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RECESSION METHODS IN MONOTONE VARIATIONAL HEMIVARIATIONAL INEQUALITIES

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Dedicated to Professor Ky Fan on his eightieth birthday

1. Introduction, notations and definitions

Throughout the paper we use standard notations except special symbols introduced when they are defined. All spaces considered are Banach spaces whose norms are always denoted by $\|\cdot\|$. For any space V we consider its dual space $V^\star$ equipped with the strong topology. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V and $V^\star$. Let $f : V \to \mathbb{R} \cup \{\infty\}$ be an *extended-real-valued function*. Identifying extended-real-valued functions with their epigraphs

$$\operatorname{epi} f = \{(x,\alpha) \mid x \in V, \alpha \in \mathbb{R} \text{ and } \alpha \geq f(x)\}$$

is a standard tool in convex analysis and in one-sided optimization theory. Also, those functions with closed epigraphs are precisely the lower semicontinuous functions on V, and as usual,

$$\operatorname{dom} f := \{x \in V \mid f(x) < \infty\}.$$

We say that f is *proper* if $\operatorname{dom} f$ is nonempty. In this case $\limsup f(x)$ and $\liminf f(x)$ denote the *upper* and *lower limits* of such (scalar) functions in the classical sense. Depending on context, the symbols $x \xrightarrow{\mathrm{s}} y$ and $x \rightharpoonup y$ mean,

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respectively, that x tends to y with respect to the strong topology and the weak topology on V.

The theory of variational inequalities is now well established and one of the most famous result which has greatly contributed to its development is the Fan minimax principle (see [9] for the original proof). It should be observed that this principle is an immediate consequence of a two-function minimax theorem proved independently by Ben-El-Mechaiekh, Deguire & Granas [6], Yen [27] and Simons [24]:

Theorem 1. *Let C be a nonempty compact convex subset of a topological vector space and let $f, g : C \times C \to \mathbb{R}$ be two functions satisfying:*

(i) *$g \leq f$ on $C \times C$;*

(ii) *f is quasiconcave in its first variable and g is lower semicontinuous in its second variable.*

Then

$$\min_{y \in C} \sup_{x \in C} g(x, y) \leq \sup_{x \in C} f(x, x).$$

As another immediate consequence of the preceding theorem we also have:

Corollary 2. *Let C be a nonempty bounded closed convex subset of a real reflexive Banach space V. Let $f, g : C \times C \to \mathbb{R}$ be two functions satisfying:*

(i) *$g(x, y) \leq f(x, y)$ for all $x, y \in C$;*

(ii) *for each $y \in C$, the function $x \mapsto f(x, y)$ is concave;*

(iii) *for each $x \in C$, the function $y \mapsto g(x, y)$ is lower semicontinuous;*

(iv) *$f(x, x) \leq 0$ for all $x \in C$.*

Then there exists $u \in C$ such that $g(v, u) \leq 0$ for all $v \in C$.

Indeed, either this minimax principle or one of its equivalent forms has been used by many authors in order to obtain existence theorems applicable to several classes of variational inequalities.

Our aim in this paper is to use this two-function minimax theorem in a new direction of research, namely, in the field of hemivariational inequalities, theory introduced and developed by P. D. Panagiotopoulos [21] since the early 80s. The aim of this theory is the treatment of nonconvex, nonsmooth energy problems in mechanics. Since variational inequalities are based on the notion of convexity and are formulated for monotone multivalued boundary conditions and/or constitutive laws, they fail to apply to the problems listed above. The theory of variational hemivariational inequalities has been employed by P. D. Panagiatopoulos and his collaborators as a very efficient tool to describe the behavior of several complex structures, such as for instance the delamination problem in laminated composites, where the interaction between the laminae due to the binding material is described by means of a nonmonotone, possibly multivalued law. Such laws express a variety of limit phenomena related to the discontinuous loss of resistance at the surface. For complete references on the origin of the theory of hemivariational inequalities, a basic reference is the book of P. D. Panagiotopoulos [20].

