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APPLICATIONS OF A THEOREM CONCERNING SETS WITH CONNECTED SECTIONS

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Dedicated to Professor Ky Fan, with my greatest admiration and esteem

As the reader can notice, the title of the present paper differs from that of [3] only because the term *connected* replaces the term *convex*. This is not casual. Indeed, it remains our aim to show, by means of a series of further applications, the usefulness of our recent Theorem 2.3 of [6] which, in a certain sense, can be regarded as a "connected" version of the famous Theorems 1′ and 2 of [3].

In the sequel, given a product space $X \times Y$, we denote by p_X and p_Y the projections from $X \times Y$ onto X and Y, respectively. Moreover, if $A \subseteq X \times Y$, then for every $x \in X$ and $y \in Y$, we put

$$A_x = \{v \in Y : (x, v) \in A\} \quad \text{and} \quad A^y = \{u \in X : (u, y) \in A\}.$$

Also, when, in proper settings, they will appear, the symbols $\overline{B}$, $\mathrm{int}(B)$, ∂B, $\mathrm{aff}(B)$, and $\mathrm{ri}(B)$ will denote, respectively, the closure, the interior, the boundary, the affine hull, and the relative interior (that is, the interior in $\mathrm{aff}(B)$) of the set B.

For the reader's convenience, we recall the statement of Theorem 2.3 of [6]:

Theorem 1 ([6], Theorem 2.3). *Let X, Y be two topological spaces, with Y admitting a continuous bijection onto* $[0,1]$*, and let S, T be two subsets of $X \times Y$, with S connected and, for each $x \in X$, T_x connected. Moreover, assume*

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that either T^y is open for each $y \in Y$, or Y is compact and T is closed. Then at least one of the following assertions holds:

(α) $p_X(T) \neq X$.
(β) $p_Y(S) \neq Y$ *and* $\{y \in Y : (p_X(S) \times \{y\}) \cap T = \emptyset\} \neq \emptyset$.
(γ) $S \cap T \neq \emptyset$.

Let us also recall the following result which is useful to recognize the connectedness of a given set in a product space.

PROPOSITION 1 ([6], Theorem 2.4). *Let X, Y be two topological spaces and let S be a subset of $X \times Y$. Assume that at least one of the following four sets of conditions is satisfied:*

(γ_1) *$p_Y(S)$ is connected, S^y is connected for each $y \in Y$, and S_x is open for each $x \in X$;*
(γ_2) *$p_Y(S)$ is connected, X is compact, S is closed, and S^y is connected for each $y \in Y$;*
(γ_3) *$p_X(S)$ is connected, S_x is connected for each $x \in X$, and S^y is open for each $y \in Y$;*
(γ_4) *$p_X(S)$ is connected, Y is compact, S is closed and S_x is connected for each $x \in X$.*

Under such hypotheses, S is connected.

Then, thanks to Proposition 1, we have the following particular case of Theorem 1:

THEOREM 2 ([6], Theorem 2.5). *Let X, Y be two topological spaces, with Y admitting a continuous bijection onto $[0,1]$, and let S, T be two subsets of $X \times Y$. Assume that at least one of the following eight sets of conditions is satisfied:*

(δ_1) *$p_Y(S)$ is connected, S^y is connected for each $y \in Y$, S_x is open for each $x \in X$, T_x is connected for each $x \in X$, and T^y is open for each $y \in Y$;*
(δ_2) *$p_Y(S)$ is connected, Y is compact, S^y is connected for each $y \in Y$, S_x is open for each $x \in X$, T is closed, and T_x is connected for each $x \in X$;*
(δ_3) *$p_Y(S)$ is connected, X is compact, S is closed, S^y is connected for each $y \in Y$, T_x is connected for each $x \in X$, and T^y is open for each $y \in Y$;*
(δ_4) *$p_Y(S)$ is connected, X and Y are compact, S and T are closed, S^y is connected for each $y \in Y$, and T_x is connected for each $x \in X$;*
(δ_5) *$p_X(S)$ is connected, S_x and T_x are connected for each $x \in X$, and S^y and T^y are open for each $y \in Y$;*

(δ_6) *$p_X(S)$ is connected, Y is compact, S_x is connected for each $x \in X$, S^y is open for each $y \in Y$, T is closed, and T_x is connected for each $x \in X$;*

(δ_7) *$p_X(S)$ is connected, Y is compact, S is closed, S_x and T_x are connected for each $x \in X$, and T^y is open for each $y \in Y$;*

(δ_8) *$p_X(S)$ is connected, Y is compact, S and T are closed, and S_x and T_x are connected for each $x \in X$.*

Then at least one of the following assertions holds:

(α) $p_X(T) \neq X$.

