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ON A QUASILINEAR PROBLEM AT STRONG RESONANCE

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

1. Introduction

This paper deals with a class of nonlinear problems at strong resonance involving the p-Laplace operator. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and let $f(x,u)$ be a bounded continuous function. We are concerned with the quasilinear problem at resonance

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2}u + f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $p > 1$, $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator and $\lambda_1 > 0$ is the "first eigenvalue" of $-\Delta_p$ with zero Dirichlet boundary conditions (see [3]).

When $p = 2$ problem (1) becomes the semilinear problem

$$\begin{cases} -\Delta u = \lambda_1 u + f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

(λ_1 denotes now the principal eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions) and has been extensively studied in the past years, after the work [11]. For example, if $f(x,s) = b(s) - h(x)$ and $b(s) \to b^+$, resp. b^-, as $s \to \infty$,

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resp. $-\infty$, a solution of (2) exists whenever h satisfies the *Landesman–Lazer condition*

$$b^- \int_\Omega \phi_1(x)\,dx < \int_\Omega h(x)\phi_1(x)\,dx < b^+ \int_\Omega \phi_1(x)\,dx,$$

where $\phi_1 > 0$ denotes the (normalized) eigenfunction associated with λ_1.

This result has been extended to the quasilinear case in [7] (see also [4, 9] for some former partial results), proving that the Landesman–Lazer condition suffices for the existence of solutions of (1).

Problem (1), or (2), is said to be at *strong resonance* when $b^+ = b^- = 0$ or, more generally, when $f(x,s) \to 0$ as $|s| \to \infty$. Semilinear problems at strong resonance like (2) have also been studied (see for example [5, 6, 8]). On the contrary, nothing is known for quasilinear problems at strong resonance and the purpose of this paper is to study a class of such problems. Roughly, we consider an f such that

$$f(x,0) = 0 \quad \text{and} \quad \lim_{s\to\infty} f(x,s) = 0, \quad \text{uniformly in } x \in \Omega,$$

and show that (1) has a positive solution provided f changes sign in a suitable way. See Section 2 for precise statements. We also prove a multiplicity result, see Theorem 2.4.

Unlike the previous works on this topic, we employ here a new approach, based on global bifurcation. Using the techniques of [2] (see also [1]) we show that there is a continuum $S \subset \mathbb{R} \times C(\overline{\Omega})$ of positive solutions (λ, u) of

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

which branches off from the trivial solution and blows up at infinity as $\lambda \to \lambda_1$. By suitable estimates we prove that S meets the set $\{\lambda_1\} \times C(\overline{\Omega})$, yelding a positive solution of (1).

2. Statement of the results

In the sequel we shall always assume that $f \in C(\Omega \times \mathbb{R}^+)$ is such that $f(x,0) = 0$ for all $x \in \Omega$. To simplify the notation, the dependence on x will be hereafter eliminated (all the limits are understood to hold uniformly in x).

We will deal with problem (P_λ), which is meant as a nonlinear perturbation of the homogeneous problem

$$(3) \qquad \begin{cases} -\Delta_p u = \lambda|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Let us recall that there exists a unique $\lambda = \lambda_1$ such that (3) has a positive solution φ_1 (see [3]). Moreover, λ_1 has the following variational characterization:

$$\lambda_1 = \inf\left\{ \int_\Omega |\nabla u|^p\,dx : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p\,dx = 1\right\}. \tag{4}$$

The existence of positive solutions of (1) will be established under appropriate sign conditions on the limits

$$\lim_{s\to\infty} f(s)s = c, \tag{5}$$

$$\lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = \alpha. \tag{6}$$

We say that f satisfies $(f1^+)$, respectively $(f1^-)$, if (5) holds with $c > 0$, resp. $c < 0$.

Similarly, we say that f satisfies $(f2^+)$, respectively $(f2^-)$, if (6) holds where either $\alpha > 0$ (resp. $\alpha < 0$) or $\alpha = 0$ and there is $\delta > 0$ such that $f(s) > 0$ (respectively $f(s) < 0$) for all $s \in (0, \delta]$.

