Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 14, 1999, 151–158

FIXED POINT INDICES OF EQUIVARIANT MAPS OF CERTAIN JIANG SPACES

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Abstract. Given X a Jiang space we know that all Nielsen classes have the same index. Now let us consider X a G-space where G is a finite group which acts freely on X. In [7], we do have the notion of X to be an equivariant Jiang space and under this condition it is true that all equivariant Nielsen classes have the same index. We study the question if the weaker condition of X being just a Jiang space is sufficient for all equivariant Nielsen classes to have the same index. We show a family of spaces where all equivariant Nielsen classes have the same index. In many cases the spaces of such a family are not equivariant Jiang spaces. Finally, we also show an example of one Jiang space together with equivariant maps where the equivariant Nielsen classes have different indices.

1. Introduction

An equivariant fixed point Nielsen theory has been developed in [7] and very little has been done in terms of computing the equivariant Nielsen numbers. Besides the calculations done in [7], J. Guo in [3] was able to use a weaker hypothesis than the G-Jiang condition to obtain Jiang type results. But in this case he could only show that the equivariant Nielsen coincidence classes have index of the same sign. This work will do some computations of this sort. We deal primarily with the simple case where the spaces involved are Jiang spaces.

1991 *Mathematics Subject Classification*. Primary 55M20; Secondary 57S99.
Key words and phrases. Index, Nielsen classes, nilmanifolds, equivariant classes, Jiang spaces, equivariant maps.

We also consider the larger family of spaces which have the property that all Reidemeister classes have the same index, which we define by saying that X has the J-property.

A G-Jiang space has been defined in [7]. Certainly if a space is G-Jiang space, then all G-equivariant Nielsen classes have the same index. For the torus with the Z_2-action which gives as quotient the Klein bottle, it is not difficult to see that it is not a Z_2-Jiang space. Nevertheless one can show that all Z_2-equivariant Nielsen classes have the same index. This has motivated us to discover a family of spaces where all G-equivariant Nielsen classes have the same index. We denote by $\mathcal{JN}$ the family of all spaces X which satisfies the J-property and its fundamental group is a finitely generated torsion free nilpotent group (so isomorphic to the fundamental group of a compact nilmanifold). In fact many of such spaces are not G-Jiang spaces, as one can see by Proposition 3.2. Then we prove:

Theorem 3.4. *Let $X \in \mathcal{JN}$ and G be a finite group which acts freely on X. If the G-spaces X have the property that the fundamental group of the orbit space is torsion free, then all equivariant Nielsen classes of a given equivariant map $f : X \to X$ have the same index. Furthermore the index of each such class is $|G|$ times the index of one of the Nielsen classes of f.*

On the other hand, we also show an example of a Jiang space and equivariant maps f_k, where the equivariant Nielsen classes don't have the same index. Let Z_2 act on $S^1 \times S^2$ by $t(x,y) = (\overline{x}, -y)$ and $f_k(x,y) = (x^k, y)$. More precisely, we prove:

Theorem 4.4. *The space $S^1 \times S^2$ is a Jiang space and the map f_k has $|k-1|$ essential fixed point classes, i.e. $N(f_k) = |k-1|$ and each class has index 2. If k is odd, the map f_k has $|k-1|/2+1$ equivariant fixed point classes where $|k-1|/2-1$ have index 4 and the two remaining ones have index 2. If k is even, the map f_k has $|k|/2$ equivariant fixed point classes where one class has index 2 and the remaining ones have index 4.*

This note is divided into three sections. In Section 2 we define the isotropy group of a Nielsen class and show how to compute its cardinality by algebraic means. This is Proposition 2.2. In Section 3 we define the family $\mathcal{JN}$ of spaces X which satisfy the J-property and whose fundamental group is a finitely generated torsion free nilpotent group. Then, we prove Theorem 3.4. Finally, in Section 4, we construct some spaces with free actions which realize some algebraic data. This is Proposition 4.3. Then we construct a very simple Z_2-space and functions which have the property that the equivariant Nielsen classes have different indices. This is Theorem 4.4. In the end we make a few remarks about the relations between families of spaces related to the Jiang condition.

We would like to thank Prof. R. Brown for his invaluable help in improving the writing up of this work. We also would like to thank P. Wong for bringing to our attention the related work of J. Guo [3].

