TZ_2(A + A_\tau) - (1-\mu)Q_3 \\ &\quad - 2Z_2 - K\Lambda_1 - c^2lNh_{12}^2G_{ij}(A + A_\tau)^TZ_2A_\tau \\ &\quad + c^2l^2h_{12}^2A_\tau^TZ_2A_\tau \\ \overline{\Omega}_{55} &= -Z_1 - Z_2 - Q_1 - W\Lambda_1. \end{aligned} \tag{33}$$

Then system (9) is global exponential synchronization. The proof is obvious from Theorem 9 and we omit the details. When $\tau(t)$ in Assumption (H1) is not differentiable or μ is unknown, we have the following corollary from Theorem 9.

Corollary 13. *Suppose Assumptions (H1)–(H3) hold, if there exist four symmetric positive definite matrices P, Q_i, R_i, Z_i for each i in $(1,2)$ and four positive diagonal matrices H, K, W, V, such that the LMIs holds for each i, j $(i<j)$ in $(1,2,\ldots,N)$. Consider*

$$\begin{aligned} \widehat{\Phi}_{ij}^1 &= \widehat{\Phi}_{ij} - \begin{bmatrix}0_{n,3n} & -I_n & I_n & 0_{n,3n}\end{bmatrix}^T \\ &\quad\times Z_2\begin{bmatrix}0_{n,3n} & -I_n & I_n & 0_{n,3n}\end{bmatrix} < 0, \\ \widehat{\Phi}_{ij}^2 &= \widehat{\Phi}_{ij} - \begin{bmatrix}0_{n,3n} & I_n & 0_n & -I_n & 0_{n,2n}\end{bmatrix}^T \\ &\quad\times Z_2\begin{bmatrix}0_{n,3n} & I_n & 0_n & -I_n & 0_{n,2n}\end{bmatrix} < 0, \end{aligned} \tag{34}$$

where

$$\widehat{\Phi}_{ij}=\begin{bmatrix}\widehat{\Omega}_{11} & \widehat{\Omega}_{12} & \widehat{\Omega}_{13} & \widehat{\Omega}_{14} & Z_1 & 0 & 0 & 0\\ * & \widehat{\Omega}_{22} & \widehat{\Omega}_{23} & \widehat{\Omega}_{24} & 0 & 0 & 0 & 0\\ * & * & \widehat{\Omega}_{33} & \widehat{\Omega}_{34} & 0 & 0 & 0 & 0\\ * & * & * & \widehat{\Omega}_{44} & Z_2 & Z_2 & 0 & 0\\ * & * & * & * & \widehat{\Omega}_{55} & 0 & W\Lambda_2 & 0\\ * & * & * & * & * & -Q_2-Z_2-V\Lambda_1 & 0 & V\Lambda_2\\ * & * & * & * & * & * & -R_1-W & 0\\ * & * & * & * & * & * & * & -R_2-V\end{bmatrix}. \tag{35}$$