(β) $p_Y(S) \neq Y$ *and* $\{y \in Y : (p_X(S) \times \{y\}) \cap T = \emptyset\} \neq \emptyset$.

(γ) $S \cap T \neq \emptyset$.

Before starting with our series of applications of Theorems 1 and 2, we point out the following

Proposition 2. *Let Y be a connected topological space admitting a continuous bijection onto $[0, 1]$. Then there are exactly two distinct points $u, v \in Y$ such that the sets $Y \setminus \{u\}$ and $Y \setminus \{v\}$ are connected. Precisely, one has $\{u, v\} = \{\varphi^{-1}(0), \varphi^{-1}(1)\}$ for any continuous bijection $\varphi : Y \to [0, 1]$.*

Proof. Let φ be any continuous bijection from Y onto $[0, 1]$. Let us show that $Y \setminus \{\varphi^{-1}(0)\}$ is connected. Arguing by contradiction, assume that there are two non-empty, open, disjoint sets A, B such that $A \cup B = Y \setminus \{\varphi^{-1}(0)\}$ (note that Y turns out to be Hausdorff). Since $Y \setminus A$ and $Y \setminus B$ are two (not singletons) closed sets whose intersection (that is, $\{\varphi^{-1}(0)\}$) and union (that is, Y) are connected, it follows that they are connected too ([5], p. 133). Consequently, $\varphi(Y \setminus A)$ and $\varphi(Y \setminus B)$ are two non-degenerate subintervals of $[0, 1]$ each of which contains 0. Of course, this is against the fact that $(Y \setminus A) \cap (Y \setminus B) = \{\varphi^{-1}(0)\}$. Likewise, it is seen that $Y \setminus \{\varphi^{-1}(1)\}$ is connected. Now, let $z \in Y \setminus \{\varphi^{-1}(0), \varphi^{-1}(1)\}$. Then the sets $\varphi^{-1}([0, \varphi(z)[)$ and $\varphi^{-1}(]\varphi(z), 1])$ are non-empty and open, and their union is $Y \setminus \{z\}$. So, $Y \setminus \{z\}$ is disconnected. This completes the proof. $\square$

The points u, v in the statement of Proposition 2 will be called the *extreme points* of Y.

Now, we start with the following

Theorem 3. *Let X, Y be two topological spaces, with Y connected and admitting a continuous bijection onto $[0, 1]$, and let S be a connected subset of $X \times Y$. In addition, assume that either S^y is closed for each $y \in Y$, or S is open and Y is compact. Finally, suppose that, for each $x \in X$, the set $Y \setminus S_x$ is connected. Then, if u, v are the extreme points of Y, at least one of the following assertions holds:*

(a) *There exists* $x_0 \in X$ *such that* $S_{x_0} = Y$.
(b) $S^u = \emptyset$.
(c) $S^v = \emptyset$.

Moreover, if $S^u = \emptyset$ *(resp.* $S^v = \emptyset$*), then* $S^v = p_X(S)$ *(resp.* $S^u = p_X(S)$*).*

PROOF. Let φ be any continuous bijection from Y onto $[0, 1]$. By Proposition 2, we have $\{u, v\} = \{\varphi^{-1}(0), \varphi^{-1}(1)\}$. For instance, let $u = \varphi^{-1}(0)$ and $v = \varphi^{-1}(1)$. Assume that (b) and (c) do not hold. Then one has $u, v \in p_Y(S)$. Hence, since $p_Y(S)$ is connected, we have $\varphi(p_Y(S)) = [0, 1]$, and so $p_Y(S) = Y$. Now, put

$$T = (X \times Y) \setminus S.$$

It is seen at once that S, T satisfy the assumptions of Theorem 1. Consequently, since (β) and (γ) are violated, (α) (that is, our present (a)) does hold.

