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# A PARABOLIC LITTLEWOOD-PALEY INEQUALITY WITH APPLICATIONS TO PARABOLIC EQUATIONS

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*Dedicated to Jean Leray*

## 0. Introduction

In [2] we have used a "parabolic" version of the Littlewood-Paley inequality for the heat equation. It turns out that convolutions with the heat kernel can be replaced by convolutions with more general functions. Here we present the corresponding result. We also give its extension to parabolic equations with coefficients depending only on $t$, extension based on one rather general principle, which might be of independent interest.

The need in the parabolic version of the Littlewood-Paley inequality can be seen from the following. In $\mathbb{R}^d$ consider the simplest stochastic Cauchy problem

$$du(t,x) = \frac{1}{2}\Delta u(t,x)\,dt + g(t,x)\,dw_t, \qquad t>0, \quad u(0,x)=0,$$

where $w_t$ is a one-dimensional Wiener process. The solution of this problem is known to be

$$u(t,x) = \int_0^t T_{t-s}g(s,\cdot)(x)\,dw_t,$$

where

$$(0.1) \qquad T_t h(x) = t^{-d/2}\phi(x/\sqrt{t}) * g(x), \qquad \phi(x) = \frac{1}{(2\pi)^{d/2}}e^{-\frac{1}{2}|x|^2}$$

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If $g$ is non-random ($\mathbb{E}$ is used for mathematical expectation), then

$$\mathbb{E}\int_0^T \|\nabla u(t,\cdot)\|^p_{L_p(\mathbb{R})}\,dt$$
$$= N(d,p)\int_0^T\int_{\mathbb{R}^d}\left[\int_0^t |\nabla T_{t-s}g(s,\cdot)(x)|^2\,ds\right]^{p/2} dx\,dt$$
$$= N(d,p)\int_0^T\int_{\mathbb{R}^d}\left[\int_a^t |\Phi_{t-s}g(s,\cdot)(x)|^2\,\frac{ds}{t-s}\right]^{p/2} dx\,dt,$$

where $\Phi_t g(x) := t^{-d/2}(\nabla\phi)(x/\sqrt{t}\,) * g(x)$, and in order to prove that $u \in H^1_p$ we have to estimate the last integral.

## 1. A generalization of the Littlewood-Paley inequality

Fix a constant $K \in (0,\infty)$ and let $\psi(x)$ be a $C^\infty(\mathbb{R}^d)$ integrable function such that

$$(1.1)\qquad \begin{aligned}&\int_{\mathbb{R}^d}\psi\,dx = 0,\\ &\|\psi\|_{L_1(\mathbb{R}^d)} + \|\,|x|\psi\|_{L_1(\mathbb{R}^d)} + \|\nabla\psi\|_{L_1(\mathbb{R}^d)} + \|x\cdot\nabla\psi\|_{L_1(\mathbb{R}^d)} \le K.\end{aligned}$$

For example any first order partial derivative of $\phi$ from (0.1) satisfies (1.1) with some $K$. Define $\Psi_t g(x) := t^{-d/2}\psi(x/\sqrt{t}\,) * g(x)$, so that the above operator $\Phi$ is a particular case of $\Psi$.

The classical Littlewood-Paley inequality (see, for instance, Chapter 1 in [3]) says that for any $p\in(1,\infty)$ and $g\in L_p$ we have

$$(1.2)\qquad \int_{\mathbb{R}^d}\left[\int_0^\infty |\Psi_t g(x)|^2\,\frac{dt}{t}\right]^{p/2} dx \le N\|g\|_p^p,$$

where the constant $N$ depends only on $d, p$.

Here we want to generalize this fact by proving the following result.

