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NONLINEAR STURM–LIOUVILLE PROBLEMS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Ky Fan

1. Introduction

In this paper we study a two-point boundary value problem for the nonlinear system

$$-(A(t)x'(t))' = f(t, x(t), x'(t)), \qquad t \in I, \tag{1.1}$$

where $A : I = [0,1] \to M_n(\mathbb{R})$ is a continuous matrix-valued function from [0,1] to the space of all $n \times n$ matrices over $\mathbb{R}$, $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying the *Carathéodory conditions* (C1)–(C3):

- (C1) for a.e. $t \in I$, the mapping $(x, y) \mapsto f(t, x, y)$ is continuous;
- (C2) for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, the mapping $t \mapsto f(t, x, y)$ is measurable;
- (C3) for every $r > 0$, there exists $g_r \in L^1(I, \mathbb{R}_+)$ such that, for every x, y with $|x| \leq r$, $|y| \leq r$ and a.e. $t \in I$,

$$|f(t, x, y)| \leq g_r(t),$$

and $x(t)$ satisfies the following boundary conditions:

$$x(0) - A_0x'(0) = 0, \qquad x(1) + A_1x'(1) = 0, \tag{BC}$$

where A_0 and A_1 are $n \times n$ matrices.

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By a solution to the above problem we mean a function $x(t) \in C^1(I, \mathbb{R}^n)$ for which Ax' is absolutely continuous and which satisfies (1.1) for almost every $t \in I$ and satisfies the boundary conditions (BC). Let $(\,\cdot\,,\,\cdot\,)$ denote the inner product in $\mathbb{R}^n$, $\|\cdot\|_p$ denote the L^p norm in $L^p(I, \mathbb{R}^n)$, and

$$\|x\|_{C^k} = \max_{\alpha \le k} \sup_{t \in I} |x^{(\alpha)}(t)|$$

for $x \in C^k(I, \mathbb{R}^n)$. We shall refer to $\|\cdot\|_{C^k}$ as the C^k-*norm.* For a matrix B, we write $B > 0$ if B is positive definite, and $B \ge 0$ if $B > 0$ or $B = 0$.

In the next section, we find the Green matrix for our problem subject to suitable conditions on A, A_0 and A_1. Some technicalities will be necessary because the function f need not be continuous. Moreover, f may not be Lebesgue integrable and the matrix-valued function $A(t)$ is not necessarily symmetric, hence the existence of the Green matrix is not trivial. We present the existence and uniqueness theorems for our problem under suitable conditions in Sections 3 and 4.

2. Green matrix

We first define the Green matrix for our problem. We assume that $A(t)$ is a continuous $n \times n$ matrix-valued function on I with $A(t)$ invertible for all $t \in I$ and that A_0, A_1 are $n \times n$ matrices. We denote by T the matrix $\int_0^1 A^{-1}(s)\,ds + A_0 A^{-1}(0) + A_1 A^{-1}(1)$ and assume that T is invertible. We call the following matrix $G(t,s)$ the *Green matrix* for our problem:

$$G(t,s) = -H(t-s)\int_s^t A^{-1}(u)du + \left(\int_0^t A^{-1}(u)du\right)B(s) + C(s),$$

where

$$H(t-s) = \begin{cases} 1 & \text{if } t \ge s, \\ 0 & \text{if } t < s, \end{cases}$$

$$B(s) = I_{n\times n} - T^{-1}\left[A_0 A^{-1}(0) + \int_0^s A^{-1}(u)du\right], \qquad C(s) = A_0 A^{-1}(0) B(s).$$

Here $I_{n\times n}$ denotes the $n \times n$ identity matrix.

LEMMA 2.1. *The Green matrix $G(t,s)$ has the following properties:*

(a) *for any fixed $s \in I$, $G(t,s)$ is a continuous function of t;*

(b) *$\frac{\partial}{\partial t} G(t,s)$ is a continuous function of t except at the point $t = s$, and*

$$\frac{\partial}{\partial t} G(t,s) = -H(t-s)A^{-1}(t) + A^{-1}(t)B(s) \qquad \text{for } t \ne s;$$

(c) *there exists a positive number k such that*

$$\sup_{s,t\in I} |G_{i,j}(t,s)| \le k, \qquad \sup_{s\ne t\in I} \left|\frac{\partial}{\partial t} G_{i,j}(t,s)\right| \le k,$$

for all i, j, where $G_{i,j}(t,s), 1 \le i, j \le n$, are the entries of $G(t,s)$;

(d) *if* $0 < s < 1$, *then*

$$G(0,s) - A_0 \frac{\partial}{\partial t} G(0,s) = 0 \quad \text{and} \quad G(1,s) + A_1 \frac{\partial}{\partial t} G(1,s) = 0;$$

(e) $\dfrac{\partial}{\partial t}\left(A(t) \dfrac{\partial}{\partial t} G(t,s)\right) = 0$ *for all* $t \neq s$.

Proof. The conclusions (a), (b), (c) are immediate consequences of our assumptions, so we only need to prove (d) and (e).

(d) If $0 < s < 1$, then

$$G(0,s) - A_0 \frac{\partial}{\partial t} G(0,s) = C(s) - A_0 A^{-1}(0) B(s) = 0,$$

and

$$\begin{aligned} G(1,s) + A_1 \frac{\partial}{\partial t} G(1,s) = & -\int_s^1 A^{-1}(u)\,du + \left(\int_0^1 A^{-1}(u)\,du\right) B(s) \\ & + A_0 A^{-1}(0) B(s) + A_1(-A^{-1}(1) + A^{-1}(1) B(s)) \\ = & \, TB(s) - A_1 A^{-1}(1) - \int_s^1 A^{-1}(u)\,du \\ = & \, T - A_0 A^{-1}(0) - \int_0^s A^{-1}(u)\,du - A_1 A^{-1}(1) \\ & - \int_s^1 A^{-1}(u)\,du \\ = & \, 0. \end{aligned}$$

(e) Since

$$A(t) \frac{\partial}{\partial t} G(t,s) = -H(t-s) I_{n\times n} + B(s) = \begin{cases} -I_{n\times n} + B(s) & \text{if } t > s, \\ B(s) & \text{if } t < s, \end{cases}$$

we have

$$\frac{\partial}{\partial t}\left(A(t) \frac{\partial}{\partial t} G(t,s)\right) = 0$$

for all $t \neq s$. □

The proof of the following lemma is based on the Leibniz rule for differentiation of integrals.