Now, assume that $S^u = \emptyset$. Let $x \in p_X(S)$. Since $Y \setminus S_x$ is connected, $[0, 1] \setminus \varphi(S_x)$ turns out to be a proper subinterval of $[0, 1]$ containing 0. Consequently, $1 \in \varphi(S_x)$, that is, $v \in S_x$, and so $x \in S^v$, as desired. The claim with the roles of u, v interchanged is proved in a similar way. $\square$

In particular, applying Theorem 3, we get

THEOREM 4. *Let* X *be a compact topological space,* $Y \subseteq \mathbb{R}$ *an interval, and* S *a closed subset of* $X \times Y$ *such that* $Y \setminus S_x$ *is connected for each* $x \in X$*, and* S^y *is connected for each* $y \in Y$*. Then either* $p_Y(S) \neq Y$*, or* $S_{x_0} = Y$ *for some* $x_0 \in X$*.*

PROOF. Suppose that $p_Y(S) = Y$. Owing to the compactness of X, to get our conclusion it suffices to show that the family $\{S^y\}_{y \in Y}$ has the finite intersection property. So, let $y_1 < y_2 < \ldots < y_n$ be n points in Y. Thanks to Proposition 1 (case (γ_2)), the set $S \cap (X \times [y_1, y_n])$ is connected. Then, applying Theorem 3 in an obvious way, we get $x^* \in X$ such that $[y_1, y_n] \subseteq S_{x^*}$. Hence, $x^* \in \bigcap_{i=1}^n S^{y_i}$, as desired. $\square$

REMARK 1. Theorem 2 is particularly useful when the sections S^y are such that after removing suitable subsets from them, they remain connected. In fact, in such a case, generally either we are allowed to require the connectedness of the sections T_x only for particular points $x \in X$, or we can bring out some suitable qualitative property of $S \cap T$. We now indicate two specific situations. For the first of them, we need the following

PROPOSITION 3. *Let* E *be a Hausdorff topological vector space,* $A \subseteq E$ *an infinite-dimensional closed affine manifold,* $\Omega \subseteq A$ *a convex set whose interior in* A *is non-empty, and* $K \subseteq E$ *a relatively compact set. Then the set* $\Omega \setminus K$ *is connected.*

Proof. We first prove the proposition in the case where $A = E$. Let $x, y \in \mathrm{int}(\Omega) \setminus \overline{K}$. Fix a closed circled neighbourhood V of the origin such that

$$V + V \subseteq ((\mathrm{int}(\Omega) \setminus \overline{K}) - x) \cap ((\mathrm{int}(\Omega) \setminus \overline{K}) - y).$$

Observe, in particular, that V is connected. Since E is infinite-dimensional, V is not compact. Consequently, there is a net $\{y_\alpha\}$ in V having no cluster point in E. We claim that, for some α, the segment joining x and $y + y_\alpha$ does not meet $\overline{K}$.

On the contrary, assume that, for each α, there is $\lambda_\alpha \in [0, 1]$ such that $\lambda_\alpha(y + y_\alpha) + (1 - \lambda_\alpha)x \in \overline{K}$. Now, consider a $\delta > 0$ such that $\delta(y - x) \in V$. Thanks to our previous choices, it is seen that $\lambda_\alpha > \delta$. Since $\overline{K}$ is compact, the net $\{\lambda_\alpha(y + y_\alpha) + (1 - \lambda_\alpha)x\}$ admits a subnet, say $\{\lambda_{\alpha_\beta}(y + y_{\alpha_\beta}) + (1 - \lambda_{\alpha_\beta})x\}$, converging to a point $z \in \overline{K}$. On the other hand, also the net $\{\lambda_{\alpha_\beta}\}$ admits a subnet, say $\{\lambda_{\alpha_{\beta_\gamma}}\}$, converging to a point $\lambda \in [\delta, 1]$. Consequently, $z - (1 - \lambda)x$ is the limit of $\{\lambda_{\alpha_{\beta_\gamma}}(y + y_{\alpha_{\beta_\gamma}})\}$. Hence, $\lambda^{-1}(z - (1 - \lambda)x) - y$ is the limit of $\{y_{\alpha_{\beta_\gamma}}\}$, and so it is a cluster point of $\{y_\alpha\}$, a contradiction.