Let us introduce further definitions. A functional $j : V \to \mathbb{R}$ is said to be *locally Lipschitz* if for each $x \in V$, there exist a neighbourhood $\mathcal{N}(x)$ of x and a constant $k_{\mathcal{N}} > 0$ such that

$$|j(u) - j(v)| \le k_{\mathcal{N}} \|u - v\|, \qquad \forall u, v \in \mathcal{N}.$$

We recall that an operator $A : V \to V^*$ is said to be *monotone* if

$$\langle Au - Av, u - v\rangle \ge 0 \qquad \forall u, v \in V,$$

while A is declared *hemicontinuous* if the functional $t \mapsto \langle A(u + tv), w\rangle$ is continuous on $[0, 1]$ for all $u, v, w \in V$.

Let K be a nonempty, closed and convex subset of a reflexive Banach space V. In order to simplify some computations we will assume that $0 \in K$. Throughout the paper, we assume that the assumptions $(\mathcal{H})$ described below are satisfied:

- (H_1) $A : V \to V^*$ is a monotone and hemicontinuous operator;
- (H_2) $j : V \to \mathbb{R}$ is a locally Lipschitz function;
- (H_3) $\Phi : V \to \mathbb{R} \cup \{\infty\}$ is a proper, convex and lower semicontinuous function satisfying $\Phi(0) = 0$;
- (H_4) f is given in V^*.

For a locally Lipschitzian functional $j : V \to \mathbb{R}$, we denote by $j^\circ(u; v)$ the *Clarke generalized directional derivative* of j at u in the direction v, that is,

$$j^\circ(u; v) := \limsup_{\substack{\lambda \to 0^+ \\ w \to u}} \frac{j(w + \lambda v) - j(w)}{\lambda}.$$

Recall also at this point that

$$\overline{\partial} j(x) := \{x^* \in V^* \mid j^\circ(x; y) \ge \langle x^*, y\rangle \ \forall y \in V\}$$

denotes the *generalized Clarke subdifferential* [8].

Let K be a closed convex subset of V. A vector y is called a *direction of recession* in K at x if for each positive t the vector $x + ty$ lies in K. The directions of recession are independent of x and form a closed convex cone called the *recession cone* of K:

$$K_\infty := \bigcap_{t>0} \left[\frac{K - x_0}{t}\right],$$

where x_0 is arbitrary chosen in K. Equivalently, this amounts to saying that x belongs to K_∞ if and only if there exist sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq K$ such that $\lim_{n\to\infty} t_n = \infty$ and $x = \lim_{n\to\infty} t_n^{-1} x_n$. The notion of recession cone has been used to sharpen many of the classical results of convexity theory (see e.g. R. T. Rockafellar [23]).

Let $\Phi : V \to \mathbb{R}\cup\{\infty\}$ be a proper, convex and lower semicontinuous function. Then the behavior at infinity of Φ can be described by the *recession* function Φ_∞ of Φ which is defined by the formula

$$\Phi_\infty(x) := \lim_{t\to\infty} \frac{\Phi(x_0 + tx) - \Phi(x_0)}{t},$$

where x_0 is taken arbitrary such that $\Phi(x_0) < \infty$. It is easily observed that

$$\operatorname{epi}\Phi_\infty = \{\operatorname{epi}\Phi\}_\infty.$$

In order to cover the case of nonconvex functionals, C. Baiocchi, G. Buttazzo, F. Gastaldi and F. Tomarelli [5] have introduced a more general concept of recession function. Let $\Psi : V \to \mathbb{R}$ be any functional. The recession function associated with Ψ is defined by

$$\Psi^\infty(x) := \inf\{\liminf_{n\to\infty} \Psi(t_n x_n)/t_n \mid t_n \to \infty,\ x_n \xrightarrow{\text{s}} x\}.$$

By a *variational hemivariational inequality* H.V.I.(A, f, j, Φ, k) we mean the problem

H.V.I. (A, f, j, Φ, K): Find $u \in K$ such that

$$\langle Au - f, v - u\rangle + j^\circ(u; v - u) + \Phi(v) - \Phi(u) \geq 0, \qquad \forall v \in K.$$

Using Corollary 2, we begin with proving an existence theorem for the class of problems H.V.I.(A, f, j, Φ, K) involving a bounded set of constraints.