Lemma 2.2. *Let* $y \in C(I, \mathbb{R}^n)$ *and* $x(t) = \int_0^1 G(t,s) y(s)\,ds$. *Then*

(a) $x, Ax' \in C^1(I, \mathbb{R}^n)$;

(b) $-(A(t)x'(t))' = y(t)$ *for all* $t \in I$;

(c) $x(0) - A_0 x'(0) = 0$ *and* $x(1) + A_1 x'(1) = 0$.

PROOF. For each $t \in I$, we have

$$\begin{aligned}
x'(t) &= \frac{d}{dt}\int_0^1 G(t,s)y(s)\,ds \\
&= \frac{d}{dt}\left(\int_0^t G(t,s)y(s)\,ds + \int_t^1 G(t,s)y(s)\,ds\right) \\
&= \int_0^t \frac{\partial}{\partial t}G(t,s)y(s)\,ds + G(t,t-)y(t) \\
&\quad + \int_t^1 \frac{\partial}{\partial t}G(t,s)y(s)\,ds - G(t,t+)y(t) \\
&= \int_0^t \frac{\partial}{\partial t}G(t,s)y(s)\,ds + \int_t^1 \frac{\partial}{\partial t}G(t,s)y(s)\,ds \\
&= \int_0^1 \frac{\partial}{\partial t}G(t,s)y(s)\,ds,
\end{aligned}$$

and so

$$A(t)x'(t) = \int_0^1 \left(A(t)\frac{\partial}{\partial t}G(t,s)\right)y(s)\,ds.$$

Thus, we have

$$\begin{aligned}
(A(t)x'(t))' &= \frac{d}{dt}\int_0^1 \left(A(t)\frac{\partial}{\partial t}G(t,s)\right)y(s)\,ds \\
&= \frac{d}{dt}\left[\int_0^t + \int_t^1\right]\left(A(t)\frac{\partial}{\partial t}G(t,s)\right)y(s)\,ds \\
&= \int_0^t \frac{\partial}{\partial t}\left(A(t)\frac{\partial}{\partial t}G(t,s)\right)y(s)\,ds + \left(A(t)\frac{\partial}{\partial t}G(t,t-)\right)y(t) \\
&\quad + \int_t^1 \frac{\partial}{\partial t}\left(A(t)\frac{\partial}{\partial t}G(t,s)\right)y(s)\,ds - \left(A(t)\frac{\partial}{\partial t}G(t,t+)\right)y(t) \\
&= A(t)\left(\frac{\partial}{\partial t}G(t,t-) - \frac{\partial}{\partial t}G(t,t+)\right)y(t) \\
&= A(t)(-A^{-1}(t))y(t) = -y(t),
\end{aligned}$$

which proves (b).

Since $(A(t)x'(t))' = -y(t)$, we have $Ax' \in C^1(I, \mathbb{R}^n)$. Since $A^{-1} \in C(I, M_n(\mathbb{R}))$, it follows that $x' \in C(I, \mathbb{R}^n)$, which proves (a).

Finally, we have

$$\begin{aligned}
x(0) - A_0x'(0) &= \int_0^1 G(0,s)y(s)\,ds - A_0\int_0^1 \frac{\partial}{\partial t}G(0,s)y(s)\,ds \\
&= \int_0^1 \left[G(0,s) - A_0\frac{\partial}{\partial t}G(0,s)\right]y(s)\,ds = 0
\end{aligned}$$

by Lemma 2.1(d), (c). Similarly,

$$
\begin{aligned}
x(1)+A_1x'(1) &= \int_0^1 G(1,s)y(s)ds + A_1\int_0^1 \frac{\partial}{\partial t}G(1,s)y(s)\,ds \\
&= \int_0^1 \left[G(1,s)+A_1\frac{\partial}{\partial t}G(1,s)\right]y(s)\,ds = 0,
\end{aligned}
$$

which proves (c). □

LEMMA 2.3. *Let* $y \in L^1(I,\mathbb{R}^n)$ *and* $x(t)=\int_0^1 G(t,s)y(s)\,ds$. *Then*

(a) $x \in C^1(I,\mathbb{R}^n)$ *and* Ax' *is absolutely continuous;*
(b) $-(A(t)x'(t))' = y(t)$ *a.e. on* I*;*
(c) $x(0)-A_0x'(0)=0$ *and* $x(1)+A_1x'(1)=0$.

PROOF. Since $y\in L^1(I,\mathbb{R}^n)$, there exists a sequence of $C(I,\mathbb{R}^n)$ functions $\{y_m\}_{m=1}^\infty$ such that $y_m \to y$ in $L^1(I,\mathbb{R}^n)$. For each $m\in\mathbb{N}$, let $x_m(t)=\int_0^1 G(t,s)y_m(s)\,ds$. Then, by Lemma 2.2, $x_m\in C^1(I,\mathbb{R}^n)$, Ax'_m is continuously differentiable and $x_m(0)-A_0x'_m(0)=0$, $x_m(1)+A_1x'_m(1)=0$.

Now, for each $t\in I$,

$$
|x_m(t)-x(t)| = \left|\int_0^1 G(t,s)(y_m(s)-y(s))\,ds\right| \le nk\|y_m-y\|_1 \to 0 \quad \text{as } m\to\infty.
$$

That is, $x_m\to x$ uniformly, where k is the constant in Lemma 2.1(c).