2. The isotropy subgroup of a Nielsen class

In [7, Section 2], a notion of equivariant Nielsen class is defined for a G-map $f : X \to X$ where G is a compact Lie Group. When we look at the particular case where G is finite and acts freely on X, we have by means of [7, Proposition 2.4] that $x, y \in \mathrm{Fix}(f)$ belong to the same equivariant Nielsen class if and only if

(a) $y = \sigma x$ for some $\sigma \in G$, or
(b) there exists a path $\alpha : I \to X$ such that $\alpha(0) = x$, $\alpha(1) = \sigma'(y)$ for some $\sigma' \in G$ where $\alpha \approx f \circ \alpha$ (rel endpoints).

Let X be a space where a finite group G acts freely. Denote by $\overline{X}$ the orbit space and $p : X \to \overline{X}$ the projection. In order to study the indices of the equivariant Nielsen classes of f we define below a certain subgroup associated with a (non-equivariant) Nielsen class of f.

Definition 2.1. Given $f : X \to X$ an equivariant map and $x_0 \in \mathrm{Fix}(f)$, let $G_{f,x_0} = \{\alpha \in G \mid x_0 \text{ and } \alpha x_0 \text{ are Nielsen related}\}$.

One can easily see that G_{f,x_0} is a subgroup of G. Also if two points x_0, x_1 are Nielsen related then $G_{f,x_0} = G_{f,x_1}$. So, from now on we will write $G_{f,F}$, where F denotes a Nielsen class, for the subgroup $G_{f,x}$ where x belongs to F.

Now we will compute the cardinality of $G_{f,F}$ in terms of the fundamental group $\pi_1(\overline{X})$ and the induced homomorphism $\overline{f}_\# : \pi_1(\overline{X}, \overline{x}_0) \to \pi_1(\overline{X}, \overline{x}_0)$, where $\overline{f} : \overline{X} \to \overline{X}$ is the map induced by f on the orbit space.

Proposition 2.2. *Let $F \subset \mathrm{Fix}(f)$ be a Nielsen class and $\theta \in \pi_1(X, x_0)$ be a representative of the Reidemeister class which corresponds to F. Then the cardinality of $G_{f,F}$, also denoted by $\varphi(\theta)$, is:*

$$|G_{f,F}| = \# \frac{C_\theta(\overline{f})}{C_\theta(\overline{f}) \cap \Gamma_1},$$

where

$$C_\theta(\overline{f}) = \{\alpha \in \pi_1(\overline{X}, \overline{x}_0) \mid \alpha\theta\overline{f}_\#(\alpha)^{-1} = \theta\},$$

and

$$\Gamma_1 = p_\#(\pi_1(X, x_0)) \subset \pi_1(\overline{X}, \overline{x}_0).$$

Proof. Let $x \in F$ and $g \in G_{f,F}$. We have that $x \sim gx$ (Nielsen related). Let λ be a path from x to gx such that $\lambda \sim f(\lambda)$, rel $\{x, gx\}$. If γ is a path from

$\overline{x}_0$ to $\overline{x} = p(x)$, we define a loop based in $\overline{x}_0$ by $\alpha = \gamma * p(\lambda) * \gamma^{-1}$. Then we have:

$$\begin{aligned}\overline{f}_\#(\alpha) &= \overline{f}_\#(\gamma) * \overline{f}_\#(p(\lambda)) * \overline{f}_\#(\gamma)^{-1} = \overline{f}_\#(\gamma) * p(f(\lambda)) * \overline{f}_\#(\gamma)^{-1} \\ &= \overline{f}_\#(\gamma).\gamma^{-1}.\gamma.p(\lambda).\gamma^{-1}.\gamma\overline{f}_\#(\gamma^{-1}) = \theta^{-1}\alpha\theta,\end{aligned}$$

or $\theta = \alpha\theta\overline{f}_\#(\alpha)^{-1}$.

By taking the class of α in the quotient $C_\theta(\overline{f})/C_\theta(\overline{f}) \cap \Gamma_1$, we define a map between $G_{f,F}$ and $C_\theta(\overline{f})/C_\theta(\overline{f}) \cap \Gamma_1$. It is a routine argument to show that this map is independent of the choice of the path λ and is a bijection. So the equality follows. □

Corollary 2.3. *If $\pi_1(\overline{X})$ is abelian then C_θ is independent of θ and consequently $G_{f,F}$ is independent of the Nielsen class.*

Corollary 2.4. *If X is a Jiang space and $\pi_1(\overline{X})$ is abelian, then all equivariant Nielsen classes have the same index.*

Corollary 2.5. *Under the hypotheses of either Corollary 2.3 or 2.4 we have that the equivariant Nielsen class which contains a Nielsen class F has index $[G : G_{f,F}].i(F)$.*

3. Certains spaces with the J-property

We will start with the definition of the J-property.