A first existence result is

THEOREM 2.1. *Problem* (1) *has a positive solution provided that f satisfies either* $(f1^-)$–$(f2^+)$, *or* $(f1^+)$–$(f2^-)$.

Instead of $(f1^-)$ we can require that

$$\text{there exists } s_0 > 0 \text{ such that } f(s_0) + \lambda_1 s_0^{p-1} < 0. \tag{$f3$}$$

THEOREM 2.2. *Problem* (1) *has a positive solution provided that f satisfies* $(f2^+)$ *and* $(f3)$.

By a limiting argument we can also handle the case in which

$$\lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = \infty. \tag{$f4$}$$

THEOREM 2.3. *Problem* (1) *has a positive solution provided f satisfies* $(f4)$ *and either* $(f1^-)$ *or* $(f3)$.

In general, problem (1) has no solution if we merely assume $(f1^+)$ and $(f2^+)$ or $(f4)$: it suffices to consider the case when $f(s) > 0$ for every $s > 0$. In contrast, the following multiplicity result can be proved.

THEOREM 2.4. *Suppose that f satisfies* $(f1^+)$ *and* $(f3)$. *Then* (1) *has at least two positive solutions provided that either* $(f2^+)$ *or* $(f4)$ *holds.*

Actually, some of the above results hold in a greater generality (see Remarks 4.1).

The proofs of these theorems are postponed until Section 4, while Section 3 is devoted to some preliminary lemmas concerning problem (P_λ).

3. Preliminary lemmas

In this section we deal with problem (P_λ). Actually, since we are looking for positive solutions of (P_λ), we can consider the problem

$$(\widetilde{P}_\lambda) \qquad \begin{cases} -\Delta_p u = g_\lambda(u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where

$$g_\lambda(s) = \begin{cases} \lambda s^{p-1} + f(s) & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases}$$

By the maximum principle [13] it follows that if (λ, u) is a nontrivial solution of $(\widetilde{P}_\lambda)$ then $u > 0$; hence (λ, u) is a solution of (P_λ). Problem $(\widetilde{P}_\lambda)$ is suited to be handled by the degree-theoretic arguments of [2] and [1]. Precisely, let us consider the Banach space

$$X = \{u \in C(\overline{\Omega}) : u(x) = 0 \text{ on } \partial\Omega\}$$

endowed with the norm $\|\cdot\|_\infty$ and set

$$\begin{aligned} \Sigma &= \mathrm{cl}\,\{(\lambda, u) \in \mathbb{R} \times X \mid u \neq 0 \text{ is a solution of } (\widetilde{P}_\lambda)\} \\ &= \mathrm{cl}\,\{(\lambda, u) \in \mathbb{R} \times X \mid u > 0 \text{ is a solution of } (P_\lambda)\}, \end{aligned}$$

where $\mathrm{cl}(A)$ denotes the closure of A. The behaviour of f at $s = 0$ and $s = \infty$ allows us to use the bifurcation results of [2] and [1] yielding

Lemma 3.1. (i) *If* (6) *holds then* $\lambda_0 = \lambda_1 - \alpha$ *is a bifurcation point from the trivial solution and the only one. Precisely, there exists an unbounded continuum* (*i.e. closed connected sets, maximal with respect to the inclusion*) $\Sigma_0 \subset \Sigma$ *branching off from* $(\lambda_0, 0)$.

(ii) *If* $\lim_{s\to\infty} f(s) = 0$ *then* $\lambda_\infty = \lambda_1$ *is a bifurcation point from infinity, and the only one. Precisely, there exists an unbounded continuum* $\Sigma_\infty \subset \Sigma$ *branching off from* (λ_1, ∞).

Let us recall that λ_∞ is a *bifurcation from infinity* if there exist $(\lambda_n, u_n) \in \Sigma$ such that $\lambda_n \to \lambda_\infty$ and $\|u_n\|_\infty \to \infty$.

We anticipate that in all theorems but Theorem 2.3 we shall show that $\Sigma_0 = \Sigma_\infty$. For this, some estimates are in order.

Lemma 3.2. *Let* $\gamma \in \mathbb{R}$ *and* $\varrho > 0$ *be such that* $f(\varrho) + \gamma\varrho^{p-1} < 0$. *If* $(\lambda, u) \in \Sigma$ *and* $\|u\|_\infty = \varrho$ *then* $\lambda > \gamma$.