Lemma 3. *Let C be a nonempty bounded closed convex subset of V such that $0 \in C$. Assume that the assumptions $(\mathcal{H})$ are satisfied. Then there exists $u \in C$ such that*

$$\langle Au - f, v - u\rangle + j^\circ(u; v - u) + \Phi(v) - \Phi(u) \geq 0, \qquad \forall v \in C.$$

Proof. Set

$$f(x, y) := \langle Ay - f, y - x\rangle - j^\circ(y; x - y) + \Phi(y) - \Phi(x),$$
$$g(x, y) := \langle Ax - f, y - x\rangle - j^\circ(y; x - y) + \Phi(y) - \Phi(x).$$

The monotonicity of A yields

$$g(x, y) \leq f(x, y).$$

For each $y \in C$, the map $x \mapsto j^\circ(y; x)$ is convex [8; Proposition 2.1.1] and thus $x \mapsto f(x, y)$ is concave. The map $(x, y) \mapsto j^\circ(x; y)$ is upper semicontinuous as a function of (x, y) [8; Proposition 2.1.1] and therefore the function $y \mapsto g(x, y)$ is lower semicontinuous.

We have $f(x,x) = 0$ for all $x \in C$. Hence by virtue of Corollary 2, we obtain the existence of $u \in C$ such that $g(v,u) \le 0$ for all $v \in C$. Equivalently, there exists $u \in C$ such that

$$\langle Av - f, v - u\rangle + j^\circ(u; v-u) + \Phi(v) - \Phi(u) \ge 0, \qquad \forall v \in C. \tag{1.1}$$

Let $y \in C$ be arbitrary. We set $v := ty + (1-t)u$, $0 < t < 1$. By putting v in (1.1), we obtain

$$\begin{aligned}\langle A(u + t(y-u)), t(y-u)\rangle + j^\circ(u; t(y-u)) + \Phi(u + t(y-u)) - \Phi(u)&\\ \ge \langle f, t(y-u)\rangle.&\end{aligned} \tag{1.2}$$

Dividing (1.2) by $t > 0$ and using the convexity of Φ, we derive

$$\langle A(u + t(y-u)), y-u\rangle + j^\circ(u; y-u) + \Phi(y) - \Phi(u) \ge \langle f, y-u\rangle. \tag{1.3}$$

Taking the limit as $t \to 0$ and using the hemicontinuity of A, (1.3) yields

$$\langle Au, y-u\rangle + j^\circ(u; y-u) + \Phi(y) - \Phi(u) \ge \langle f, y-u\rangle.$$

This ends the proof of the lemma. $\square$

In [13], a specific recession function was introduced which will play an important role in our study. We set $\Psi(x) := -j^\circ(x; -x)$ and denote by J°_∞ the recession function associated with Ψ, that is,

$$\begin{aligned}J^\circ_\infty(x) &= \liminf_{\substack{t\to\infty\\ y \xrightarrow{s} x}} \Psi(ty)/t = \inf\{\liminf_{n\to\infty} \Psi(t_n y_n)/t_n \mid t_n \to \infty,\ y_n \xrightarrow{s} x\}\\ &= \inf\{\liminf_{n\to\infty} -j^\circ(t_n y_n; -y_n) \mid t_n \to \infty,\ y_n \xrightarrow{s} x\}\\ &= \inf\{\liminf_{n\to\infty} -(-j)^\circ(t_n y_n; y_n) \mid t_n \to \infty,\ y_n \xrightarrow{s} x\}.\end{aligned}$$

Brézis & Nirenberg [7] introduced a *recession function* associated with a given nonlinear operator $A : V \to V^*$. They defined

$$\begin{aligned}r_A(u) :=&\ \liminf\{\langle A(tv), v\rangle \mid t \to \infty,\ v \xrightarrow{s} u\}\\ =&\ \inf\{\liminf_{n\to\infty} \langle A(t_n v_n), v_n\rangle \mid t_n \to \infty,\ v_n \xrightarrow{s} u\}.\end{aligned}$$

If we set $\Psi(u) := \langle Au, u\rangle$, then we can see that $r_A(u) = \Psi^\infty(u)$.

2. The main result

Let us introduce the set $R(A, f, j, \Phi, K)$ of *asymptotic directions*:

$$\begin{aligned}R(A, f, j, \Phi, K) := \{&w \in K_\infty \mid \exists u_n \in K,\ t_n := \|u_n\| \to \infty,\\ &w_n := u_n/\|u_n\| \rightharpoonup w \text{ and } \langle Au_n - f, u_n\rangle - j^\circ(u_n; -u_n) + \Phi(u_n) \le 0\}.\end{aligned}$$

The study of the properties of the recession set by means of recession tools as those defined in Section 1 constitutes what is called the recession analysis.