With

$$\begin{aligned}
\widehat{\Omega}_{11}={}& -PC-C^TP+\sum_{i=1}^{2}Q_i-clPA_\tau-clA_\tau^TP-Z_1\\
&+C^T\left(h_1^2Z_1+h_{12}^2Z_2\right)\left(C+clA_\tau\right)\\
&+c^2l^2A_\tau^T\left(h_1^2Z_1+h_{12}^2Z_2\right)A_\tau\\
&+clC^T\left(h_1^2Z_1+h_{12}^2Z_2\right)A_\tau-H\Lambda_1\\
\widehat{\Omega}_{12}={}& -C^T\left(h_1^2Z_1+h_{12}^2Z_2\right)D+PD\\
&-clA_\tau^T\left(h_1^2Z_1+h_{12}^2Z_2\right)D+H\Lambda_2\\
\widehat{\Omega}_{13}={}& -C^T\left(h_1^2Z_1+h_{12}^2Z_2\right)B\\
&-clA_\tau^T\left(h_1^2Z_1+h_{12}^2Z_2\right)B+PB\\
\widehat{\Omega}_{14}={}& C^T\left(h_1^2Z_1+h_{12}^2Z_2\right)\left(cNG_{ij}\left(A+A_\tau\right)-clA_\tau\right)\\
&-cNG_{ij}P\left(A+A_\tau\right)+clPA_\tau\\
&-c^2lNG_{ij}A_\tau^T\left(h_1^2Z_1+h_{12}^2Z_2\right)\left(A_\tau+A\right)\\
&-c^2l^2A_\tau^T\left(h_1^2Z_1+h_{12}^2Z_2\right)A_\tau\\
\widehat{\Omega}_{22}={}& D^T\left(h_1^2Z_1+h_{12}^2Z_2\right)D+\sum_{i=1}^{2}R_i-H\\
\widehat{\Omega}_{23}={}& D^T\left(h_1^2Z_1+h_{12}^2Z_2\right)B\\
\widehat{\Omega}_{24}={}& D^T\left(h_1^2Z_1+h_{12}^2Z_2\right)\left(-cNG_{ij}\left(A+A_\tau\right)+clA_\tau\right)\\
&+clD^T\left(h_1^2Z_1+h_{12}^2Z_2\right)A_\tau\\
\widehat{\Omega}_{33}={}& B^T\left(h_1^2Z_1+h_{12}^2Z_2\right)B-K\\
\widehat{\Omega}_{34}={}& -cNG_{ij}B^T\left(h_1^2Z_1+h_{12}^2Z_2\right)\left(A+A_\tau\right)+K\Lambda_2\\
&+clB^T\left(h_1^2Z_1+h_{12}^2Z_2\right)A_\tau\\
\widehat{\Omega}_{44}={}& c^2NG_{ij}^2\left(A+A_\tau\right)^T\left(h_1^2Z_1+h_{12}^2Z_2\right)\left(A+A_\tau\right)\\
&-2Z_2-K\Lambda_1\\
&-c^2lNG_{ij}\left(A+A_\tau\right)^T\left(h_1^2Z_1+h_{12}^2Z_2\right)A_\tau\\
&+c^2l^2A_\tau^T\left(h_1^2Z_1+h_{12}^2Z_2\right)A_\tau\\
\widehat{\Omega}_{55}={}& -Z_1-Z_2-Q_1-W\Lambda_1
\end{aligned} \tag{36}$$

then system (9) *is global exponential synchronization.*

Remark 14. Theorem 9 can be applied to both slow and fast time-varying delays only if μ is known. But when $\tau(t)$ is not differentiable or μ is unknown, Theorem 9 fails to work; however, Corollary 13 can check the synchronization of system (9) instead.

4. Numerical Examples

In this section, two examples are provided to illustrate the effectiveness of the conclusion.

Example 1. Consider a lower-dimensional network model with 5 nodes, where each node is a simple three-dimensional stable linear system. To simplify, we assume that $C = \mathrm{diag}(1,2,3)$, $D = B = A_\tau = 0$, $I(t) = 0$, and the inner-coupling matrix is $A = \mathrm{diag}(1,1,1)$, and the outer-coupling matrix is defined as

$$G=\begin{bmatrix}-2 & 1 & 0 & 0 & 1\\ 1 & -3 & 1 & 1 & 0\\ 0 & 1 & -2 & 1 & 0\\ 0 & 1 & 1 & -3 & 1\\ 1 & 0 & 0 & 1 & -2\end{bmatrix}. \tag{37}$$

We compare the admissible delay upper bounds for τ and the number of variables with those from other references; it is clear from Table 1 that the delay is a little larger and the number of variables is less than those from [10, 12].