THEOREM 1.1. *Let $H$ be a separable Hilbert space, $p\in[2,\infty)$, $-\infty \le a < b \le \infty$ and $g \in L_p((a,b)\times\mathbb{R}^d, H)$. Then*

$$(1.3)\qquad \int_{\mathbb{R}^d}\int_a^b\left[\int_a^t |\Psi_{t-s}g(s,\cdot)(x)|_H^2\,\frac{ds}{t-s}\right]^{p/2} dt\,dx \le N\int_{\mathbb{R}^d}\int_a^b |g(t,x)|_H^p\,dt\,dx,$$

*where the constant $N$ depends only on $d, p$ and $K$.*

REMARK 1.1. The Littlewood-Paley inequality (1.2) follows directly from (1.3) if $p \geq 2$. Indeed, take $a = 0$, $b = 2$, $g(s, x) = g(x)$. Then the left-hand side of (1.3) equals

$$\begin{aligned}\int_{\mathbb{R}^d}\int_0^2 \left[\int_0^t |\Psi_s g(t-s,\cdot)(x)|_H^2 \frac{ds}{s}\right]^{p/2} dt\,dx \\ \geq \int_{\mathbb{R}^d}\int_1^2 \left[\int_0^t |\Psi_s g(x)|_H^2 \frac{ds}{s}\right]^{p/2} dt\,dx \\ \geq \int_{\mathbb{R}^d} \left[\int_0^1 |\Psi_s g(x)|_H^2 \frac{ds}{s}\right]^{p/2} dx.\end{aligned}$$

Thus, from (1.3) it follows that

$$\int_{\mathbb{R}^d} \left[\int_0^1 |\Psi_s g(x)|_H^2 \frac{ds}{s}\right]^{p/2} dx \leq N \int_{\mathbb{R}^d} |g(x)|_H^p\,dx,$$

and a standard argument based on self-similarity allows us to replace the upper limit 1 by infinity keeping the *same* constant $N$ on the right.

This gives (1.2) for $p \geq 2$ for Hilbert space valued $g$. It is a standard fact that from the Hilbert-space version of (1.2) for $p \geq 2$ the same inequality follows for $p \in (1, 2)$ by duality.

It is also well known that having (1.2) proved for $p \in (1, 2]$, the case $p \in [2, \infty)$ can be treated by duality. In this sense the cases $p \in (1, 2]$ and $p \in [2, \infty)$ are equivalent if we are only dealing with (1.2). The general inequality (1.3) does not exibit this property.

REMARK 1.2. For $p \in (1, 2)$ estimate (1.3) is not true even if $d = 1$, $H = \mathbb{R}$. Indeed, take $a = 0$, a finite $b$, $\psi = \phi'$, $g(s, x) = \phi(x)e^{-\lambda s}$ where $\lambda > 0$. Then

$$\begin{aligned}\Psi_t g(s, x) &= \sqrt{t}\,\frac{\partial}{\partial x}[t^{-1/2}\phi(x/\sqrt{t}) * \phi(x)]e^{-\lambda s} \\ &= \sqrt{t}\,\frac{\partial}{\partial x}[(t+1)^{-1/2}\phi(x/\sqrt{t+1})]e^{-\lambda s} \\ &= \sqrt{t}\,e^{-\lambda s}(t+1)^{-1}\phi'(x/\sqrt{t+1}),\end{aligned}$$

and as is easy to see the limit as $\lambda$ goes to infinity of the product of the left-hand side of (1.3) and $\lambda^{p/2}$ equals

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}}\int_0^b \left[\lambda\int_0^t \frac{x^2}{2\pi(t-s+1)^3} e^{-2\lambda s} e^{-\frac{1}{t-s+1}x^2}\,ds\right]^{p/2} dt\,dx. \tag{1.4}$$