If we let $z(t)=\int_0^1 \frac{\partial}{\partial t}G(t,s)y(s)\,ds$, then for each $t\in I$, we have

$$
\begin{aligned}
|x'_m(t)-z(t)| &= \left|\int_0^1 \frac{\partial}{\partial t}G(t,s)(y_m-y)(s)\,ds\right| \\
&= \left|\left(\int_0^t + \int_t^1\right)\frac{\partial}{\partial t}G(t,s)(y_m-y)(s)\,ds\right| \\
&\le nk\left(\int_0^t + \int_t^1\right)|y_m-y|(s)\,ds \\
&= nk\|y_m-y\|_1 \to 0 \qquad \text{as } m\to\infty.
\end{aligned}
$$

Thus, $x'_m\to z$ uniformly.

Since $x_m\in C^1(I,\mathbb{R}^n)$ for all $m\in\mathbb{N}$ and $x'_m\to z$ uniformly and $x_m(t)\to x(t)$ for all $t\in I$, it follows that $x'(t)=z(t)$ and $x\in C^1(I,\mathbb{R}^n)$. Moreover, $x'(t)=\int_0^1 \frac{\partial}{\partial t}G(t,s)y(s)\,ds$. Since $x_m(0)-A_0x'_m(0)=0$ and $x_m(1)+A_1x'_m(1)=0$, letting $m\to\infty$, we get (c).

Finally, since Ax'_m is continuously differentiable for all $m\in\mathbb{N}$,

$$
-A(t)x'_m(t)+A(0)x'_m(0) = -\int_0^t (A(s)x'_m(s))'\,ds = \int_0^t y_m(s)\,ds
$$

for all $t \in I$. Then, since $y_m \to y$ in $L^1(I, \mathbb{R}^n)$, we obtain

$$-A(t)x'(t) + A(0)x'(0) = \int_0^t y(s)\,ds$$

for all $t \in I$. Consequently, Ax' is absolutely continuous and

$$-(A(t)x'(t))' = y(t) \qquad \text{a.e. on } I.$$

The proof is complete. □

3. Existence theorems

The existence theorems in this paper are based on the following nonlinear alternative theorem of A. Granas [2].

LEMMA 3.1. *Assume that U is a relatively open subset of a convex set K in a Banach space E. Let $N : \overline{U} \to K$ be a compact map, and assume that $0 \in U$. Then either*

(1) *N has a fixed point in $\overline{U}$,*

or

(2) *there is a point $u \in \partial U$ and a number $\lambda \in (0,1)$ such that $u = \lambda N u$.*

We shall apply Lemma 3.1 with $E = C^1(I, \mathbb{R}^n)$ equipped with the C^1-norm, $K = C^1_{\mathrm{B}}(I, R^n) = \{x \in E : x(0) - A_0 x'(0) = 0,\ x(1) + A_1 x'(1) = 0\}$, and with $N : K \to K$ being the mapping defined by

$$(Nx)(t) = \int_0^1 G(t,s) f(s, x(s), x'(s))\,ds$$

for all $t \in I$ and $x \in K$. That N is completely continuous is known when f is continuous [7]; we shall establish this fact for f satisfying the Carathéodory conditions.

LEMMA 3.2. *N is completely continuous.*

PROOF. Let Z be any bounded set in K. Then there is a constant $r > 0$ such that $\|x\|_{C^1} \le r$ for all $x \in Z$. Since f satisfies the Carathéodory conditions, there is a Lebesgue integrable function g_r such that

$$|f(s, x(s), x'(s))| \le g_r(s)$$

for almost every $s \in I$ and for all $x \in Z$. Hence

$$|(Nx)(t)| = \left| \int_0^1 G(t,s) f(s, x(s), x'(s))\,ds \right| \le nk \|g_r\|_1$$

and

$$|(Nx)'(t)| = \left| \int_0^1 \frac{\partial}{\partial t} G(t,s) f(s, x(s), x'(s))\,ds \right| \le nk \|g_r\|_1$$

for all $t \in I$ and $x \in Z$, where k is the constant in Lemma 2.1(c). Therefore $N(Z)$ is uniformly bounded in K.

Since $G(t,s)$ is continuous on $[0,1]^2$, it is also uniformly continuous. Therefore, for any $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that, for $t_1, t_2, s_1, s_2 \in [0,1]$,

$$|G(t_1,s_1) - G(t_2,s_2)| < \frac{\epsilon}{n(\|g_r\|_1 + 1)}$$

whenever $|t_1 - t_2|, |s_1 - s_2| < \delta$. Hence, if $|t_1 - t_2| < \delta$ and $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned} |(Nx)(t_1) - (Nx)(t_2)| &\le n \int_0^1 |G(t_1,s) - G(t_2,s)| \cdot |f(s,x(s),x'(s))|\, ds \\ &\le \frac{\epsilon}{\|g_r\|_1 + 1} \int_0^1 |f(s,x(s),x'(s))|\, ds \\ &\le \frac{\epsilon}{\|g_r\|_1 + 1} \|g_r\|_1 < \epsilon \end{aligned}$$

for all $x \in Z$. Therefore $\{Nx : x \in Z\}$ is equicontinuous.

Finally, for any $\epsilon > 0$, since $g_r \in L^1(I, \mathbb{R}_+)$, there is $\delta_1 = \delta_1(\epsilon) > 0$ such that, for any set $\Omega \subset [0,1]$ with $|\Omega| < 3\delta_1$,

$$\int_\Omega g_r(s)\, ds < \frac{\epsilon}{6kn^2},$$

where k is the constant in Lemma 2.1(c). Since $\frac{\partial}{\partial t}G(t,s)$ is uniformly continuous on $S_1 = \{(t,s) \in [0,1]^2 : s \le t - \delta_1\}$, there is $\delta_2 = \delta_2(\epsilon) > 0$ such that for $(t_1,s_1), (t_2,s_2) \in S_1$,

$$\left| \frac{\partial}{\partial t}G(t_1,s_1) - \frac{\partial}{\partial t}G(t_2,s_2) \right| < \frac{\epsilon}{3n(\|g_r\|_1 + 1)}$$

whenever $|t_1 - t_2|, |s_1 - s_2| < \delta_2$. Similarly, there is $\delta_3 = \delta_3(\epsilon) > 0$ such that for $(t_1,s_1), (t_2,s_2) \in \{(t,s) \in [0,1]^2 : s \ge t + \delta_1\}$,