Definition 3.1. A space X is said to have the *J-property*, if for every map $f : X \to X$ all Reidemeister classes have the same index.

Observe that if X satisfies the J-property and the number of Reidemeister classes is infinite, then all Nielsen classes have index zero.

In many cases X satisfies the J-property but it is not a G-Jiang space.

Proposition 3.2. *Let X be a compact nilmanifold where G acts freely. If the extension $1 \to \pi_1(X) \to \pi_1(\overline{X}) \to G \to 1$ is not central, then X is not a G-Jiang space.*

Proof. Let $J_G(\mathrm{id}) \subset \pi_1(X, x_0)$ be the set of loops which are obtained by an equivariant self-homotopy of the identity map on X. Every such homotopy factors through the quotient space. So the projection provides a natural map $p_\# : J_G(\mathrm{id}_X) \to J(\mathrm{id}_{\overline{X}})$ which is injective. But since X is a $K(\pi, 1)$ space, we know that $J(\mathrm{id}_{\overline{X}}) = \mathrm{center}(\pi_1(\overline{X}))$ (see [B, Corollary 12, p. 103]). But the center of $\pi_1(\overline{X})$ cannot contain $\pi_1(X)$ since the extension is not central by hypothesis. So $J_G(\mathrm{id})$ is different from $\pi_1(X, x_0)$ and the result follows. □

Remarks. (a) This, in particular, shows that the Torus, with the action which gives as orbit space the Klein bottle, is not a G-Jiang space.

(b) The family of compact Nilmanifolds satisfies the J-property. For a larger family see [2].

Proposition 3.3. *Let $f : X \to X$ be an equivariant map and $\theta \in \pi_1(X)$. We have that $C_\theta(f) \subset C_\theta(\overline{f})$ and $C_\theta(f)$ is a normal subgroup of $C_\theta(\overline{f})$. Furthermore the quotient group is a torsion group.*

Proof. By abuse of notation let us denote by θ an element of $\pi_1(X)$ and the corresponding element of $\pi_1(\overline{X})$ under the natural inclusion. It is clear that $C_\theta(f) \subset C_\theta(\overline{f})$. Next we show that $C_\theta(f)$ is a normal subgroup of $C_\theta(\overline{f})$. For, let α and β satisfying $\alpha\theta\overline{f}(\alpha^{-1}) = \theta$ and $\beta\theta f(\beta^{-1}) = \theta$, respectively. So we have $\alpha\beta\alpha^{-1}\theta\overline{f}(\alpha)f(\beta^{-1})\overline{f}(\alpha^{-1}) = \alpha\beta\alpha^{-1}\alpha\theta f(\beta^{-1})\theta^{-1}\alpha^{-1}\theta = \alpha\beta\theta\theta^{-1}\beta^{-1}\alpha^{-1}\theta = \theta$. Therefore the subgroup is normal.

Since $C_\theta(\overline{f}) \cap \pi_1(X) = C_\theta(f)$, the quotient group injects in the finite group G and the last part follows. □

Now we come to the main theorem of this section. We define, by $\mathcal{JN}$, the family of all spaces X which satisfies the J-property and its fundamental group is a finitely generated torsion free nilpotent group (so isomorphic to the fundamental group of a compact nilmanifold).

Theorem 3.4. *Let $X \in \mathcal{JN}$ and G be a finite group which acts freely on X. If the G-spaces X have the property that the fundamental group of the orbit space is torsion free, then all equivariant Nielsen classes of a given equivariant map $f : X \to X$ have the same index. Furthermore the index of each such class is $|G|$ times the index of one of the Nielsen classes of f.*

Proof. Let $\theta \in \pi_1(X)$ and $f : X \to X$. We can assume that $N(f) \neq 0$, otherwise the result is clear. Since f satisfies the J-property, we have $\#R(f) < \infty$. Since $\pi_1(X)$ is a finitely generated torsion free nilpotent group, by Theorem 2.3 of [2] we get $\#R(f) < \infty$ implies that $\mathrm{Fix}(f_\#) = 1$. By Proposition 2.1 of [2] we have that $C_\theta(f) = 1$. By Proposition 3.3 above follows that $C_\theta(\overline{f})$ is a torsion group and, at the same time, a subgroup of $\pi_1(\overline{X})$. Since $\pi_1(\overline{X})$ is torsion free group, it follows that $C_\theta(\overline{f}) = 1$. From Proposition 2.2 of Section 2 and the fact that the space X satisfies the J-property, the result follows. □