Example 2. We consider the following DNNs:

$$\dot{x}_i(t)=-Cx_i(t)+Df\left(x_i(t)\right)+Bf\left(x_i\left(t-\tau(t)\right)\right)+I(t). \tag{38}$$

Let us see the synchronized states of a chaotic system: Chua's circuit [19].

Table 1: Admissible delay upper bound τ for different c.

Methods	Theorem 1 in [12]	Theorem 1 in [10]	Our result
$c = 0.3$	1.345	1.632	1.666
$c = 0.4$	0.950	0.998	1.000
$c = 0.5$	0.731	0.732	0.898
The number of variables	More than 11	7	6

The dynamics of Chua's circuit is

$$\begin{aligned}\frac{dx_1}{dt} &= 10\left(-x_1 + x_2 - f\left(x_1\right)\right)\\ \frac{dx_2}{dt} &= x_1 - x_2 + x_3 \\ \frac{dx_3}{dt} &= -18x_2,\end{aligned} \tag{39}$$

where

$$f\left(x_1\right) = bx_1 + 0.5\left(d-b\right)\left(\left|x_1+1\right| - \left|x_1-1\right|\right), \quad d = -\frac{4}{3}, \quad b = -\frac{3}{4}. \tag{40}$$

Here we choose Chua's circuit (39) as the uncoupled system. Figure 1 illustrates the chaotic trajectories of system (39). The coupling time-varying delay $\tau(t) = 0.03(2 + \sin(40t) + \cos^2(80t))$ and the coupling strength $c = 1$, then we consider dynamic networks consisting of three linearly coupled identical DNNs with couplings as

$$\begin{aligned}\dot{x}_i(t) = & -Cx_i(t) + Df\left(x_i(t)\right) + Bf\left(x_i\left(t-\tau(t)\right)\right) + I(t)\\ & + c\sum_{j=1, j\neq i}^{N} G_{ij}A\left[x_j\left(t-\tau(t)\right) - x_i\left(t-\tau(t)\right)\right]\\ & + c\sum_{j=1, j\neq i}^{N} G_{ij}A_\tau\left[x_j(t-\tau) - x_i(t)\right]\end{aligned} \tag{41}$$

for $i = 1, 2, 3$. Let

$$C = \begin{bmatrix} 10 & -10 & 0 \\ -1 & 1 & -1 \\ 0 & 18 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} -10 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$G = \begin{bmatrix} -4 & 1 & 3 \\ 2 & -4 & 2 \\ 3 & 1 & -4 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{42}$$
$$A_\tau = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0 & 0.5 & 0 \\ 0.3 & 0 & 0.5 \end{bmatrix},$$

$I(t) = B = 0$, $N = 3$, and $l = 4$.

By calculating we have $h_1 = 0.06$, $h_2 = 0.12$, and $\mu = 3.6$, and the activation functions $f(x_i)$ satisfy Assumption (H3).

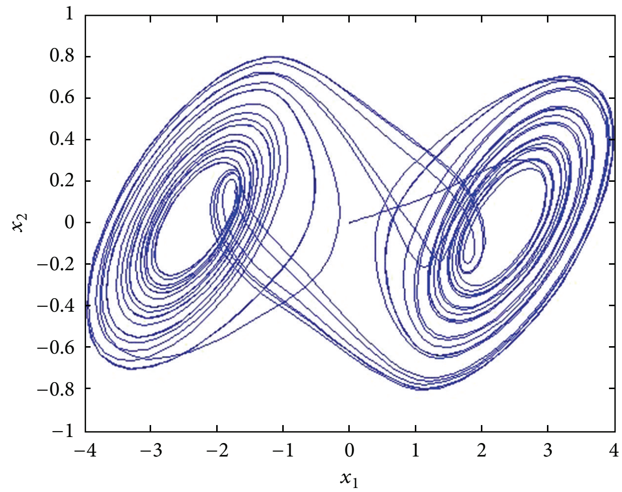

Figure 1: The chaotic trajectories of system (39).