$$\left| \frac{\partial}{\partial t}G(t_1,s_1) - \frac{\partial}{\partial t}G(t_2,s_2) \right| < \frac{\epsilon}{3n(\|g_r\|_1 + 1)}$$

whenever $|t_1 - t_2|, |s_1 - s_2| < \delta_3$. Letting $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, for $0 \le t_2 - t_1 < \delta$ and $t_1, t_2 \in [0,1]$ we have

$$\begin{aligned} I &= |(Nx)'(t_1) - (Nx)'(t_2)| \\ &\le n \int_0^1 \left| \frac{\partial}{\partial t}G(t_1,s) - \frac{\partial}{\partial t}G(t_2,s) \right| |f(s,x(s),x'(s))|\, ds = I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 = n\int_0^{\max\{t_1-\delta_1,0\}} \left|\frac{\partial}{\partial t}G(t_1,s) - \frac{\partial}{\partial t}G(t_2,s)\right| |f(s,x(s),x'(s))|\,ds,$$
$$I_2 = n\int_{\max\{t_1-\delta_1,0\}}^{\min\{t_2+\delta_1,1\}} \left|\frac{\partial}{\partial t}G(t_1,s) - \frac{\partial}{\partial t}G(t_2,s)\right| |f(s,x(s),x'(s))|\,ds,$$
$$I_3 = n\int_{\min\{t_2+\delta_1,1\}}^{1} \left|\frac{\partial}{\partial t}G(t_1,s) - \frac{\partial}{\partial t}G(t_2,s)\right| |f(s,x(s),x'(s))|\,ds.$$

Without loss of generality we may assume that $\delta_1 \le t_1 \le t_2 \le 1-\delta_1$. We have

$$\begin{aligned}
I_1 &= n\int_0^{t_1-\delta_1} \left|\frac{\partial}{\partial t}G(t_1,s) - \frac{\partial}{\partial t}G(t_2,s)\right| |f(s,x(s),x'(s))|\,ds \\
&\le \frac{\epsilon}{3(\|g_r\|_1+1)}\int_0^1 |f(s,x(s),x'(s))|\,ds \\
&\le \frac{\epsilon}{3(\|g_r\|_1+1)}\|g_r\|_1 \le \frac{\epsilon}{3}; \\
I_3 &= n\int_{t_2+\delta_1}^{1} \left|\frac{\partial}{\partial t}G(t_1,s) - \frac{\partial}{\partial t}G(t_2,s)\right| |f(s,x(s),x'(s))|\,ds \\
&\le \frac{\epsilon}{3(\|g_r\|_1+1)}\int_0^1 |f(s,x(s),x'(s))|\,ds \\
&\le \frac{\epsilon}{3(\|g_r\|_1+1)}\|g_r\|_1 \le \frac{\epsilon}{3};
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= n\int_{t_1-\delta_1}^{t_2+\delta_1} \left|\frac{\partial}{\partial t}G(t_1,s) - \frac{\partial}{\partial t}G(t_2,s)\right| |f(s,x(s),x'(s))|\,ds \\
&\le 2n^2k\int_{t_1-\delta_1}^{t_2+\delta_1} |f(s,x(s),x'(s))|\,ds \\
&\le 2n^2k\int_{t_1-\delta_1}^{t_2+\delta_1} g_r(s)\,ds < \frac{2n^2k\epsilon}{6n^2k} = \frac{\epsilon}{3}.
\end{aligned}$$

It follows that $I < \epsilon$ and $\{(Nx)' : x \in Z\}$ is equicontinuous. Therefore, by the Ascoli theorem, $N(Z)$ is relatively compact in K, which establishes the lemma.$\square$

Now, let λ be in $[0,1]$ and $S(\lambda)$ be the set of C^1 functions $x : I \to \mathbb{R}^n$ which satisfy

$$(3.1_\lambda) \qquad \begin{cases} -(A(t)x'(t))' = \lambda f(t,x(t),x'(t)); \\ x(0) - A_0x'(0) = 0, \\ x(1) + A_1x'(1) = 0. \end{cases}$$

Then we have the following lemma.

Lemma 3.3. *If there is $r > 0$ such that for each $\lambda \in (0,1)$, we have*

$$\|x\| \leq r, \qquad \|x'\| \leq r$$

for all $x \in S(\lambda)$, then $S(1)$ is not empty.

Proof. Let $U = \{x \in K : \|x\|_{C^1} < r+1\}$. Then $N : \overline{U} \to K$ is a compact map by Lemma 3.2. Suppose that there is a point $u \in \partial U$ and a number $\lambda \in (0,1)$ such that $u = \lambda N u$. Then $u(t) = \lambda \int_0^1 G(t,s) f(s,u(s),u'(s))\,ds$, $\|u\|_{C^1} = r+1$. Now $f(s,u(s),u'(s))$ is a Lebesgue integrable function by (C3) and therefore, by Lemma 2.3, $u(t) = \lambda \int_0^1 G(t,s) f(s,u(s),u'(s))\,ds \in S(\lambda)$. This implies $\|u\|_{C^1} \leq r$, which is a contradiction. Therefore, it follows from Lemma 3.1 that N has a fixed point in $\overline{U}$. Repeating the argument with λ replaced by 1 we infer that $S(1)$ is not empty. □

We now establish our main results as follows:

Theorem 3.1. *Assume that $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying the Carathéodory conditions, $A : I \to M_n(\mathbb{R})$ a continuous matrix-valued function, and A_0, A_1 $n \times n$ matrices satisfying the following conditions* (1)–(3)*:*

(1) *there exists a positive number μ such that $(\xi, A(t)\xi) \geq \mu|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $t \in I$;*
(2) *$\int_0^1 A^{-1}(s)\,ds + A_0 A^{-1}(0) + A_1 A^{-1}(1)$ is invertible;*
(3) *one of the following conditions holds:*
  (i) *$A_0 = A_1 = 0$;*
  (ii) *$A_0 = 0$ and $A^\top(1) A_1 > 0$;*
  (iii) *$A_1 = 0$ and $A^\top(0) A_0 > 0$,*
  where $A^\top$ is the transpose of A.