Furthermore, for $\lambda \geq 1$,

$$\int_0^t \frac{\lambda x^2}{2\pi(t-s+1)^3} e^{-2\lambda s} e^{-\frac{1}{t-s+1}x^2}\, ds$$

$$= \int_0^{t/2} + \int_{t/2}^t$$

$$\leq \frac{Nx^2}{(t+2)^3} e^{-\frac{1}{t+1}x^2} \int_0^{t/2} \lambda e^{-2\lambda s}\, ds + e^{-\lambda t} N x^2 e^{-\frac{2}{t+2}x^2}$$

$$\leq \frac{Nx^2}{(t+2)^3} e^{-\frac{2}{t+2}x^2}.$$

This allows us to evaluate the limit in (1.4) by using the dominated convergence theorem. We see that the limit equals

$$\int_{\mathbb{R}} \int_0^b \left[ \frac{x^2}{4\pi(t+1)^3} e^{-\frac{1}{t+1}x^2} \right]^{p/2} dt\, dx, \tag{1.5}$$

which is finite and non-zero.

Thus the left-hand side of (1.3) is of order $\lambda^{-p/2}$. At the same time the right-hand side of (1.3) is of order $\lambda^{-1}$, and $\lambda^{-p/2}$ is much bigger than $\lambda^{-1}$ if $p \in (1,2)$ and $\lambda \to \infty$.

REMARK 1.3. There is one more feature of inequality (1.3) which distinguishes it from (1.2). It is well known that inequality (1.2) is reversible under mild conditions on $\psi$. This is not the case for (1.3) (unless $p = 2$), which can be seen from the above remark if $p > 2$ and $b = \infty$ (in this case the above argument is still valid) since the integral in (1.5) is finite, and as $\lambda \to \infty$ the left-hand side of (1.3) is much smaller than its right-hand side.

PROOF OF THEOREM 1.1. First note that by considering $g(t,x)I_{(a,b)}(t)$ instead of $g(t,x)$ we can reduce the general case to the one in which $a = -\infty$ and $b = \infty$. Therefore, only this case is considered below. Next, for $p = 2$ application of the Fourier transformation shows that the left-hand side in (1.3) equals

$$\int_{\mathbb{R}^d} \int_{-\infty}^{\infty} \int_{-\infty}^{t} |\tilde{\psi}(\xi\sqrt{t-s})|^2 |\tilde{g}(s,\xi)|_H^2 \frac{ds}{t-s}\, dt\, d\xi$$

$$= \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} \left[ \int_0^\infty |\tilde{\psi}(\xi\sqrt{t})|^2 \frac{dt}{t} \right] |\tilde{g}(s,\xi)|_H^2\, ds\, d\xi.$$

Here $\tilde{\psi}(0) = 0$ and

$$|\tilde{\psi}(\xi)| \leq |\xi|\, |\nabla\tilde{\psi}(\xi)| \leq N(d)|\xi| \int_{\mathbb{R}^d} |x|\, |\psi(x)|\, dx,$$

$$|\xi|\,|\tilde{\psi}(\xi)| \leq N(d) \int_{\mathbb{R}^d} |\nabla\psi(x)|\,dx,$$

so that

$$\int_0^\infty |\tilde{\psi}(\xi\sqrt{t})|^2 \frac{dt}{t} \leq N(d,K),$$

$$\int_{\mathbb{R}^d}\int_{-\infty}^{\infty}\left[\int_0^\infty |\tilde{\psi}(\xi\sqrt{t})|^2 \frac{dt}{t}\right]|\tilde{g}(s,\xi)|_H^2\,ds\,d\xi \leq N\int_{\mathbb{R}^d}\int_{-\infty}^{\infty}|\tilde{g}(s,\xi)|_H^2\,ds\,d\xi,$$

which in turn equals the right-hand side of (1.3). This proves (1.3) for $p=2$ and shows that if we introduce the operator $P$ by the formula

$$Pg(t,x) = \left[\int_{-\infty}^t |\Psi_{t-s}g(s,\cdot)(x)|_H^2 \frac{ds}{t-s}\right]^{1/2},$$

then $P$ is bounded from $L_2(\mathbb{R}^{d+1},H)$ into $L_2(\mathbb{R}^{d+1},\mathbb{R})$.