Then let α be such that the segment, say $S(x, y + y_\alpha)$, joining x and $y + y_\alpha$ does not meet $\overline{K}$. Since $\mathrm{int}(\Omega)$ is convex, we have $S(x, y + y_\alpha) \subseteq \mathrm{int}(\Omega) \setminus \overline{K}$. Therefore, $S(x, y + y_\alpha) \cup (y + V)$ is a connected subset of $\mathrm{int}(\Omega) \setminus \overline{K}$ containing x and y. This shows that $\mathrm{int}(\Omega) \setminus \overline{K}$ is connected. Now, taking into account that $\overline{\Omega} = \overline{\mathrm{int}(\Omega)}$, we have

$$\mathrm{int}(\Omega) \setminus \overline{K} \subseteq \Omega \setminus K \subseteq \overline{\mathrm{int}(\Omega) \setminus \overline{K}}$$

and so $\Omega \setminus K$ is connected.

Finally, to prove our proposition when $A \neq E$, it suffices to observe that, since A is closed, $K \cap A$ is relatively compact in A and that A is affinely homeomorphic to an infinite-dimensional Hausdorff topological vector space. □

We then have

Theorem 5. *Let X be a non-empty set in a Hausdorff topological vector space E, K a relatively compact subset of E, Y a connected topological space admitting a continuous bijection onto $[0, 1]$, and S, T two subsets of $X \times Y$. Assume that:*

(i) *S^y is convex, $\mathrm{aff}(S^y)$ is infinite-dimensional and closed in E, $\mathrm{ri}(S^y)$ is non-empty for each $y \in p_Y(S)$, and S_x is open in Y for each $x \in X \setminus K$;*
(ii) *T_x is connected for each $x \in X \setminus K$;*
(iii) *either $T^y \setminus K$ is open in $X \setminus K$ for each $y \in Y$, or Y is compact and $T \setminus (K \times Y)$ is closed in $(X \setminus K) \times Y$.*

Then at least one of the following assertions holds:

(a) $X \setminus (K \cup p_X(T)) \neq \emptyset$.

(b) $p_Y(S) \neq Y$.

(c) *For every set* $V \subseteq X \times Y$ *such that* V^y *is relatively compact in* E *for each* $y \in Y$ *and* V_x *is closed in* Y *for each* $x \in X \setminus K$*, one has* $(S \setminus (V \cup (K \times Y))) \cap T \neq \emptyset$.

Proof. Assume that (a) and (b) do not hold. Let V be as in (c). Then, by Proposition 3, $(S \setminus (V \cup (K \times Y)))^y$ is non-empty and connected for each $y \in Y$, and $(S \setminus (V \cup (K \times Y)))_x$ is open for each $x \in X \setminus K$. Hence, since Y is connected, the sets $S \setminus (V \cup (K \times Y))$ and $T \setminus (K \times Y)$ satisfy either (δ_1) or (δ_2) of Theorem 2, applied taking $(X \setminus K) \times Y$ as product space. So, $(S \setminus (V \cup (K \times Y))) \cap T \neq \emptyset$. □

The other situation to which we alluded in Remark 1 involves the covering dimension in $\mathbb{R}^n$. So, for each set $A \subseteq \mathbb{R}^n$, we denote by $\dim(A)$ its covering dimension ([2], p. 54).

Theorem 6. *Let* $X \subseteq \mathbb{R}^n$ *be a non-empty set,* Y *a connected topological space admitting a continuous bijection onto* $[0, 1]$*, and* S, T *two subsets of* $X \times Y$*. Assume that:*

(i) S^y *is connected and open in* $\mathbb{R}^n$ *for each* $y \in Y$*, and* S_x *is open in* Y *for each* $x \in X$*;*

(ii) T_x *is connected for each* $x \in X$*;*

(iii) *either* T^y *is open in* X *for each* $y \in Y$*, or* Y *is compact and* T *is closed in* $X \times Y$*.*

Then at least one of the following assertions holds:

(a) $p_X(T) \neq X$.

(b) $p_Y(S) \neq Y$.

(c) *For every set* $V \subseteq X \times Y$ *such that* $\dim(V^y) \leq n - 2$ *for each* $y \in Y$ *and* V_x *is closed in* Y *for each* $x \in X$*, one has* $(S \setminus V) \cap T \neq \emptyset$.