Proof. We argue by contradiction and assume that $\lambda \leq \gamma$. Let $x_0 \in \Omega$ be such that $u(x_0) = \varrho$. Then there exists $r > 0$ such that

$$-\Delta_p u(x) = \lambda u(x)^{p-1} + f(u(x)) \leq \gamma u(x)^{p-1} + f(u(x)) < 0$$

for all $x \in B_r(x_0) \subset \Omega$. Now, by the strong maximum principle [13], we obtain $u(x) = \varrho$ for all $x \in \overline{B_r(x_0)}$. This proves that $\{x \in \Omega : u(x) = \varrho\}$ is open. But it is also closed and hence is all Ω, a contradiction. □

From the preceding lemma we infer:

Corollary 3.3. (i) *If* $(f2^{\pm})$ *holds then there exists* $\Lambda > 0$ *such that* $\Sigma \subset (-\Lambda, \infty) \times X$.

(ii) *If* $(f3)$ *holds then, for* $\lambda \le \lambda_1$, *problem* (P_λ) *has no positive solution* u *such that* $\|u\|_\infty = s_0$.

Proof. (i) Let $\Lambda > 0$ be such that $f(s) < \Lambda s^{p-1}$ for all $s > 0$. Then Lemma 3.2 applies with $\gamma = -\Lambda$ and all $\varrho > 0$. Hence $(\lambda, u) \in \Sigma$ implies that $\lambda > -\Lambda$.

(ii) If $(f3)$ holds then $f(s_0) < -\lambda s_0^{p-1}$ for all $\lambda \le \lambda_1$ and Lemma 3.2 implies that $\|u\|_\infty \ne s_0$ whenever $(\lambda, u) \in \Sigma$ and $\lambda \le \lambda_1$. □

Remark 3.4. If $(f4)$ holds, f is bounded and $\lambda < 0$, we set

$$\varrho(\lambda) = \inf\{r > 0 : f(s) < -\lambda s^{p-1} \text{ for all } s \ge r\}.$$

Then $\lim_{\lambda \to -\infty} \varrho(\lambda) = 0$ and $f(\varrho) + \lambda \varrho^{p-1} < 0$ for all $\varrho \in (\varrho(\lambda), \infty)$. Hence if $(\lambda, u) \in \Sigma$, Lemma 3.2 yields $\|u\|_\infty < \varrho(\lambda)$.

Moreover, by (4) we infer

Lemma 3.5. *There exists* $\Lambda^* > 0$ *such that* $\Sigma \subset (-\infty, \Lambda^*) \times X$.

Proof. Let $\Lambda^* > 0$ be such that $\Lambda^* s^{p-1} + f(s) > L s^{p-1}$ for all $s > 0$, with $L > \lambda_1$. If $(\lambda, u) \in \Sigma$ with $\lambda \ge \Lambda^*$ it follows that u is an upper solution of the problem

$$\begin{cases} -\Delta_p u = L|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Then, using $t\varphi_1$ as lower solution with $t > 0$ sufficient small, we would obtain a positive solution of this problem; i.e. a positive eigenfunction of $-\Delta_p$ with associated eigenvalue $L > \lambda_1$. But this is not possible [3, Proposition 2]. □

When Corollary 3.3(i) and Lemma 3.5 apply it follows immediately that $\Sigma_0 = \Sigma_\infty$:

Lemma 3.6. *If* $(f2^{\pm})$ *holds and* $\lim_{s \to \infty} f(s) = 0$ *then there is a continuum* $S \subset \Sigma$ *bifurcating from infinity at* $\lambda = \lambda_1$ *and from zero at* $\lambda = \lambda_0$. *Moreover,* $S \subset (-\Lambda, \Lambda^*) \times X$.

The remainder of this section is devoted to the behaviour of S near the bifurcation points. Recall that a bifurcation is said *subcritical* or *supercritical* provided S is on the left, respectively on the right, in a deleted neighbourhood of the bifurcation point.