We now make use of the parabolic sharp inequality

$$||f||_{L_p(\mathbb{R}^d)} \leq N||f^{\#}||_{L_p(\mathbb{R}^d)}$$

(see [1]), where

$$f^{\#}(t,x) := \sup \inf_{f_0\in\mathbb{R}} \frac{1}{|Q(r)|}\int_{(t_0,x_0)+Q(r)} |f(s,y)-f_0|\,ds\,dy,$$

$$Q(r) = (0,r^2)\times\{x : |x|\leq r\}$$

and the supremum is taken over all $r>0$ and $(t_0,x_0)$ such that $(t,x)\in(t_0,x_0)+Q(r)$. Since $f^{\#}$ is less than the parabolic maximal function of $f$, which has $L_2$-norm controlled by that of $f$, we see that $P^{\#}$ is a bounded operator from $L_2(\mathbb{R}^{d+1},H)$ into $L_2(\mathbb{R}^{d+1},\mathbb{R})$. If we show that it is also a bounded operator from $L_\infty(\mathbb{R}^{d+1},H)$ into $L_\infty(\mathbb{R}^{d+1},\mathbb{R})$, then by the Marcinkiewicz interpolation theorem it is a bounded operator from $L_p(\mathbb{R}^{d+1},H)$ into $L_p(\mathbb{R}^{d+1},\mathbb{R})$ for any $p\in(2,\infty)$, and owing to the above sharp inequality this will end the proof of the theorem.

In other words, the parabolic version of the Stampacchia interpolation theorem is available, and we will prove our theorem if we prove that $P$ is a bounded operator from $L_\infty(\mathbb{R}^{d+1},H)$ into $BMO(\mathbb{R}^{d+1},\mathbb{R})$. More precisely, it suffices to show that if $|g(t,x)|_H \leq 1$ for all $(t,x)\in\mathbb{R}^{d+1}$, then for any $r>0$ and $(t_0,x_0)\in\mathbb{R}^{d+1}$ there exist a constant $g_0\in\mathbb{R}$ (depending on $g,r,t_0,x_0$) and an absolute constant $N$ such that

$$\int_{(t_0,x_0)+Q(r)} |Pg(t,x)-g_0|^2\,dt\,dx \leq N|Q(r)|. \tag{1.6}$$

Observe that if $c$ is a constant $\neq 0$, then $\Psi_t h(c\,\cdot)(x) = \Psi_{tc^2} h(cx)$, and

$$
\begin{aligned}
Pg(c^2\cdot, c\,\cdot)(t,x) &= \Big[\int_{-\infty}^{t} |\Psi_{(t-s)c^2} g(c^2 s,\cdot)(cx)|_H^2 \,\frac{ds}{t-s}\Big]^{1/2} \\
&= \Big[\int_{-\infty}^{tc^2} |\Psi_{tc^2-s} g(s,\cdot)(cx)|_H^2 \,\frac{ds}{tc^2-s}\Big]^{1/2} \\
&= Pg(c^2 t, cx).
\end{aligned}
$$

This and a shift of the origin show that we only need to prove (1.6) for $t_0 = 0$, $x_0 = 0$ and $r = 1$.

Thus, take $t_0 = 0$, $x_0 = 0$, $r = 1$. Also observe that for $t \in (0,1)$,

$$
\begin{aligned}
Pg(t,x) &\le \Big[\int_{-\infty}^{-1} |\Psi_{t-s} g(s,\cdot)(x)|_H^2 \,\frac{ds}{t-s}\Big]^{1/2} + \Big[\int_{-1}^{t} |\Psi_{t-s} g(s,\cdot)(x)|_H^2 \,\frac{ds}{t-s}\Big]^{1/2} \\
&=: P_1 g(t,x) + P_2 g(t,x).
\end{aligned}
$$