This approach is now the object of intensive work: see Adly, Goeleven & Théra [1], Attouch, Chbani & Moudafi [4], Goeleven [11] and the references therein, Goeleven & Théra [13], F. Tomarelli [26].

Theorem 4. *If the set* $R(A,f,j,\Phi,K)$ *is empty then* H.V.I.(A,f,j,Φ,K) *has at least one solution.*

Proof. Set $K_n := \{v \in K \mid \|v\| \le n\}$. Using Lemma 3, we get the existence of $u_n \in K_n$ such that

$$\langle Au_n - f, v - u_n\rangle + j^\circ(u_n; v-u_n) + \Phi(v) - \Phi(u_n) \ge 0, \qquad \forall v \in K_n.$$

Claim 1. *There exists* $n_0 \in \mathbb{N}$ *such that* $\|u_{n_0}\| < n_0$.

Indeed, suppose the contrary: $\|u_n\| = n$ for each solution u_n of H.V.I.(A,f,j,Φ,K_n). On relabelling if necessary, we can assume that $w_n := u_n/\|u_n\| \rightharpoonup w$ and

$$\langle Au_n - f, v - u_n\rangle + j^\circ(u_n; v-u_n) + \Phi(v) - \Phi(u_n) \ge 0, \qquad \forall v \in K_n. \tag{2.1}$$

By taking $v = 0$ in (2.1), we have

$$\langle Au_n - f, u_n\rangle - j^\circ(u_n; -u_n) + \Phi(u_n) \le 0.$$

Therefore $w \in R(A,f,j,\Phi,K)$, which contradicts the assumptions of Theorem 4.

Claim 2. u_{n_0} *solves* H. V. I.(A,f,j,Φ,K).

Since $\|u_{n_0}\| < n_0$, we have, for each $y \in K$, the existence of an $\varepsilon > 0$ such that $u_{n_0} + \varepsilon(y - u_{n_0}) \in K_{n_0}$. It suffices to take

$$\begin{cases} \varepsilon < (n_0 - \|u_{n_0}\|)/\|y - u_{n_0}\| & \text{if } y \ne u_{n_0}, \\ \varepsilon = 1 & \text{if } y = u_{n_0}. \end{cases}$$

We have

$$\langle Au_{n_0} - f, v - u_{n_0}\rangle + j^\circ(u_{n_0}; v-u_{n_0}) + \Phi(v) - \Phi(u_{n_0}) \ge 0, \qquad \forall v \in K_{n_0}. \tag{2.2}$$

If we put $v = u_{n_0} + \varepsilon(y - u_{n_0})$ in (2.2), we obtain

$$\langle Au_{n_0} - f, \varepsilon(y - u_{n_0})\rangle + j^\circ(u_{n_0}; \varepsilon(y-u_{n_0})) + \Phi(u_{n_0} + \varepsilon(y - u_{n_0})) - \Phi(u_{n_0}) \ge 0.$$

Using the convexity of Φ, we derive

$$\varepsilon\langle Au_{n_0} - f, y - u_{n_0}\rangle + \varepsilon j^\circ(u_{n_0}; y-u_{n_0}) + \varepsilon(\Phi(y) - \Phi(u_{n_0})) \ge 0. \tag{2.3}$$

Dividing (2.3) by $\varepsilon > 0$, we finally obtain

$$\langle Au_{n_0} - f, y - u_{n_0}\rangle + j^\circ(u_{n_0}; y-u_{n_0}) + \Phi(y) - \Phi(u_{n_0}) \ge 0, \qquad \forall y \in K.$$

This completes the proof. □

We say that $R(A,f,j,\Phi,K)$ is *asymptotically compact* if the sequence $\{w_n\}_{n\in\mathbb{N}}$ which appears in the definition of this set converges strongly to w, that is, if $w_n := u_n/\|u_n\|$, $\|u_n\| \to \infty$, $u_n \in K$ and

$$\langle Au_n - f, u_n\rangle - j^\circ(u_n; -u_n) + \Phi(u_n) \le 0$$

implies that $w_n \xrightarrow{\mathrm{s}} w$.