Lemma 3.7. *Assume f satisfies* $(f2^+)$ (*respectively* $(f2^-)$) *with* $\alpha = 0$. *Then the bifurcation at* $(\lambda_0, 0)$ *is subcritical* (*resp. supercritical*).

Proof. We deal with the case when $(f2^+)$ holds. The other is proved in a similar way, with obvious changes. Suppose, by contradiction, that there exists a sequence $(\lambda_n, u_n) \in S$ such that $\lambda_n > \lambda_1$, $\lambda_n \to \lambda_1$, $\|u_n\|_\infty \to 0$, $u_n \neq 0$. Without loss of generality, $\|u_n\|_\infty \leq \delta$ and hence u_n is an upper solution of the problem

$$\begin{cases} -\Delta_p u = \lambda_n |u|^{p-2} u, & x \in \Omega, \\ \qquad u = 0, & x \in \partial\Omega. \end{cases}$$

Arguing as in the proof of Lemma 3.5, we arrive at a contradiction. $\square$

Lemma 3.8. *If* $(f1^-)$ (*respectively* $(f1^+)$) *holds, then the bifurcation from infinity is supercritical* (*resp. subcritical*).

Proof. Let u_n be a positive solution of (P_{λ_n}) with $\lambda_n \to \lambda_1$, $\|u_n\|_\infty \to \infty$. Dividing (P_λ) by $\|u_n\|_\infty^{p-1}$, we infer that $v_n = u_n \|u_n\|_\infty^{-1}$ satisfies

$$-\Delta_p v_n = \lambda_n |v_n|^{p-2} v_n + \frac{f(u_n)}{\|u_n\|^{p-1}}.$$

From the regularity theory [12] it follows that, up to a subsequence, $v_n \to v$ in $C^1(\overline{\Omega})$ and $v \in X$ has norm 1 and satisfies

$$-\Delta_p v = \lambda_1 |v|^{p-2} v, \quad x \in \Omega.$$

As a consequence, $v = \varphi_1$, with $\|\varphi_1\|_\infty = 1$.

Now we consider separately the cases where $(f1^-)$ or $(f1^+)$ hold.

Case (a). From the preceding arguments we infer that $u_n(x) = \|u_n\|_\infty v_n(x) \to \infty$ for every $x \in \Omega$. Then the Lebesgue theorem and $(f1^-)$ imply

$$\lim_{n\to\infty} \int_\Omega f(u_n(x)) u_n(x)\, dx = c\,\mathrm{meas}(\Omega) < 0. \tag{7}$$

From (4) we also deduce

$$\lambda_1 \int_\Omega |u_n|^p\, dx \leq \int_\Omega |\nabla u_n|^p\, dx = \lambda_n \int_\Omega |u_n|^p\, dx + \int_\Omega f(u_n) u_n\, dx.$$

Then from (7) it follows that $\lambda_n > \lambda_1$ for large n enough and this means that the bifurcation from infinity is supercritical.

Case (b). Suppose that $(f1^+)$ holds. Since $v_n \to \varphi_1$ in $C^1(\overline{\Omega})$ we can assume that $\frac{1}{2}\varphi_1(x) \leq v_n(x) \leq \frac{3}{2}\varphi_1(x)$ for every $x \in \Omega$. Let $\{t_n\}$ be a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{t_n}{\|u_n\|_\infty} = \infty, \quad t_n \geq \frac{3}{2}\|u_n\|_\infty, \quad \forall n \in \mathbb{N}.$$

Consider the functional I defined on

$$D(I) = \{(u,v) : u,v \in W_0^{1,p}(\Omega),\ u,v \geq 0,\ uv^{-1}, vu^{-1} \in L^\infty(\Omega)\}$$

by setting

$$I(u,v) = \left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \right\rangle - \left\langle -\Delta_p v, \frac{u^p - v^p}{v^{p-1}} \right\rangle.$$

One has

$$I(t_n\varphi_1, u_n) = (\lambda_1 - \lambda_n)\int_\Omega [t_n^p\varphi_1^p - u_n^p]\,dx - \int_\Omega f(u_n)\frac{t_n^p\varphi_1^p - u_n^p}{u_n^{p-1}}\,dx.$$