On the other hand, obviously $P_1 g \le Pg$. It follows that for any constant $g_0$,

$$
|Pg(t,x) - g_0| \le |P_1 g(t,x) - g_0| + |P_2 g(t,x)|. \tag{1.7}
$$

To estimate $P_2 g(t,x)$ we represent $g$ as $g_1 + g_2$, where $g_2 = g I_{Q(2)}$, and we notice that by the previous result

$$
\int_{Q(1)} (P_2 g_2)^2 \,dt\,dx \le \int_{\mathbb{R}^{d+1}} (P g_2)^2 \,dt\,dx \le N \|g_2\|^2_{L_2(\mathbb{R}^{d+1},H)} \le N.
$$

Furthermore, obviously for $s, |x| \le 1$ we have

$$
|\Psi_t g_1(s,\cdot)(x)|_H \le t^{-d/2} \int_{|y|\ge 1} |\psi(y/\sqrt{t})|\,dy = \int_{|y|\ge t^{-1/2}} |\psi(y)|\,dy,
$$

so that

$$
\begin{aligned}
P_2 g_1(t,x) &\le \Big[\int_0^{t+1} \Big\{\int_{|y|\ge s^{-1/2}} |\psi(y)|\,dy\Big\}^2 \frac{ds}{s}\Big]^{1/2} \\
&\le \Big[\int_0^{t+1} \Big\{\int_{\mathbb{R}^d} |y|\,|\psi(y)|\,dy\Big\}^2 ds\Big]^{1/2} \\
&= \sqrt{t+1} \int_{\mathbb{R}^d} |y|\,|\psi(y)|\,dy,
\end{aligned}
$$

whence we see that $P_2 g_1$ is just bounded on $Q(1)$.

Owing to (1.7) it only remains to find an appropriate constant $g_0$ such that

$$
\int_{Q(1)} |P_1 g(t,x) - g_0|^2 \,dt\,dx
$$

is bounded by a constant independent of $g$. Actually, it turns out that as $g_0$ one can take any particular value of $P_1 g(t,x)$ in $Q(1)$. Indeed, as we will see, the first derivatives of $P_1 g(t,x)$ are bounded on $Q(1)$.

Let us first estimate the derivatives with respect to $x$. By using the inequality $|\nabla\Psi_t h| \le \sup|h| t^{-1/2} ||\nabla\psi||_{L_1}$ and by the Minkowski inequality we obtain

$$\left|\frac{\partial}{\partial x^i} P_1 g(t,x)\right| \le \left[\int_{-\infty}^{-1} |\nabla\Psi_{t-s} g(s,\cdot)(x)|_H^2 \frac{ds}{t-s}\right]^{1/2}$$
$$\le ||\nabla\psi||_{L(\mathbb{R}^d)} \left[\int_{-\infty}^{-1} \frac{ds}{(t-s)^2}\right]^{1/2},$$

which is indeed bounded on $Q(1)$. One obtains a similar estimate for the derivative of $P_1 g$ with respect to $t$ if one notices that

$$\frac{\partial}{\partial t}\Psi_t h(x) = -\frac{d}{2} t^{-1}\Psi_t h(x) - \frac{1}{2} t^{-1}\tilde{\Psi}_t h(x),$$

where the operator $\widetilde{\Psi}$ is constructed as $\Psi$ with $x \cdot \nabla\psi(x)$ instead of $\psi$ This proves the theorem.

## 2. An application to solutions of parabolic equations

By taking $\psi = \nabla\phi$ we see that under the assumptions of Theorem 1.1,

$$\left[\int_a^t |\nabla T_{t-s} g(s,\cdot)(x)|_H^2\, ds\right]^{p/2} dt\, dx \le N \int_{\mathbb{R}^d}\int_a^b |g(t,x)|_H^p\, dt\, dx, \tag{2.1}$$

where $N = N(d,p)$ and $T_t$ is the semigroup associated with the operator $(1/2)\Delta$. This fact can be generalized.