Proof. The proof goes exactly as that of Theorem 5, with $K = \emptyset$. The only difference is that, this time, the connectedness of each $(S \setminus V)^y$ follows directly from a celebrated theorem of Mazurkiewicz ([2], p. 80). □

Proceeding in a way by now evident, we also get

Theorem 7. *Let* X*,* Y *be as in Theorem* 6*, let* $S, T \subseteq X \times Y$*, and let* $K \subseteq X$ *be such that* $\dim(K) \leq n - 2$*. Assume that:*

(i) S^y *is connected and open in* $\mathbb{R}^n$ *for each* $y \in Y$*, and* S_x *is open in* Y *for each* $x \in X \setminus K$*;*

(ii) T_x *is connected for each* $x \in X \setminus K$*;*

(iii) *either* $T^y \setminus K$ *is open in* $X \setminus K$ *for each* $y \in Y$*, or* Y *is compact and* $T \setminus (K \times Y)$ *is closed in* $(X \setminus K) \times Y$*.*

Then at least one of the following assertions holds:

(a) $X \setminus (K \cup p_X(T)) \neq \emptyset$.
(b) $p_Y(S) \neq Y$.
(c) $(S \setminus (K \times Y)) \cap T \neq \emptyset$.

Before stating our next result, we need the following

Proposition 4. *Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty open connected set and A, B two proper subsets of Ω, both closed in Ω, such that $\Omega = A \cup B$. Then* $\dim(A \cap B) \geq n - 1$.

Proof. If $\mathrm{int}(A) \cap \mathrm{int}(B) \neq \emptyset$, clearly one has $\dim(A \cap B) = n$ ([2], p. 76). So, let us assume that $\mathrm{int}(A) \cap \mathrm{int}(B) = \emptyset$. Since A, B are closed in Ω, one has

$$\Omega \setminus (A \cap B) \subseteq \mathrm{int}(A) \cup \mathrm{int}(B).$$

On the other hand, since A, B are proper subsets of Ω, both $\mathrm{int}(A)$ and $\mathrm{int}(B)$ meet $\Omega \setminus (A \cap B)$. So, $\Omega \setminus (A \cap B)$ is disconnected. At this point, our conclusion follows directly from the already quoted theorem of Mazurkiewicz. $\square$

Now, we are able to establish the following

Theorem 8. *Let $[a, b]$ be a compact real interval and T a subset of $\mathbb{R}^n \times [a, b]$ which is closed in $p_{\mathbb{R}^n}(T) \times [a, b]$. Then, for every non-empty connected subset X of $p_{\mathbb{R}^n}(T)$ which is open in* $\mathrm{aff}(X)$ *and such that T_x is connected for each $x \in X$, at least one of the following assertions holds:*

(a) $X \subseteq T^a$.
(b) $X \subseteq T^b$.
(c) *There exists some $y \in\,]a, b[$ such that* $\dim(T^y \cap X) \geq \dim(X) - 1$.

Proof. Assume that (a) and (b) do not hold. Put

$$\Gamma = X \setminus (T^a \cup T^b).$$

We distinguish two cases.

First, suppose that $\Gamma \neq \emptyset$. Note that Γ is open in $\mathrm{aff}(X)$. Now, fix a sequence $\{Y_k\}$ of (non-degenerate) compact subintervals of $]a, b[$ such that $]a, b[\,= \bigcup_{k \in \mathbb{N}} Y_k$. For each $k \in \mathbb{N}$, put $V_k = \bigcup_{y \in Y_k} T^y$. By Theorem 7.1.16 of [4], the set V_k is closed in $p_{\mathbb{R}^n}(T)$. Clearly, one has $\Gamma \subseteq \bigcup_{k \in \mathbb{N}} V_k$. Endowed with the relative topology, Γ turns out to be a Baire space. Hence, there is some $k^* \in \mathbb{N}$ such that the interior of $V_{k^*} \cap \Gamma$ in Γ, and so in $\mathrm{aff}(X)$, is non-empty. Choose a non-empty connected set $W \subseteq V_{k^*} \cap \Gamma$ which is open in $\mathrm{aff}(X)$. We claim that there exists $y_0 \in Y_{k^*}$ such that $\dim(T^{y_0} \cap W) \geq \dim(W) - 1$.