Suppose, moreover, that

(4) *there exist nonnegative numbers a, b such that $a + b < \mu$ and $g \in L^1(I, \mathbb{R}_+)$ such that for every $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have*
$$(x, f(t,x,y)) \leq a|x|^2 + b|x||y| + g(t)|x| \qquad \text{for a.e. } t \in I;$$
(5) *there exist $c \geq 0$ and $h \in L^1(I, \mathbb{R}_+)$ such that for every $x \in \mathbb{R}^n$ with $|x| \leq (\mu - a - b)^{-1}\|g\|_1$ and every $y \in \mathbb{R}^n$, we have*
$$|f(t,x,y)| \leq c|y|^2 + h(t) \qquad \text{for a.e. } t \in I.$$

Then the problem

$$\begin{cases} -(A(t)x'(t))' = f(t,x(t),x'(t)); \\ x(0) - A_0 x'(0) = 0, \\ x(1) + A_1 x'(1) = 0, \end{cases} \tag{3.2}$$

has a solution.

Proof. Consider the family of problems

$$(3.2_\lambda)\qquad \begin{cases} -(A(t)x'(t))' = \lambda f(t, x(t), x'(t)); \\ x(0) - A_0 x'(0) = 0, \\ x(1) + A_1 x'(1) = 0. \end{cases}$$

Let x be a solution of (3.2_λ) for some $\lambda \in (0,1)$. Then for a.e. $t \in I$, we have

$$\begin{aligned} -(x(t), (A(t)x'(t))') &= \lambda(x(t), f(t, x(t), x'(t))) \\ &\le a|x(t)|^2 + b|x(t)||x'(t)| + g(t)|x(t)|. \end{aligned}$$

Since

$$(A(t)x'(t), x(t))' = ((A(t)x'(t))', x(t)) + (A(t)x'(t), x'(t)) \qquad \text{a.e. on } I,$$

by integrating over I, we obtain

$$\begin{aligned} \mu\|x'\|_2^2 &\le (A(t)x'(t), x(t))|_0^1 - \int_0^1 (x(t), (A(t)x'(t))')\,dt \\ &\le (A(t)x'(t), x(t))|_0^1 + a\|x\|_2^2 + b\|x\|_2\|x'\|_2 + \|g\|_1\|x\|_{C^0} \\ &= I_1 + I_2, \end{aligned}$$

where $I_1 = (A(t)x'(t), x(t))|_0^1 = (A(1)x'(1), x(1)) - (A(0)x'(0), x(0))$ and $I_2 = a\|x\|_2^2 + b\|x\|_2\|x'\|_2 + \|g\|_1\|x\|_{C^0}$.

Now, we claim that $I_1 \le 0$. If $A_0 = A_1 = 0$, then $x(0) = x(1) = 0$, which implies $I_1 = 0$. If $A_0 = 0$ and $A^\top(1)A_1 > 0$, then $x(0) = 0$ and $x(1) = -A_1x'(1)$, which implies $I_1 = -(A(1)x'(1), A_1x'(1)) = -(x'(1), A^\top(1)A_1x'(1)) \le 0$. If $A_1 = 0$ and $A^\top(0)A_0 > 0$, then $x(1) = 0$ and $x(0) = A_0x'(0)$, which implies $I_1 = -(A(0)x'(0), A_0x'(0)) = -(x'(0), A^\top(0)A_0x'(0)) \le 0$.

Since $x(t) = x(0) + \int_0^t x'(s)\,ds = x(1) - \int_t^1 x'(s)\,ds$ for all $t \in I$, we have

$$|x(t)| \le |x(0)| + \int_0^1 |x'(s)|\,ds \quad \text{and} \quad |x(t)| \le |x(1)| + \int_0^1 |x'(s)|\,ds$$

for all $t \in I$. Hence

$$\|x\|_2 \le \|x\|_{C^0} \le \|x'\|_2,$$

so

$$\mu\|x'\|_2^2 \le I_2 \le a\|x'\|_2^2 + b\|x'\|_2^2 + \|g\|_1\|x'\|_2,$$

and consequently

$$\|x\|_{C^0} \le \|x'\|_2 \le (\mu - a - b)^{-1}\|g\|_1 \equiv r_1,$$

where r_1 is independent of x. Thus, by assumption, there exist $c \ge 0$ and $h \in L^1(I, \mathbb{R}_+)$ such that

$$|f(t, x(t), x'(t))| \le c|x'(t)|^2 + h(t)$$

for a.e. $t \in I$. Hence

$$\begin{aligned}\int_0^1 |f(t,x(t),x'(t))|\,dt &\le c\int_0^1 |x'(t)|^2\,dt + \int_0^1 h(t)\,dt \\ &\le c\|x'\|_2^2 + \|h\|_1 \le cr_1^2 + \|h\|_1 \equiv r_2,\end{aligned}$$

where r_2 is also independent of x.

Finally, let us estimate $\|x'\|_{C^0}$. If $A_0 = A_1 = 0$, then $x(0) = x(1) = 0$, and

$$x(t) = -\lambda \int_0^t A^{-1}(s) \int_0^s f(u,x(u),x'(u))\,du\,ds + \bigg(\int_0^t A^{-1}(s)\,ds\bigg)Q,$$

where $Q = \lambda(\int_0^1 A^{-1}(s)ds)^{-1}(\int_0^1 A^{-1}(s)\int_0^s f(u,x(u),x'(u))\,du\,ds)$. Hence

$$x'(t) = -\lambda A^{-1}(t)\int_0^t f(u,x(u),x'(u))du + A^{-1}(t)Q,$$

and

$$\begin{aligned}|x'(t)| &\le nM\|f(\,\cdot\,,x(\,\cdot\,),x'(\,\cdot\,))\|_1 + n^3M^2P\|f(\,\cdot\,,x(\,\cdot\,),x'(\,\cdot\,))\|_1 \\ &\le c(A,I,A_0,A_1,n)\|f(\,\cdot\,,x(\,\cdot\,),x'(\,\cdot\,))\|_1 \le cr_2 \equiv r_3\end{aligned}$$

for all $t \in I$, where

$$P = \sup_{1\le i,j\le n}\bigg|\bigg(\bigg(\int_0^1 A^{-1}(s)ds\bigg)^{-1}\bigg)_{i,j}\bigg|, \qquad M = \sup_{\substack{1\le i,j\le n\\ t\in I}} |(A^{-1})_{i,j}(t)|.$$

Thus $\|x'\|_{C^0} \le r_3$ (independent of x).