COROLLARY 5. *Suppose that the assumptions* $(\mathcal{H})$ *are satisfied. Assume that:*

(i) $R(A,f,j,\Phi,K)$ *is asymptotically compact;*

(ii) *there is a subset* W *of* $V\setminus\{0\}$ *such that* $R(A,f,j,\Phi,K)\subseteq W$ *and*

$$r_A(w) + J^\circ_\infty(w) + \Phi_\infty(w) > \langle f, w\rangle, \qquad \forall w \in W.$$

Then the problem H.V.I.(A,f,j,Φ,K) *has at least one solution.*

PROOF. Suppose by contradiction that $R(A,f,j,\Phi,K)$ is nonempty. Then we can find a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $t_n := \|u_n\| \to \infty$, $w_n := u_n/t_n \rightharpoonup w$ and

$$\langle Au_n, u_n\rangle - j^\circ(u_n; -u_n) + \Phi(u_n) \le \langle f, u_n\rangle. \tag{2.4}$$

Dividing (2.4) by $t_n > 0$, we obtain

$$\langle A(t_n w_n), w_n\rangle - j^\circ(t_n w_n; -w_n) + \frac{\Phi(t_n w_n)}{t_n} \le \langle f, w_n\rangle. \tag{2.5}$$

Passing to the liminf in (2.5), we derive

$$r_A(w) + J^\circ_\infty(w) + \Phi_\infty(w) \le \langle f, w\rangle. \tag{2.6}$$

Hence, (2.6) contradicts (ii) and the proof follows. □

3. Discussion on a robot hand grasping problem

Our main motivation in this section is given by a robot hand grasping problem which can be formulated as a variational hemivariational inequality involving monotone and singular matrices. Inequality methods in robotics were introduced by A. M. Al-Fahed, G. E. Stavroulakis & P. D. Panagiotopoulos [2], P. D. Panagiotopoulos & A. M. Al-Fahed [22] for the study of unilateral contact effects, both frictionless and frictional ones, between the fingers of a multifingered robotic hand and the manipulated object. In these papers the problem has been reduced to appropriately defined, generally nonsymmetric linear complementarity problems. However, the linear complementarity model is not accurate enough to take into account adhesive effects. More recently, a generalization of the above mentioned model which includes nonmonotone adhesive effects and nonclassical friction effects by including nonconvex yield surfaces in the linear complementarity model has been studied by D. Goeleven, G. E. Stavroulakis & P. D. Panagiotopoulos [12] and G. E. Stavroulakis, D. Goeleven & P. D. Panagiotopoulos [25]. By applying the theory presented in Section 2 we are able to consider another class of nonmonotone adhesive grippers. We note that the class we will investigate next cannot be considered by means of the theory given in [12]. Conversely, the class of adhesive effects described in [12], [25] is modelled by a specific and complex class of hemivariational inequalities which is not included in the formulation given by H.V.I.(A, f, j, Φ, K). In this sense, we enlarge the theory applicable to robot hand grasping problems.

Let us consider a rigid object which is grasped by a robotic hand with n elastic fingers. Besides the unilateral contact effects, frictional effects are also assumed on the fingertip-object areas. Only hard fingers which prevent displacements of the object in the normal and tangential directions with respect to the boundary are considered here.

The external forces or torques applied on the reference point of the rigid object are denoted by the vector

$$p = (p_1, p_2, p_3, m_1, m_2, m_3)^t$$

(for the general three-dimensional case; in two-dimensional applications p_1 and m_2, m_3 disappear). With respect to a rectangular cartesian coordinate system $Ox_1x_2x_3$, we gather the rigid body displacements and rigid rotations into the vector

$$u^0 := (u_1^0, u_2^0, u_3^0, \phi_1^0, \phi_2^0, \phi_3^0)^t,$$

where the elements u_i^0 denote the rigid body displacements while ϕ_i^0 denote the rigid body rotations. Regarding normal forces, they are gathered into the vector

$$r_N := (r_{N_1}, \dots, r_{N_n})^t,$$

while friction forces and torques are gathered into the vector r_T where

$$r_T = (r_{T11}, r_{T12}, r_{T21}, r_{T22}, \dots, r_{Tn1}, r_{Tn2})^t.$$

As described in [2], [3], the relations that govern the object-gripper system are given by the global equilibrium equations:

$$Gr = G_N r_N + G_T r_T,$$

where the equilibrium matrix G is decomposed in a $6 \times n$ submatrix G_N for the normal contact reactions and a $6 \times 2n$ submatrix G_T for the frictional forces. Compatibility conditions are given by the formula

$$u_N + d_N = u_N^0,$$

where u_N is the deformation of the fingertip, u_N^0 is the normal component of the displacement of the object boundary at points adjacent to the fingertips and d_N is the possibly nonzero initial gap between the fingertips and the object.