Arguing by contradiction, assume that $\dim(T^y \cap W) \leq \dim(W) - 2$ for each $y \in Y_{k^*}$. Put

$$S = (W \times Y_{k^*}) \setminus T.$$

Then, thanks to the theorem of Mazurkiewicz, S^y is non-empty and connected for each $y \in Y_{k^*}$. Consequently, we can apply Theorem 2 (case (δ_2)) to the sets S and $T \cap (W \times Y_{k^*})$, upon taking $W \times Y_{k^*}$ as product space. But, recalling the definition of W, we see that the conclusion of Theorem 2 does not hold, which is absurd.

So, the claimed y_0 actually exists. Observing that $W \subseteq X$ and $\dim(W) = \dim(X)$, we then have $\dim(T^{y_0} \cap X) \geq \dim(X) - 1$, which yields (c).

Now, suppose that $X \subseteq T^a \cup T^b$. In other words, $T^a \cap X$ and $T^b \cap X$ are proper subsets of X, both closed in X, whose union is X. Then, by Proposition 4, we have $\dim(T^a \cap T^b \cap X) \geq \dim(X) - 1$. But, if $x \in T^a \cap T^b \cap X$, then since T_x is connected, we have $T_x = [a, b]$, that is to say, $x \in T^y$ for each $y \in [a, b]$. Hence, in the present case, we get $\dim(T^y \cap X) \geq \dim(X) - 1$ even for each $y \in [a, b]$. This completes the proof. $\square$

Remark 2. In Theorem 8, the closedness assumption on T cannot be dropped, in general. Indeed, if T is the graph of a bijection from $\mathbb{R}^2$ onto $[0, 1]$, taking, for instance, $X = \mathbb{R}^2$, none of (a), (b), (c) holds.

Here is an application of Theorem 8 to control theory. Let b be a positive real number and let F be a given multifunction from $[0, b] \times \mathbb{R}^n$ into $\mathbb{R}^n$. We denote by $\mathcal{S}_F$ the set of all Carathéodory solutions of the problem $x' \in F(t, x), x(0) = 0$ in $[0, b]$. That is to say,

$$\mathcal{S}_F = \{u \in AC([0, b], \mathbb{R}^n) : u'(t) \in F(t, u(t)) \text{ a.e. in } [0, b],\ u(0) = 0\}$$

where, of course, $AC([0, b], \mathbb{R}^n)$ denotes the space of all absolutely continuous functions from $[0, b]$ into $\mathbb{R}^n$. For each $t \in [0, b]$, put

$$\mathcal{A}_F(t) = \{u(t) : u \in \mathcal{S}_F\}.$$

In other words, $\mathcal{A}_F(t)$ denotes the attainable set at time t. Also, put

$$V_F = \bigcup_{t \in [0,b]} \mathcal{A}_F(t).$$

Finally, set

$$C_F = \{x \in \mathbb{R}^n : \{t \in [0, b] : x \in \mathcal{A}_F(t)\} \text{ is connected}\}.$$

With these notations, we have the following

Theorem 9. *Assume that F has non-empty compact convex values and bounded range. Moreover, assume that $F(\cdot, x)$ is measurable for each $x \in \mathbb{R}^n$ and that $F(t, \cdot)$ is upper semicontinuous for a.e. $t \in [0, b]$. Then, for every non-empty connected set $X \subseteq V_F \cap C_F$ which is open in* $\operatorname{aff}(X)$ *and different from* $\{0\}$*, one has the following alternative: either*

$$X \subseteq \mathcal{A}_F(b)$$

or

$$\dim(\mathcal{A}_F(t) \cap X) \geq \dim(X) - 1$$

for some $t \in]0, b[$.

PROOF. Put

$$T = \{(x,t) \in \mathbb{R}^n \times [0,b] : x \in \mathcal{A}_F(t)\}.$$

Under our assumptions, by a well-known result (see, for instance, Theorem 7.1 of [1]), the set T turns out to be closed. Now, our conclusion follows directly from Theorem 9, taking into account that $\mathcal{A}_F(0) = \{0\}$. □

REMARK 3. On the basis of Theorem 9, it would be interesting to investigate the structure of the set C_F.