From

$$\begin{aligned}A(t)x'(t) &= A(0)x'(0) + \int_0^t (A(s)x'(s))'\,ds \\ &= A(0)x'(0) - \lambda\int_0^t f(s,x(s),x'(s))\,ds,\end{aligned}$$

if $A_1 = 0$ and $A^\top(0)A_0 > 0$, we have

$$x'(t) = A^{-1}(t)A(0)A_0^{-1}x(0) - \lambda A^{-1}(t)\int_0^t f(s,x(s),x'(s))\,ds.$$

Therefore

$$|x'(t)| \le nM_1|x(0)| + nM\|f(\,\cdot\,,x(\,\cdot\,),x'(\,\cdot\,))\|_1 \le nM_1r_1 + nMr_2 \equiv r_3,$$

where

$$M_1 = \sup_{\substack{1\le i,j\le n\\ t\in I}} |(A^{-1}(t)A(0)A_0^{-1})_{i,j}| < \infty,$$

and r_3 depends only on $n, A_0, A_1, A(t)$ and I. We can conclude that $\|x'\|_{C^0} \le r_3$. The same result can be obtained if $A_0 = 0$ and $A^\top(1)A_1 > 0$. Hence $\|x\|_{C^1} < r$ if we set $r = 1 + r_1 + r_3$. This completes the proof. $\square$

THEOREM 3.2. *Assume that $f : I\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying the Carathéodory conditions, $A : I \to M_n(\mathbb{R})$ a continuous matrix-valued function, and A_0, A_1 $n\times n$ matrices satisfying the following conditions* (1)–(3)*:*

(1) *there exists a positive number μ such that $(\xi, A(t)\xi) \geq \mu|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $t \in I$;*
(2) *$\int_0^1 A^{-1}(s)\,ds + A_0A^{-1}(0) + A_1A^{-1}(1)$ is invertible;*
(3) *$A^\top(1)A_1 > 0$ and there is a positive number ν such that*
$$(\xi, A^\top(0)A_0\xi) \geq \nu|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Suppose, moreover, that

(4) *there exist nonnegative numbers a, b such that $2a+\frac{3}{2}b < \mu$, $2a+\frac{1}{2}b+\mu \leq \nu/(n^2M^2)$ and there exists $g \in L^1(I,\mathbb{R}_+)$ such that for every $(x,y) \in \mathbb{R}^n\times\mathbb{R}^n$, we have*
$$(x, f(t,x,y)) \leq a|x|^2 + b|x||y| + g(t)|x| \qquad \text{for a.e. } t \in I,$$
where $M = \sup_{1\leq i,j\leq n}|(A_0)_{i,j}|$;
(5) *there exist $c \geq 0$ and $h \in L^1(I,\mathbb{R}_+)$ such that for every $x \in \mathbb{R}^n$ with*
$$|x| \leq \mu^{-1}\big[(4a+3b)\big(\mu-2a-\tfrac{3}{2}b\big)^{-1}+2\big]\|g\|_1,$$
and for every $y \in \mathbb{R}^n$, we have
$$|f(t,x,y)| \leq c|y|^2 + h(t) \qquad \text{for a.e. } t \in I.$$

Then problem (3.2) *has a solution.*

PROOF. Consider the family of problems (3.2_λ) and let x be a solution of (3.2_λ) for some $\lambda \in (0,1)$. From the proof of the previous theorem, we know that
$$\mu\|x'\|_2^2 \leq I_1 + I_2,$$
where $I_1 = (A(t)x'(t), x(t))|_0^1$ and $I_2 = a\|x\|_2^2 + b\|x\|_2\|x'\|_2 + \|g\|_1\|x\|_{C^0}$. From assumption (3), we have
$$I_1 = -(x'(1), A^\top(1)A_1x'(1)) - (x'(0), A^\top(0)A_0x'(0)) \leq -\nu|x'(0)|^2.$$
Since $x \in C^1(I,\mathbb{R}^n)$, we may write $x(t) = x(0) + \int_0^t x'(s)\,ds$ for all $t \in I$, so $|x(t)| \leq |x(0)| + \int_0^t |x'(s)|\,ds$ for all $t \in I$, and $\|x\|_2 \leq \|x\|_{C^0} \leq |x(0)| + \|x'\|_2$. Consequently,
$$\begin{aligned}
\mu\|x'\|_2^2 &\leq I_1 + I_2\\
&\leq -\nu|x'(0)|^2 + a\|x\|_2^2 + b\|x\|_2\|x'\|_2 + \|g\|_1\|x\|_{C^0}\\
&\leq -\nu|x'(0)|^2 + a(|x(0)| + \|x'\|_2)^2 + b(|x(0)| + \|x'\|_2)\|x'\|_2\\
&\quad + \|g\|_1\|x\|_{C^0}\\
&\leq -\nu|x'(0)|^2 + 2a(|x(0)|^2 + \|x'\|_2^2) + \tfrac{1}{2}b(|x(0)|^2 + \|x'\|_2^2)\\
&\quad + b\|x'\|_2^2 + \|g\|_1\|x\|_{C^0}\\
&\leq \big(2an^2M^2 + \tfrac{1}{2}bn^2M^2 - \nu\big)|x'(0)|^2 + \big(2a + \tfrac{3}{2}b\big)\|x'\|_2^2 + \|g\|_1\|x\|_{C^0}.
\end{aligned}$$