By means of the principle of complementary virtual work, the relationship is derived as follows: for every statically admissible pair (p, r) we have

$$p^t u^0 + r^t u_N^0 = 0,$$

where u^0 and u_N^0 are supposed to be virtual displacements.

In the direction normal to the surface, the unilateral contact effects between fingertips and object couple u_N and r_N are described as follows: if contact occurs then a nonnegative reaction arises, that is,

$$\text{if} \quad u_N + d_N = u_N^0 \quad \text{then} \quad r_N \geq 0,$$

otherwise if contact is released then a zero reaction must be considered, i.e.

$$\text{if} \quad u_N + d_N > u_N^0 \quad \text{then} \quad r_N = 0.$$

We also assume the existence of a Coulomb law of dry friction connecting the tangential (frictional) forces, exerted by the fingertips on the object with normal (contact) forces. Moreover, to achieve a linear complementarity formulation of the above-described frictional contact gripper, we introduce a piecewise linearization of the friction law by a polyhedral approximation of the friction cone from the interior. We refer to [2] and [3] for more details concerning these considerations.

A linear elastic finger behaviour is assumed next:

$$u = Fr,$$

where

$$u = (u_N, u_T)^t, \quad r = (r_N, r_T)^t, \quad F := \begin{pmatrix} F_{NN} & F_{NT} \\ F_{TN} & F_{TT} \end{pmatrix}.$$

Here F is the symmetric flexibility matrix which is composed by an $n \times n$ nonsingular normal flexibility matrix F_{NN}, the $2n \times 2n$ nonsingular tangential flexibility matrix F_{TT} and appropriate couple flexibility matrices $F_{NT} = F_{TN}^t$.

Finally, by making use of all these mechanical relations and the unilateral boundary conditions, we obtain the following linear complementarity problem (the complete calculations related to our problem are very long and out of the scope of this paper. A detailed description of this model can be found in A. M. Al-Fahed, G. E. Stavroulakis & P. D. Panagiotopoulos [3], G. E. Stavroulakis, D. Goeleven & P. D. Panagiotopoulos [25]):

$$w - Mz = b, \quad w \geq 0, \quad z \geq 0, \quad w^t z = 0,$$

with

$$\begin{aligned} w &= (y_N, \gamma)^t, \qquad z = (r_N, \lambda)^t, \\ b &= (d_N - F_{NT}(Bd_T - A^t p) - G_N^t(Ad_T - Up), -T_T^t(Bd_T - A^t p))^t, \end{aligned}$$

and

$$M := \begin{pmatrix} F_{NN} - F_{NT}(BF_{TN} + A^t G_N) - G_N^t(AF_{TN} + UG_N) & T_N^t - T_T^t(BF_{TN} + A^t G_N) \\ (F_{NT}B + G_N^t A)T_T & T_T^t B T_T \end{pmatrix}$$

with

$$\begin{aligned} B &:= (F_{TT})^{-1} - (F_{TT})^{-1} G_T^t (G_T (F_{TT})^{-1} G_T^t)^{-1} G_T (F_{TT})^{-1}, \\ A^t &:= (F_{TT})^{-1} G_T^t (G_T (F_{TT})^{-1} G_T^t)^{-1}, \\ U &:= -(G_T (F_{TT})^{-1} G_T^t)^{-1}. \end{aligned}$$

Here $y_N = u_N - u_N^0 + d_N$, $\gamma := \mu|r_N| - |r_T|$, μ is the coefficient of friction, λ is the nonnegative slipping value associated with γ, d_T is the initial tangential distance between the fingertip and the potential point of contact, the matrices T_N and T_T are the matrices of the linearized friction law (here for a vector $x = \{x_i\}$, we denote by $|x|$ the vector $\{|x_i|\}$). The linear complementarity problem is defined on $K := \mathbb{R}_+^{n+m}$. Here m is equal to $n \times l$ for the hard-finger model where l is the number of the facets of the linearized friction cone [2]. It is known [2] that M is a positive semidefinite matrix.