THEOREM 2.1. *Let $a(t)$ be a $d \times d$ symmetric matrix-valued function given on $\mathbb{R}$. Assume that $a(t) \ge I$ in matrix sense, where $I$ is the unit $d \times d$ matrix and assume that $a(t)$ is locally integrable on $\mathbb{R}$. For sufficiently regular functions $h(x)$ and $s < t$ define $T_{s,t} h(x)$ as the value at $(t,x)$ of the solution $v$ of the initial-value problem*

$$\frac{\partial v}{\partial t} = \frac{1}{2} a^{ij}(t) v_{x^i x^j}, \qquad t \ge s,$$
$$v(s,x) = h(x), \qquad x \in \mathbb{R}^d.$$

*Then the operators $T_{s,t}$ can be extended to $L_p(\mathbb{R}^d, H)$ and*

$$\int_{\mathbb{R}^d}\int_a^b \left[\int_a^t |\nabla T_{s,t} g(s,\cdot)(x)|_H^2\, ds\right]^{p/2} dt\, dx \le N \int_{\mathbb{R}^d}\int_a^b |g(t,x)|_H^p\, dt\, dx, \tag{2.2}$$

*for any $g \in L_p((a,b) \times \mathbb{R}^d, H)$ with the same constant $N$ as in (2.1).*

To prove this theorem we will apply the following general principle we were talking about in the introduction.

THEOREM 2.2. *Let $\mathcal{U}$ be a set of couples of $H$-valued functions $u = (u_1, u_2)$ of $(s,t,x) \in \mathbb{R}^{d+2} \cap \{s \le t\}$ and $\mathcal{F}$ be a set of couples of $H$-valued functions $f = (f_1, f_2)$ defined on $\mathbb{R}^{d+1}$. Assume that on $\mathcal{U}$ and $\mathcal{F}$ we are given functions $\|\cdot\|_{\mathcal{U}}, \|\cdot\|_{\mathcal{F}}$. Suppose that these "norms" in $\mathcal{U}$ and $\mathcal{F}$ are translation invariant: for any continuous $\mathbb{R}^d$-valued function $b(t)$ defined on $\mathbb{R}$,*

$$\|u(s,t,b(t)+x)\|_{\mathcal{U}} = \|u\|_{\mathcal{U}}, \qquad \|f(t,b(t)+x)\|_{\mathcal{F}} = \|f\|_{\mathcal{F}}.$$

*Assume that $\mathcal{F}$ contains $\mathcal{A}\times\mathcal{A}$, where $\mathcal{A}$ is a translation invariant set of functions $g(t,x)$ with compact support which are continuous in $(t,x)$ together with each $x$-derivative. Also assume that $\mathcal{U}$ contains the set $\mathcal{B}\times\mathcal{B}$, where $\mathcal{B}$ is a translation invariant set of functions $g(s,t,x)$ which are bounded and continuous in $(s,t,x)$ together with each $x$-derivative. Assume that if $(\Omega, \Sigma, P)$ is a probability space, and $u(\omega,s,t,x) \in \mathcal{B}^2$ for any $\omega \in \Omega$ and $u(\omega,s,t,x)$ is measurable in $\omega$ for any $(s,t,x)$ and if $\mathbb{E}\,\|u\|_{\mathcal{U}} < \infty$, then $\mathbb{E}\,u \in \mathcal{U}$ and $\|\mathbb{E}\,u\|_{\mathcal{U}} \le \mathbb{E}\,\|u\|_{\mathcal{U}}$.*

*On $\mathcal{A}^2$ define the operators*

$$R_0 : f = (f_1, f_2) \to R_0(f_1,f_2)(s,t,x) = \left(T_{t-s}f_1(s,x), \int_s^t T_{t-r}f_2(r,x)\,dr\right),$$

$$R : f = (f_1, f_2) \to R(f_1,f_2)(s,t,x) = \left(T_{s,t}f_1(s,x), \int_s^t T_{r,t}f_2(r,x)\,dr\right),$$