Since $2a + \frac{1}{2}b + \mu \le \nu/(n^2M^2)$ and $2a + \frac{3}{2}b < \mu$, we have

$$\big(\mu - 2a - \tfrac{3}{2}b\big)\|x'\|_2^2 \le \|g\|_1\|x\|_{C^0},$$

and so

$$\|x'\|_2^2 \le \big(\mu - 2a - \tfrac{3}{2}b\big)^{-1}\|g\|_1\|x\|_{C^0}.$$

Consequently,

$$\begin{aligned}
\mu\|x\|_{C^0}^2 &\le 2[\mu|x(0)|^2 + \mu\|x'\|_2^2] \\
&\le 2\big[\mu n^2M^2|x'(0)|^2 + \big(2an^2M^2 + \tfrac{1}{2}bn^2M^2 - \nu\big)|x'(0)|^2 \\
&\quad + \big(2a + \tfrac{3}{2}b\big)\|x'\|_2^2 + \|g\|_1\|x\|_{C^0}\big] \\
&\le (4a + 3b)\|x'\|_2^2 + 2\|g\|_1\|x\|_{C^0} \\
&\le \big[(4a + 3b)\big(\mu - 2a - \tfrac{3}{2}b\big)^{-1} + 2\big]\|g\|_1\|x\|_{C^0}.
\end{aligned}$$

Thus $\|x\|_{C^0} \le \mu^{-1}[(4a + 3b)\big(\mu - 2a - \frac{3}{2}b\big)^{-1} + 2]\|g\|_1 \equiv r_1$ (independent of x). Then, by assumption, we have $c \ge 0$ and $h \in L^1(I, \mathbb{R}_+)$ such that

$$|f(t, x(t), x'(t))| \le c|x'(t)|^2 + h(t)$$

for a.e. $t \in I$, which implies

$$\begin{aligned}
\int_0^1 |f(t, x(t), x'(t))|\,dt &\le c\int_0^1 |x'(t)|^2\,dt + \int_0^1 h(t)\,dt \\
&\le c\big(\mu - 2a - \tfrac{3}{2}b\big)^{-1}\|g\|_1\|x\|_{C^0} + \|h\|_1 \\
&\le c\big(\mu - 2a - \tfrac{3}{2}b\big)^{-1}\|g\|_1 r_1 + \|h\|_1 \equiv r_2,
\end{aligned}$$

where r_2 is also independent of x.

From

$$\begin{aligned}
A(t)x'(t) &= A(0)x'(0) + \int_0^t (A(s)x'(s))'\,ds \\
&= A(0)x'(0) - \lambda\int_0^t f(s, x(s), x'(s))\,ds,
\end{aligned}$$

we have

$$x'(t) = A^{-1}(t)A(0)A_0^{-1}x(0) - \lambda A^{-1}(t)\int_0^t f(s, x(s), x'(s))\,ds.$$

Therefore

$$|x'(t)| \le nM_1|x(0)| + nM\|f(\,\cdot\,, x(\,\cdot\,), x'(\,\cdot\,))\|_1 \le nM_1r_1 + nMr_2 \equiv r_3,$$

where

$$M_1 = \sup_{\substack{1\le i,j\le n\\ t\in I}} |(A^{-1}(t)A(0)A_0^{-1})_{i,j}| < \infty, \qquad M = \sup_{\substack{1\le i,j\le n\\ t\in I}} |(A^{-1})_{i,j}(t)|$$

and r_3 depends only on $n, A_0, A_1, A(t)$, and I.

Thus, $\|x'\|_{C^0} \le r_3$ and eventually we get $\|x\|_{C^1} < r$ if we set $r = 1 + r_1 + r_3$, which completes the proof. □

REMARK. If condition (3) of Theorem 3.2 is replaced by

(3′) $A^\top(0)A_0 > 0$ and there is a positive number ν such that $(\xi, A^\top(1)A_1\xi) \ge \nu|\xi|^2$ for all $\xi \in \mathbb{R}^n$,

and M (in condition (4)) is redefined as $M = \sup_{1\le i,j\le n} |(A_1)_{i,j}|$, then the same conclusion holds.

4. Uniqueness theorems

THEOREM 4.1. *Assume that $f : I\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying the Carathéodory conditions, $A : I \to M_n(\mathbb{R})$ a continuous matrix-valued function, and A_0, A_1 $n \times n$ matrices satisfying the following conditions* (1)–(3):

(1) *there exists a positive number μ such that $(\xi, A(t)\xi) \ge \mu|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $t \in I$;*
(2) $\int_0^1 A^{-1}(s)\,ds + A_0A^{-1}(0) + A_1A^{-1}(1)$ *is invertible;*
(3) *one of the following conditions holds:*
  (i) $A_0 = A_1 = 0$;
  (ii) $A_0 = 0$ *and* $A^\top(1)A_1 > 0$;
  (iii) $A_1 = 0$ *and* $A^\top(0)A_0 > 0$,
  where $A^\top$ is the transpose of A.

Suppose, moreover, that

(4) *there exist nonnegative numbers a, b such that $a + b < \mu$ and*

$$(x-u, f(t,x,y) - f(t,u,v)) \le a|x-u|^2 + b|x-u||y-v|$$

for every $x, y, u, v \in \mathbb{R}^n$ and a.e. $t \in I$;
(5) *there exist $c \ge 0$ and $h \in L^1(I, \mathbb{R}_+)$ such that for every $x \in \mathbb{R}^n$ with $|x| \le (\mu - a - b)^{-1}\|f(t,0,0)\|_1$ and every $y \in \mathbb{R}^n$, we have*

$$|f(t,x,y)| \le c|y|^2 + h(t) \qquad \text{for a.e. } t \in I.$$

Then problem (3.2) *has a unique solution.*

PROOF. By (4) with $u = v = 0$, we have

$$(x, f(t,x,y)) \le a|x|^2 + b|x||y| + |f(t,0,0)||x|$$

for every $x, y \in \mathbb{R}^n$ and a.e. $t \in I$, which together with Theorem 3.1 implies the existence. Now, if x and u are two solutions of (3.2), we have

$$\begin{aligned} &-((x-u)(t), A(t)x'(t) - A(t)u'(t)) \\ &\qquad\qquad = ((x-u)(t), f(t,x(t),x'(t)) - f(t,u(t),u'(t))) \end{aligned}$$

for a.e. $t \in I$.