The next result, another application of Theorem 2, concerns the existence of Nash equilibrium points.

THEOREM 10. *Let X be a Hausdorff compact topological space, Y an arc, and f, g two continuous real functions on $X \times Y$ such that, for each $\lambda \in \mathbb{R}$, $x_0 \in X$, $y_0 \in Y$, the sets $\{x \in X : f(x, y_0) \geq \lambda\}$ and $\{y \in Y : g(x_0, y) \geq \lambda\}$ are connected. Then there exists $(x^*, y^*) \in X \times Y$ such that*

$$f(x^*, y^*) = \max_{x \in X} f(x, y^*) \quad \textit{and} \quad g(x^*, y^*) = \max_{y \in Y} g(x^*, y).$$

PROOF. For each $x \in X$, $y \in Y$, put

$$\alpha(x) = \max_{v \in Y} g(x, v) \quad \text{and} \quad \beta(y) = \max_{u \in X} f(u, y).$$

Next, consider the sets

$$S = \{(x, y) \in X \times Y : f(x, y) = \beta(y)\}$$

and

$$T = \{(x, y) \in X \times Y : g(x, y) = \alpha(x)\}.$$

The continuity of f and g readily implies that S and T are closed. On the other hand, for each $x \in X$, $y \in Y$, one has

$$S^y = \bigcap_{n \in \mathbb{N}} \{u \in X : f(u, y) \geq \beta(y) - 1/n\}$$

and

$$T_x = \bigcap_{n \in \mathbb{N}} \{v \in Y : g(x, v) \geq \alpha(x) - 1/n\}.$$

So, by a classical result (see, for instance, [5], p. 170), S^y and T_x are connected (and non-empty, of course). Consequently, thanks to Theorem 2 (case (δ_4)), one has $S \cap T \neq \emptyset$. Plainly, any point in $S \cap T$ satisfies our conclusion. □

REMARK 4. Compare Theorem 10 with Theorem 4 of [3].

The next result, suggested by the new approach recently proposed in [7], is about the existence of zeros for certain operators.

Theorem 11. *Let V be a topological space, X a real topological vector space (with topological dual X^*), and $\Phi : V \to X^*$ an operator such that the set $\{x \in X : v \to \langle \Phi(v), x\rangle$ is continuous$\}$ is dense in X. Assume that there are a continuous function $u : [0,1] \to V$, a continuous function $\alpha : [0,1] \to \mathbb{R}$, a lower semicontinuous function $f : X \to [0,1]$ and an upper semicontinuous function $g : X \to [0,1]$, with $f(x) \leq g(x)$ for all $x \in X$, such that $\langle \Phi(u(y)), x\rangle \neq \alpha(y)$ for every $(x, y) \in X \times [0,1]$ satisfying $y \in [f(x), g(x)]$. Then the operator Φ vanishes at some point of V.*

Proof. Put

$$S = \{(x, y) \in X \times [0,1] : \langle \Phi(u(y)), x\rangle = \alpha(y)\}$$

and

$$T = \{(x, y) \in X \times [0,1] : y \in [f(x), g(x)]\}.$$

Arguing by contradiction, assume that $\Phi(v) \neq 0$ for all $v \in V$. In particular, this implies that $p_{[0,1]}(S) = [0,1]$. Also, observe that T is closed and $S \cap T = \emptyset$. Then, in view of Theorem 1, S must be disconnected. At this point, we can apply Theorem 1 and Proposition 1 of [7] to the operator $\Phi \circ u$, and so $\Phi(u(y)) = 0$ for some $y \in [0,1]$, a contradiction. □

We conclude with an application of Theorem 1 to compact mappings in Banach spaces. First, we need the following

Proposition 5. *Let X be a topological space, $Y \subseteq \mathbb{R}$ a compact interval, and $f : X \times Y \to \mathbb{R}$ an upper semicontinuous function such that $f(\cdot, y)$ is continuous for each $y \in Y$. Moreover, let $\lambda \in \mathbb{R}$ be such that*

$$\{y \in Y : f(x, y) > \lambda\} \neq \emptyset$$

and

$$\inf\{y \in Y : f(x, y) \geq \lambda\} = \inf\{y \in Y : f(x, y) > \lambda\}$$

for each $x \in X$. Then the function $x \to \inf\{y \in Y : f(x, y) \geq \lambda\}$ is continuous.