Remark. The hypothesis that the fundamental group of the orbit space is torsion free is equivalent to saying that it is isomorphic to the fundamental group of an infranilmanifold (for details about infranilmanifolds, in particular for properties of its fundamental group, see [5])

4. Examples

Let G be a finite group which acts on an abelian group A. We consider the very simple algebraic situation where $A \approx Z$, $G \approx Z_2$ and the action $w : Z_2 \to$

$\mathrm{Aut}(Z)$ is the only non-trivial one. Let $H = Z \rtimes Z_2$, be the semi-direct product of Z by Z_2 with respect to the action w above, and k an integer different from 1. Denote by $\langle k\rangle$ the integer $k-2$ if $k > 1$ and $|k|$ if $k \leq 0$

Proposition 4.1. *Let $\psi : Z \rtimes Z_2 \to Z \rtimes Z_2$ be the homomorphism defined by $\psi(n,1) = (kn,1)$ and $\psi(0,t) = (0,t)$ where $Z \lhd Z \rtimes Z_2$ is a normal subgroup. Let $\theta \in Z$ and $0 \leq \theta \leq \langle k\rangle$. Then the cardinality of $C_\theta(\psi)/C_\theta(\psi) \cap H$ is 2, if k is odd and θ is either 0 or $|k-1|/2$, or if k is even and θ is 0. Otherwise the cardinality is 1.*

Proof. It is easy to see that ψ defines a homomorphism. For every $\theta \in Z$ we have: $C_\theta(\psi) = \{(m,\xi) \mid (m,\xi)(\theta,1)(km,\xi)^{-1} = (\theta,1)\}$. So

$$\begin{aligned} (m,1)(\theta,1)(-km,1) &= (k\theta,1), \quad \xi = 1,\\ (m,t)(\theta,1)(km,t) &= (\theta,1), \quad \xi = t. \end{aligned}$$

The solution of the first equation is always $m = 0$. Recall that $k \neq 1$. So let us solve the second equation:

$$(m-\theta-km,1) = (\theta,1) \Leftrightarrow m-\theta-km = -\theta+(1-k)m = \theta \Leftrightarrow (1-k)m = 2\theta.$$

Now, by a simple divisibility argument, the last equation has a unique solution if and only if θ is as stated in the proposition. So the result follows. $\square$

Corollary 4.2. *If either $k \geq 3$ or $k \leq -2$ then there are two Reidemeister classes θ_1, θ_2 such that $\varphi(\theta_1) = 1$ and $\varphi(\theta_2) = 2$.*

Now let $w : G \to \mathrm{Aut}(A)$ be a given action of G on the abelian group A and X a G-space such that $\pi_1(X,x_0) = A$, where $x_0 \in X$ is fixed by G. Suppose that the induced action of G on $\pi_1(X,x_0)$ is w. Let Y be a simply connected space such that G acts freely.

Proposition 4.3. *The action of G on $X \times Y$ given by $g(x,y) = (gx,gy)$ where G acts on X and Y as above is free. The quotient space W has $\pi_1(W) = A \rtimes G$, i.e., the semi-direct product of A by G with respect to the action w.*

Proof. Since G acts freely on $X \times Y$, we have the covering space

$$X \times Y \xrightarrow{p} W,$$

and consequently the short exact sequence

$$0 \to \pi_1(X) \to \pi_1(W) \to G \to 1 \quad \text{or} \quad 0 \to A \to \pi_1(W) \to G \to 1.$$