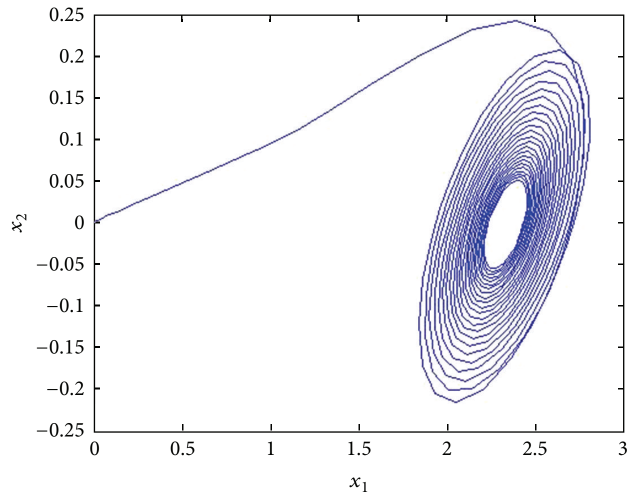

Figure 2: The synchronized state of system (39).

If the system reaches synchronization, we have the following synchronized state equation:

$$\begin{aligned}\dot{s}(t) = & -Cs(t) + Df\left(s(t)\right) + Bf\left(s\left(t-\tau(t)\right)\right)\\ & + lA_\tau\left(s\left(t-\tau(t)\right) - s(t)\right).\end{aligned} \tag{43}$$

By calculating other variables in Theorem 9 using Matlab toolbox, we realize the system synchronization, and the total error is given as $\text{error}(t) = \sum_{i=1}^{3}\sqrt{[x_{1i} - x_{2i}]^2 + [x_{2i} - x_{3i}]^2}$. Figures 2 and 3 show the synchronized state of system (39) and the synchronous error, and Figures 4, 5, and 6 show the curves of state variables $x_{i1}(t)$, $x_{i2}(t)$, and $x_{i3}(t)$ with the initial value randomly chosen from $[0, 1]$.

5. Conclusion

In this paper, global synchronization problem is investigated for general complex networks with coupling delays. Two

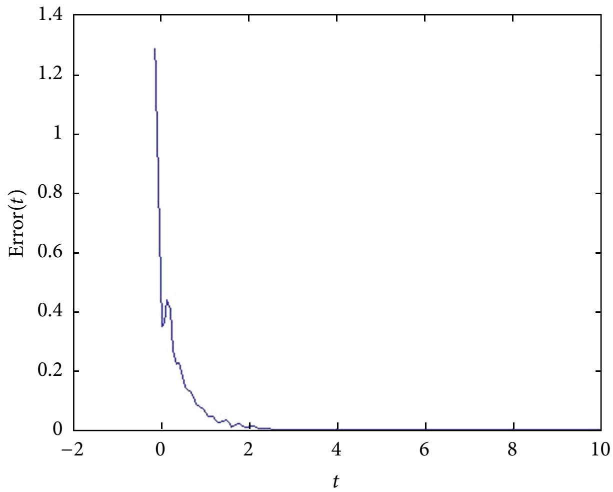

FIGURE 3: The synchronous error of system (39).

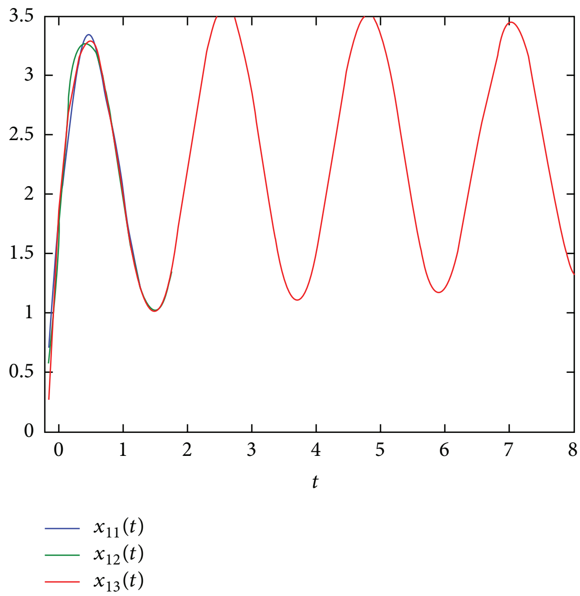

FIGURE 4: The synchronized trajectory of $x_{i1}(t)$, $i = 1, 2, 3$.