Moreover, it is known (see [10]) that $I \geq 0$. Hence it follows that

$$\int_\Omega f(u_n)\frac{t_n^p\varphi_1^p - u_n^p}{u_n^{p-1}}\,dx \leq (\lambda_1 - \lambda_n)\int_\Omega [t_n^p\varphi_1^p - u_n^p]\,dx.$$

We claim that the left hand side of this inequality tends to ∞ provided $(f1^+)$ holds. Indeed, we have

$$\begin{aligned}\int_\Omega & f(u_n)\frac{t_n^p\varphi_1^p - u_n^p}{u_n^{p-1}}\,dx \\ &= \left(\frac{t_n}{\|u_n\|_\infty}\right)^p \int_\Omega f(u_n)u_n\left(\frac{\|u_n\|_\infty\varphi_1}{u_n}\right)^p dx - \int_\Omega f(u_n)u_n\,dx.\end{aligned}$$

Now, since $\|u_n\|_\infty u_n(x)^{-1}\varphi_1(x) = v_n(x)^{-1}\varphi_1(x) \leq 2$ for every $x \in \Omega$, we deduce from $(f1^+)$ that

$$\lim_{n\to\infty}\int_\Omega f(u_n)u_n\left(\frac{\|u_n\|_\infty\varphi_1}{u_n}\right)^p dx = c\,\mathrm{meas}(\Omega) > 0,$$

which, together with (7), gives

$$\lim_{n\to\infty}\int_\Omega f(u_n)u_n\frac{t_n^p\varphi_1^p - u_n^p}{u_n^{p-1}}\,dx = \infty,$$

proving the claim. Therefore, for n large enough,

$$0 < (\lambda_1 - \lambda_n)\int_\Omega [t_n^p\varphi_1^p - u_n^p]\,dx.$$

Recalling that $t_n\varphi_1(x) \geq \frac{3}{2}\|u_n\|_\infty\varphi_1(x) \geq u_n(x)$ for every $x \in \Omega$, this implies that $\lambda_1 > \lambda_n$ and thus the bifurcation is subcritical in this case. $\square$

4. Proof of Theorems

Proof of Theorem 2.1. First suppose $(f1^-)$ and $(f2^+)$. Then Lemma 3.6 applies and yields a continuum $S \subset \Sigma$ which connects $(\lambda_1 - \alpha, 0)$ and (λ_1, ∞). By Lemma 3.8, S emanates from the right of (λ_1, ∞) and hence there exists $(\lambda, u) \in S \setminus \{0\}$ with $\lambda > \lambda_1$. Moreover, there also exists $(\lambda, u) \in S \setminus \{0\}$ with $\lambda < \lambda_1$. If $\alpha > 0$ this is immediate because then the bifurcation takes place at $\lambda_0 = \lambda_1 - \alpha$; if $\alpha = 0$ the claim holds true because the bifurcation is subcritical (see Lemma 3.7).

Since S is connected it follows that there exists $u \neq 0$ such that $(\lambda_1, u) \in S$, yielding a positive solution of (1).

If f satisfies $(f1^+)$ and $(f2^-)$ the proof is similar. □

Proof of Theorem 2.2. Consider the unbounded continuum Σ_0 branching off from $(\lambda_0, 0)$ (see Lemma 3.1(i)). As in the proof of Theorem 2.1 assumption $(f2^+)$ implies that there is $(\lambda, u) \in \Sigma_0 \setminus \{0\}$ with $\lambda < \lambda_1$. Taking into account that Σ_0 is connected and unbounded and using Corollary 3.3(i), (ii), one infers that Σ_0 meets the set $\{\lambda_1\} \times X$ and the result follows. □

Proof of Theorem 2.3. Let $f_n \in C(\Omega \times \mathbb{R}^+)$ be a sequence of functions such that $f_n(s) = f(s)$ for $s \geq 1$ and satisfying

$$\lim_{s \to 0^+} \frac{f_n(s)}{s^{p-1}} = n.$$

If $(f1^-)$ (respectively $(f3)$) holds then we can use Theorem 2.1 (respectively Theorem 2.2) to find positive solutions u_n of the approximated problems

$$\begin{cases} -\Delta_p u = \lambda_1 u^{p-1} + f_n(u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