The map $Y \to (X \times Y)$ defined by $y \mapsto (x_0,y)$ is certainly a G-map. Let $\overline{Y}$ be the quotient of Y by G. So we have the induced map $\overline{Y} \to W$ which gives us a splitting $G \to \pi_1(W)$ of the short exact sequence. It remains to show that the action of G on A is the one given by w. Let $\alpha : I \to X$ be a loop which represents an element of $\pi_1(X)$. So (α, c), where c is the constant path $c(t) = y_0 \in Y$, $t \in I$, represents the image of α in $\pi_1(X \times Y)$. For $g \in G$, consider a path λ in Y such that $\lambda(0) = y_0$ and $\lambda(1) = g.y_0$. So $(p(\lambda), c)$ represents an element of $\pi_1(W)$ which projects on g under the map $\pi_1(W) \to G$. We must compute $gp(\alpha)g^{-1}$. This loop in W lifts to the loop $(x_0, \lambda) * (g(\alpha), g.y_0) * (x_0, \lambda^{-1})$. This last loop is homotopic to the loop $(g(\alpha), c)$ and the result follows. □

Now consider the space $X = S^1$, $Y = S^2$ and Z_2-actions on S^1 and S^2 respectively as follows: $t(x) = \overline{x}$ is the conjugation of the complex number for $x \in S^1$ and $t(y) = -y$, $y \in S^2$. Let $f_k : S^1 \times S^2 \to S^1 \times S^2$ be the map $(x, y) \mapsto (x^k, y)$ where k is any integer greater than 3 or less than -2. The map f_k is certainly a Z_2-equivariant map for any value of k.

THEOREM 4.4. *The space $S^1 \times S^2$ is a Jiang space and the map f_k has $|k-1|$ essential fixed point classes, i.e. $N(f_k) = |k - 1|$, and each class has index 2. If k is odd, the map f_k has $|k - 1|/2 + 1$ equivariant fixed point classes where $|k - 1|/2 - 1$ have index 4 and the two remaining ones have index 2. If k is even, the map f_k has $|k|/2$ equivariant fixed point classes where one class has index 2 and the remaining ones have index 4.*

PROOF. Let us compute $\mathrm{Fix}(f_k)$.

$$f_k(x, y) = (x, y) \Leftrightarrow x^k = x \Leftrightarrow \begin{cases} x = 1, \xi, \xi^2, \dots, \xi^{k-2}, & \text{if } k \geq 2, \\ \text{or} & \\ x = 1, \xi, \xi^2, \dots, \xi^{|k|}, & \text{if } k \leq 0, \end{cases}$$

where ξ is a $|k-1|$-primitive root of the unity. So $\mathrm{Fix}(f) = \{1, \xi, \xi^2, \dots, \xi^l\} \times S^2$ where l is either $k - 2$ or $|k|$. Call $F_i = \{\xi^i\} \times S^2$. The Reidemeister class which represents F_i is given by $[i - 1]$. By Proposition 4.3, we have that $\pi_1(W) = Z \rtimes Z_2$ where W is the quotient of $S^1 \times S^2$ by the action given above. So, by Propositions 2.2 and 4.1, we know when $\varphi([i])$ is either 1 or 2. The cases where $\varphi([i]) = 1$ means that the equivariant class which contains the Nielsen class has index the order of the group, which is 2, times the index of a Nielsen class. Therefore index 4. The cases where $\varphi[i] = 2$ mean that the Nielsen class is itself the equivariant Nielsen class. Therefore it has index 2. So the result follows. □

REMARKS. (a) This example shows that we can have a finite cover which has the property that all Nielsen classes of a map f have the same index, but it is not true for the map $\overline{f}$. In our particular example for $k = 5$ we have $p(\mathrm{Fix}(f)) = \mathrm{Fix}(\overline{f})$, where p is the projection, $N(\overline{f}) = 3$ and the Nielsen classes of $\overline{f}$ have indices 1, 2 and 1.

(b) Let us consider the following three properties of one space X:

(1) X is a Jiang space.

(2) X is a G-equivariant Jiang space.

(3) $\overline{X}$, the orbit space, is a Jiang space.

We have the following results:

(i) (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) , clear.

(ii) (2) $\not\Rightarrow$ (3) by [6]. Of course the sphere S^2 having the projective plane RP^2 as orbit space, also shows that (2) $\not\Rightarrow$ (3). Neverthless by using [6] we get an example with the property that the free actions on the odd dimensional spheres are orientation preserving and the orbit space is not a Jiang space.

(iii) (1) $\not\Rightarrow$ (2) by the first remark at the end of Proposition 3.2. So (1) $\not\Rightarrow$ (3). Finally (2) and (3) certainly imply that all equivariant Nielsen classes have the same index but (1) doesn't, as a result of Theorem 4.4.

(c) This Section 4 gives a procedure for building many other examples from an algebraic data.

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Manuscript received July 23, 1999

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