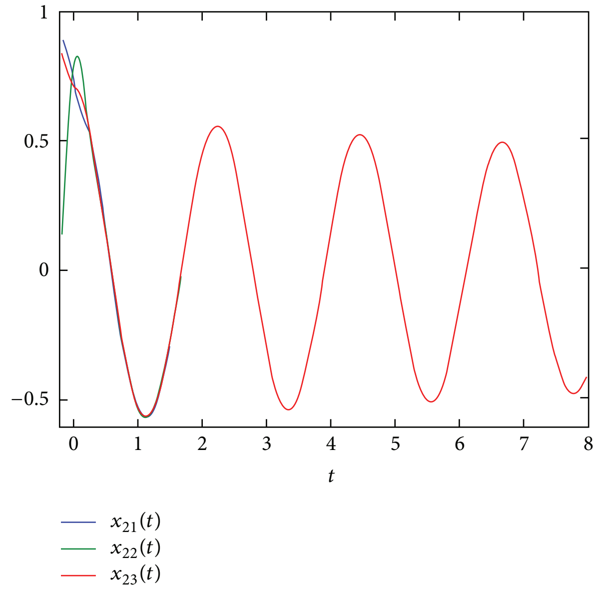

FIGURE 5: The synchronized trajectory of $x_{i2}(t)$, $i = 1, 2, 3$.

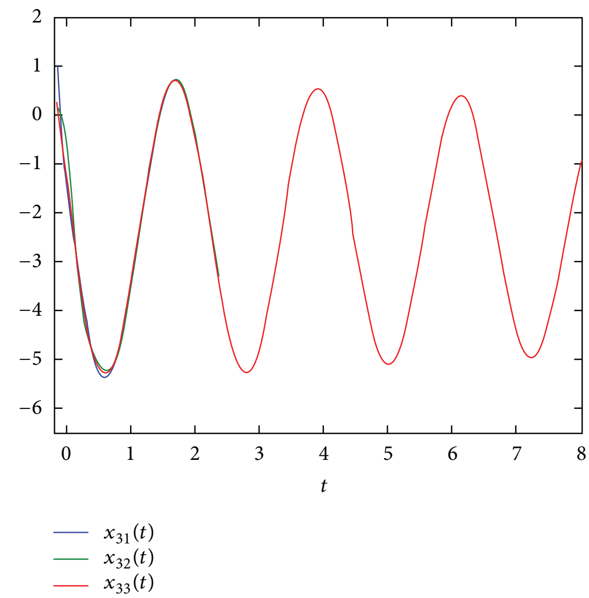

FIGURE 6: The synchronized trajectory of $x_{i3}(t)$, $i = 1, 2, 3$.

novel theorems are obtained about synchronization of the coupled systems by employing Lyapunov-Krasovskii functional and convex combination techniques. We find that the derivative of the time-varying delay can be any given value, although most of the former results are based on assumption that the derivative of the time-varying delay should be less than one. The condition obtained in this paper is expressed in the form of linear matrix inequalities, which have less variables and are easy to be computed and checked by resorting to Matlab LMI toolbox. The proposed network model may shed some new lights on the synchronization with one delay coupling. Furthermore, there are abundant dynamical behaviors in arrays of coupled systems with different coupling schemes and they deserve to be further studied in the future. Finally, two examples are given to illustrate the usefulness of the derived methods by the simulation results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of the paper.

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