Proof. For each $x \in X$, put

$$F(x) = \{y \in Y : f(x, y) \geq \lambda\} \quad \text{and} \quad G(x) = \{y \in Y : f(x, y) > \lambda\}.$$

Our assumptions imply that the multifunction F is upper semicontinuous ([4], Theorem 7.1.16) and that the multifunction G is lower semicontinuous (in fact, its fibers are open). Consequently, the multifunction $x \to [\inf F(x), \sup F(x)]$ is upper semicontinuous and the multifunction $x \to [\inf G(x), \sup G(x)]$ is lower semicontinuous ([4], Theorem 7.3.17). This readily implies that the function

$x \to \inf F(x)$ (resp. $x \to \sup F(x)$) is lower (resp. upper) semicontinuous and that the function $x \to \inf G(x)$ (resp. $x \to \sup G(x)$) is upper (resp. lower) semicontinuous. The proof is complete. $\square$

Remark 5. It is clear from the proof that Proposition 5 is still true replacing, in the assumptions and in the conclusion, "inf" by "sup".

Theorem 12. *Let E be a Banach space, $[a, b]$ a compact real interval, Ω a non-empty open bounded subset of E, and f a continuous function from $\overline{\Omega} \times [a, b]$ into E, with relatively compact range. Assume that $f(x, y) \neq x$ for all $(x, y) \in \partial\Omega \times [a, b]$ and that the Leray–Schauder index of $f(\cdot, a)$ is not zero. Then, for every lower semicontinuous function $\varphi : \Omega \to [a, b]$ and every upper semicontinuous function $\psi : \Omega \to [a, b]$ with $\varphi(x) \leq \psi(x)$ for all $x \in \Omega$, there exist $x^* \in \Omega$ and $y^* \in [\varphi(x^*), \psi(x^*)]$ such that $f(x^*, y^*) = x^*$.*

In addition, if for some sequence $\{\lambda_n\}$ of positive real numbers with $\inf_{n\in\mathbb{N}} \lambda_n = 0$, *one has*

$$\inf\{y \in [a, b] : \|f(x, y) - x\| \geq \lambda_n\} = \inf\{y \in [a, b] : \|f(x, y) - x\| > \lambda_n\}$$

for each $x \in \Omega$ and $n \in \mathbb{N}$ for which

$$\{y \in [a, b] : \|f(x, y) - x\| > \lambda_n\} \neq \emptyset,$$

then there exists $x_0 \in \Omega$ such that $f(x_0, y) = x_0$ for all $y \in [a, b]$.

Proof. Thanks to the classical Leray–Schauder continuation principle (see, for instance, [8], Theorem 14.C), there exists a compact connected set $S \subseteq \Omega \times [a, b]$ such that $p_{[a,b]}(S) = [a, b]$ and $f(x, y) = x$ for all $(x, y) \in S$. Let φ, ψ be as in the statement. Put

$$T = \{(x, y) \in \Omega \times [a, b] : y \in [\varphi(x), \psi(x)]\}.$$

Then, in view of Theorem 1, one has $S \cap T \neq \emptyset$, which yields the first conclusion of the theorem.

Now, assume that there is some $\{\lambda_n\}$ as in the statement. For each $n \in \mathbb{N}$, put

$$V_n = \{(x, y) \in \Omega \times [a, b] : \|f(x, y) - x\| > \lambda_n\}.$$

Observe that $p_\Omega(V_n) \neq \Omega$. Indeed, if $p_\Omega(V_n) = \Omega$, then in view of Proposition 5, the function $x \to \inf\{y \in [a, b] : \|f(x, y) - x\| \geq \lambda_n\}$ would be continuous in Ω, and so, by Theorem 1 again, its graph should meet S, which is clearly absurd. Then pick $x_n \in \Omega$ such that $\|f(x_n, y) - x_n\| \leq \lambda_n$ for all $y \in [a, b]$. Since $f(\overline{\Omega} \times [a, b])$ is relatively compact and $\inf_{n\in\mathbb{N}} \lambda_n = 0$, the sequence $\{x_n\}$ admits some convergent subsequence. Plainly, the limit of such a subsequence satisfies the second conclusion of the theorem. $\square$

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