We claim that there are constants $a, b > 0$ such that $a \leq \|u_n\|_\infty \leq b$. The upper bound follows by repeating the arguments used in the proof of Lemma 3.8 (Case (a)), with λ_1 instead of λ_n. As for the lower bound, we shall closely follow the proof of Lemma 3.8 (Case (b)) and thus we shall be sketchy. Suppose, by contradiction, that $\|u_n\|_\infty \to 0$. From $I(\varphi_1, u_n) \geq 0$ it follows by direct calculation that

$$\int_\Omega f(u_n) \frac{\varphi_1^p - u_n^p}{u_n^{p-1}}\, dx \leq 0.$$

Since $u_n \to 0$ and $(f4)$ holds, we find a contradiction, proving the claim. Finally, the uniform bound allows us to pass to the limit yielding a positive solution of (1). □

Proof of Theorem 2.4. Consider the continuum S connecting $(\lambda_0, 0)$ and (λ_1, ∞). A first positive solution u_1 of (1), with $\|u_1\|_\infty < s_0$, can be found using

Theorem 2.3. Since $(f1^+)$ holds, the bifurcation from infinity is now subcritical and hence (1) has a second positive solution u_2 with $\|u_2\|_\infty > s_0$. □

Remarks 4.1.

1. Minor changes would allow us to substitute the assumption $\lim_{s\to\infty} f(s) = 0$ with the slightly more general $\lim_{s\to\infty} f(s)s^{1-p} = 0$, as well as to permit that c and α depend on x.
2 In Theorem 2.2 we do not require $f(s) \to 0$ as $s \to \infty$; it suffices to assume that f is bounded.
3 The results of Section 3 allow us to describe the bifurcation diagram of (P_λ). In particular, in the case covered by Theorem 2.3, Remark 3.4 shows that the projection of Σ_∞ on the λ axis contains $(-\infty, 0)$ and hence (P_λ) has positive solutions for all $\lambda < 0$. Moreover, along Σ_∞ one has that $\|u\|_\infty \to 0$ for all $(\lambda, u) \in \Sigma_\infty$ with $\lambda \to -\infty$.

References

[1] A. Ambrosetti, J. Garcia Azorero and I. Peral, *Multiplicity results for some nonlinear elliptic equations*, J. Funct. Anal. (to appear).

[2] A. Ambrosetti and P. Hess, *Positive solutions of asymptotically linear elliptic eigenvalue problems*, J. Math. Anal. Appl. **73** (1980), 411–422.

[3] A. Anane, *Simplicité et isolation de la première valeur propre du p-Laplacien avec poids*, C. R. Acad. Sci. Paris **305** (1987), 725–728.

[4] A. Anane and J. P. Gossez, *Strongly nonlinear elliptic problems near resonance: a variational approach*, Comm. Partial Differential Equations **15** (1990), 1141–1159.

[5] D. Arcoya and A. Cañada, *Critical point theorems and applications to nonlinear boundary value problems*, Nonlinear Anal. **14** (1990), 393–411.

[6] D. Arcoya and D. G. Costa, *Nontrivial solutions for a strongly resonant problem*, Differential Integral Equations **8** (1995), 151–159.

[7] D. Arcoya and L. Orsina, *Landesman–Lazer conditions and quasilinear elliptic equations*, Nonlinear Anal. (to appear).

[8] P. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity*, Nonlinear Anal. **7** (1983), 981–1012.

[9] L. Boccardo, P. Drábek and M. Kučera, *Landesman–Lazer conditions for strongly nonlinear boundary value problems*, Comment. Math. Univ. Carolin. **30** (1989), 411–427.

[10] J. I. Díaz and J. E. Saa, *Uniqueness of nonnegative solutions for elliptic nonlinear diffusion equations with a general perturbation term*, Proceedings of the VII CEDYA (1985), Santander.

[11] E. M. Landesman and A. C. Lazer, *Nonlinear perturbations of linear elliptic problems at resonance*, J. Math. Mech. **19** (1970), 609–623.

[12] P. Tolksdorff, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.

[13] J. L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.

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