*and assume that for any $f \in \mathcal{A}^2$ we have $R_0 f \in \mathcal{B}^2$ and*

$$\|R_0 f\|_{\mathcal{U}} \le M\|f\|_{\mathcal{F}},$$

*where $M$ is a finite constant. Then for any $f \in \mathcal{A}^2$ we have $Rf \in \mathcal{U}$ and*

$$\|Rf\|_{\mathcal{U}} \le M\|f\|_{\mathcal{F}}.$$

PROOF. Take a probability space $(\Omega, \Sigma, P)$ carrying a $d$-dimensional Wiener process $w_t$ defined for all $t \in (-\infty, \infty)$. Define $\sigma(t) = \sqrt{a(t) - I}$ and define the random process

$$b(t) = \int_0^t \sigma(r)\,dw_r.$$

Also take a $d$-dimensional Wiener process $B_t$ independent of $w_t$. It is well known that for any $s < t$ the random vectors

$$b(t) - b(s) + B_t - B_s \qquad \text{and} \qquad \int_s^t \sqrt{a(r)}\,dw_r$$

have the same Gaussian distribution and that for bounded non-random functions $h$ we have

$$T_{t-s}h(s,x) = \mathbb{E}\,h(s,x+B_t-B_s),$$
$$T_{s,t}h(s,x) = \mathbb{E}\,h\Big(s,x+\int_s^t \sqrt{a(r)}\,dw_r\Big).$$

Next for any functions $h(t,x)$ and $g(s,t,x)$ define

$$h^{\pm b}(t,x) = h(t,x\pm b(t)),$$
$$g^{\pm b}(s,t,x) = g(s,t,x\pm b(t)).$$

Then for $f \in \mathcal{A}^2$ we have

$$\begin{aligned}||\mathbb{E}\,[R_0 f^{-b}]^b||_{\mathcal{U}} &\le \mathbb{E}\,||\,[R_0 f^{-b}]^b||_{\mathcal{U}} = \mathbb{E}\,||R_0 f^{-b}||_{\mathcal{U}}\\ &\le M\mathbb{E}\,||f^{-b}||_{\mathcal{F}} = M\mathbb{E}\,||f||_{\mathcal{F}} = M||f||_{\mathcal{F}}.\end{aligned}$$

It only remains to check that

$$\mathbb{E}\,[R_0 f^{-b}]^b = Rf. \tag{2.3}$$

But

$$\begin{aligned}\mathbb{E}\,[T_{t-s}h^{-b}(s,x)]^b &= \mathbb{E}\,T_{t-s}h^{-b}(s,x+b(t))\\ &= \mathbb{E}\,h(s,x+b(t)-b(s)+B_t-B_s)\\ &= \mathbb{E}\,h\Big(s,x+\int_s^t \sqrt{a(r)}\,dw_r\Big),\end{aligned}$$

and the last expression coincides with $T_{s,t}h(s,x)$. This certainly proves (2.3) and hence the theorem. □

PROOF OF THEOREM 2.1. The fact that the operators $T_{s,t}$ can be extended to $L_p(\mathbb{R}^d,H)$ is well known. To derive estimate (2.2) from Theorem 2.2 and (2.1) it suffices to take $\mathcal{F}$ to be the set of all couples of measurable $H$-valued functions $f=(f_1(t,x),0)$ such that

$$||f||_{\mathcal{F}} := \Big[\int_{\mathbb{R}^d}\int_a^b |f_1(t,x)|_H^p\,dt\,dx\Big]^{1/p} < \infty,$$

and $\mathcal{U}$ to be the set of all couples of measurable $H$-valued functions $u=(u_1(s,t,x),0)$ such that

$$||u||_{\mathcal{U}} := \Big[\int_{\mathbb{R}^d}\int_a^b \Big(\int_a^t |\nabla u_1(s,t,x)|_H^2\,ds\Big)^{p/2} dt\,dx\Big]^{1/p} < \infty.$$

□

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