Since $A(x-u)'$ is absolutely continuous and $x-u \in C^1(I,\mathbb{R}^n)$, we have

$$\begin{aligned}&(A(t)(x-u)'(t),(x-u)(t))' \\ &\qquad = ((A(t)(x-u)'(t))',(x-u)(t)) + (A(t)(x-u)'(t),(x-u)'(t))\end{aligned}$$

for a.e. $t \in I$. Thus, by condition (1) and by integrating over I, we have

$$\begin{aligned}\mu\|x'-u'\|_2^2 &\le (A(t)(x-u)'(t),(x-u)(t))|_0^1 \\ &\quad + \int_0^1 |(f(t,x(t),x'(t)) - f(t,u(t),u'(t)),(x-u)(t))|\,dt \\ &\le (A(t)(x-u)'(t),(x-u)(t))|_0^1 + a\|x-u\|_2^2 + b\|x-u\|_2\|x'-u'\|_2 \\ &= I_1 + I_2,\end{aligned}$$

where

$$I_1 = (A(t)(x-u)'(t),(x-u)(t))|_0^1, \quad I_2 = a\|x-u\|_2^2 + b\|x-u\|_2\|x'-u'\|_2.$$

Now, we claim $I_1 \le 0$. If $A_0 = A_1 = 0$, then $x(0) = x(1) = u(0) = u(1)$, thus $I_1 = 0$. If $A_0 = 0$ and $A^\top(1)A_1 > 0$, then $x(0) = u(0)$ and so

$$I_1 = (A(1)(x-u)'(1),(x-u)(1)) = -((x-u)'(1), A^\top(1)A_1(x-u)'(1)) \le 0.$$

If $A_1 = 0$ and $A^\top(0)A_0 > 0$, then $x(1) = u(1)$ and so

$$I_1 = -(A(0)(x-u)'(0),(x-u)(0)) = -((x-u)'(0), A^\top(0)A_0(x-u)'(0)) \le 0.$$

Finally, since $x-u \in C^1(I,\mathbb{R}^n)$, we have

$$|x(t)-u(t)| \le |x(0)-u(0)| + \int_0^1 |(x'-u')(s)|\,ds$$

and

$$|x(t)-u(t)| \le |x(1)-u(1)| + \int_0^1 |(x'-u')(s)|\,ds,$$

for all $t \in I$. If $A_0 = 0$ or $A_1 = 0$, then $x(0) = u(0) = 0$ or $x(1) = u(1) = 0$, thus we always have $\|x-u\|_2 \le \|x'-u'\|_2$ and this implies $I_2 \le (a+b)\|x'-u'\|_2^2$. Hence $\mu\|x'-u'\|_2^2 \le (a+b)\|x'-u'\|_2^2$. Thus, if $a+b < \mu$, we have $x' = u'$ and so $x = u$ for $x(0) = u(0)$ or $x(1) = u(1)$. □

Similarly, corresponding to Theorem 3.2, we have the following uniqueness theorem.

THEOREM 4.2. *Assume that $f : I\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying the Carathéodory conditions, $A : I \to M_n(\mathbb{R})$ a continuous matrix-valued function, and A_0, A_1 $n\times n$ matrices satisfying the following conditions* (1)–(3)*:*

(1) *there exists a positive number μ such that $(\xi, A(t)\xi) \ge \mu|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $t \in I$;*
(2) $\int_0^1 A^{-1}(s)\,ds + A_0A^{-1}(0) + A_1A^{-1}(1)$ *is invertible;*

(3) $A^\top(1)A_1 > 0$ *and there is a positive number* ν *such that*

$$(\xi, A^\top(0)A_0\xi) \geq \nu|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^n.$$

Suppose, moreover, that

(4) *there exist nonnegative numbers* a, b *such that* $2a+\frac{3}{2}b < \mu$, $2a+\frac{1}{2}b+\mu \leq \nu/(n^2M^2)$, *and*

$$(x-u, f(t,x,y) - f(t,u,v)) \leq a|x-u| + b|y-v|$$

for every $x, y, u, v \in \mathbb{R}^n$ *and a.e.* $t \in I$, *where* $M = \sup_{1\leq i,j\leq n} |(A_0)_{i,j}|$;

(5) *there exist* $c \geq 0$ *and* $h \in L^1(I, \mathbb{R}_+)$ *such that for every* $x \in \mathbb{R}^n$ *with*

$$|x| \leq \mu^{-1}\big[(4a+3b)\big(\mu - 2a - \tfrac{3}{2}b\big)^{-1} + 2\big]\|f(t,0,0)\|_1,$$

and for every $y \in \mathbb{R}^n$, *we have*

$$|f(t,x,y)| \leq c|y|^2 + h(t) \qquad \textit{for a.e. } t \in I.$$

Then problem (3.2) *has a unique solution.*

5. Remarks

(1) If $A = I_{n\times n}, A_0, A_1 \geq 0$, then conditions (1), (2) of Lemma 2.1 hold.

(2) If $A = I_{n\times n}, A_0 = A_1 = 0$, Theorem 3.1 reduces to a result of J. Mawhin [8]. For related work with $A = I_{n\times n}$, we refer to [3] and [5]–[8].

(3) Suppose we have more general (nonhomogeneous) boundary conditions

$$y(0) - A_0y'(0) = r_0, \qquad y(1) + A_1y'(1) = r_1,$$

for given vectors r_0, r_1 in $\mathbb{R}^n$ and the equation

$$-(Ay')' = g(t, y, y'),$$

where $\int_0^1 A^{-1}(s)\,ds + A_0A^{-1}(0) + A_1A^{-1}(1)$ is invertible together with $A(t)$ for all $t \in I$. This problem reduces to the homogeneous problem

$$\begin{cases} -(Ax')' = f(t,x,x'); \\ x(0) - A_0x'(0) = 0, \\ x(1) + A_1x'(1) = 0, \end{cases}$$

by a transformation $x(t) = y(t) + v_1 + (\int_0^t A^{-1}(s)\,ds)v_2$, where v_1 and v_2 are suitably chosen in $\mathbb{R}^n$.

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