We now extend the previous model to cover nonclassical fingertip-object interaction effects by using the theory derived in Section 2. The possibility of existing adhesive and unilateral effects with nonconvex yielding characteristics on the fingertips will be included in our previous model (unilateral contact part and adhesive part will be denoted by subscripts "u" and "a" respectively). In the spirit of a decomposition in unilateral and adhesive parts, the notation y_{N_u}, r_{N_u}, γ_u, λ_u should be used in our previous relations. Nevertheless, for notational simplicity, we avoided showing all these indices untill now that the classical frictional contact problem has been formulated. We know that the unilateral effects can be described by a linear complementarity problem which is also equivalent to the following variational inequality:

$$z_\mathrm{u} \in K: \quad w_\mathrm{u}^t(v - z_\mathrm{u}) \geq 0, \quad \forall v \in K. \tag{3.1}$$

Let us consider the case where adhesive effects can be represented by nonconvex yield surfaces in the (r_N, λ) space. Let

$$r_N = r_{N_\mathrm{u}}, \quad y_N = y_{N_\mathrm{u}} + y_{N_\mathrm{a}}, \quad \lambda = \lambda_\mathrm{u}, \quad \gamma = \gamma_\mathrm{u} + \gamma_\mathrm{a}.$$

Let y_{N_a} and γ_a be derived by

$$w_\mathrm{a} = w_{\mathrm{a},1} + w_{\mathrm{a},2},$$

where

$$(3.2) \qquad w_{\mathrm{a},1} \in -\partial\Phi(r_N, \lambda),$$

$$(3.3) \qquad w_{\mathrm{a},2} \in -\overline{\partial} j(r_N, \lambda).$$

Here $\Phi : \mathbb{R}^{n+m} \to \mathbb{R}$ is a proper convex and lower semicontinuous functional and $\partial\Phi$ denotes its convex subdifferential and $j : \mathbb{R}^{n+m} \to \mathbb{R}$ is a locally Lipschitz functional and $\overline{\partial} j$ denotes its Clarke's subdifferential. Law (3.2) contains as a special case the case of adhesive rotational contact, as well as the case of zigzag rotational and tangential law. See P. D. Panagiotopoulos [21] and J. J. Moreau & P. D. Panagiotopoulos [18] for more details. Law (3.2) is a well-known convex superpotential [17] which can be used to describe monotone tangential law.

We have

$$\Phi(v) - \Phi(z) + w_{\mathrm{a},1}^t (v - z) \geq 0, \qquad \forall v \in \mathbb{R}^{n+m},$$

and

$$j^\circ(z; v - z) + w_{\mathrm{a},2}^t (v - z) \geq 0, \qquad \forall v \in \mathbb{R}^{n+m}.$$

We have

$$w^t(v - u) = w_{\mathrm{a}}^t (v - u) + w_{\mathrm{u}}^t (v - u)$$

and thus our problem can be described by the following variational hemivariational inequality:

$$(3.4) \quad z \in K\colon \quad (Mz - b)^t (v - u) + \Phi(v) - \Phi(u) + j^\circ(u; v - u) \geq 0, \quad \forall v \in K.$$

It is clear that $R(M, b, j, \Phi, K)$ is asymptotically compact since our problem is stated in a finite-dimensional space. Moreover, we can prove (see [13] for more details) that

$$R(M, b, j, \Phi, K) \subseteq \mathrm{Ker}(M + M^t) \cap K \setminus \{0\}.$$

Therefore, a sufficient condition for the solvability of problem (3.4) is

$$\langle b, e \rangle < \Phi_\infty(e) + J_\infty^\circ(e), \qquad \forall e \in \mathrm{Ker}(M + M^t) \cap K \setminus \{0\}.$$

Remark 6.

(i) For a thorough discussion concerning the role of such a condition in mechanics we refer to [1], [4], [5], [11], [25].

(ii) Several results concerning the estimation of the recession functional J_∞° are stated in [13].

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