Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 16, 2000, 93–101

R_δ-SET OF SOLUTIONS TO A BOUNDARY VALUE PROBLEM

Valter Šeda

Abstract. In the paper a sufficient condition for the existence of an R_δ-set of solutions to a generalized boundary value problem on a compact interval is established. The proof is based on the Browder-Gupta theorem on the existence of an R_δ-set of solutions of an operator equation and on the relation between boundary value problems and Fredholm operators. Similar result is obtained by means of the Vidossich theorem.

1. Browder–Gupta and Vidossich theorems

In the theory of differential equations the Peano phenomenon of the existence of a continuum of solutions of the initial value problem for ordinary differential systems is well-known. This phenomenon has been studied in a less or more general setting by many authors (see e.g. [1]–[3], [6]–[7], [9]–[11], [14], [15], [22]) and one of its abstract versions is the Browder–Gupta theorem (Theorem 7 in [6, p. 394]) which has been improved by L. Górniewicz in [7, pp. 347–349]. The Górniewicz theorem will be presented here as Proposition 1. First of all let us recall the following notions. Let X be a metric space, $(E, \|\cdot\|)$ a real Banach space. A nonempty subset $A \subset X$ is called a *retract of* X if there exists a retraction $r : X \to A$, i.e. r is continuous and $r(x) = x$ for every $x \in A$.

2000 *Mathematics Subject Classification.* 34B15, 47H15, 47N20, 54C10.

Key words and phrases. Compact R_δ-set, generalized boundary value problem, Fredholm operator, proper mapping, Browder-Gupta theorem, Vidossich theorem.

Supported by grant no. 1/7176/20 of the Scientific Grant Agency VEGA of Slovak Republic.

A nonempty compact subset B of X is called a *compact absolute retract* if and only if for any metric space Y and for any homeomorphism $h : B \to Y$ the set $h(B)$ is a retract of Y. A nonempty convex compact subset of the space E is a compact absolute retract.

A nonempty subset C of X is a *compact R_δ-set in the space X* if C is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

A connected set $D \subset X$ is assumed to be nonempty. A compact R_δ-set is a special case of a compact connected set.

Proposition 1. *Let X be a metric space, $(E, \|\cdot\|)$ a real Banach space and $f : X \to E$ a proper map, i.e. f is continuous and for every compact $K \subset E$ the set $f^{-1}(K)$ is compact. Assume further that there exists a sequence of positive numbers ε_k with the property $\lim_{k\to\infty} \varepsilon_k = 0$ and a sequence of proper maps $f_k : X \to E$, $k = 1, 2, \dots$ such that the following conditions are satisfied:*

(i) *$\|f_k(x) - f(x)\| < \varepsilon_k$ for every $x \in X$,*

(ii) *for any $u \in E$ such that $\|u\| \le \varepsilon_k$ the equation*

$$f_k(x) = u \tag{1}$$

has exactly one solution.

Then the set $S = f^{-1}(0)$ is a compact R_δ-set.

Our considerations will be also based on the Vidossich theorem (Theorem 2.2 in [22, pp. 606–607]) which has been improved by S. Szufla in [14, p. 972]. The Szufla theorem is given here in a weaker modification as

Proposition 2. *Let X be a metric space, $(E, \|\cdot\|)$ a real Banach space and $f : X \to E$ a proper map. Assume further that there exists a sequence of positive numbers ε_k such that $\lim_{k\to\infty} \varepsilon_k = 0$, a positive number r and a sequence of proper maps $f_k : X \to E$, $k = 1, 2, \dots$ with the following properties:*

(i) *$\|f_k(x) - f(x)\| < \varepsilon_k$ for every $x \in X$,*

(ii') *for any $u \in E$ such that $\|u\| \le r$ the set of all solutions of the equation (1) is connected.*

Then the set $S = f^{-1}(0)$ is compact and connected.

From now on let X and Y be two real Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $q \in Y$ be an element. In the whole paper $B(q, \varepsilon)$ will mean the closed ball centered at q with radius r. The following theorem represents a special case of the Górniewicz theorem.

Theorem 1. *Suppose that there exists a map $f = A + B : X \to Y$ and a sequence of mappings $f_k = A_k + B_k : X \to Y$, $k = 1, 2, \dots$ such that*

(A.1) *A and A_k, $k = 1, 2, \dots$ are linear bounded Fredholm operators of index zero,* (A.2) *B and B_k, $k = 1, 2, \dots$ are completely continuous,*

(A.3) *there exists a sequence of positive numbers ε_k with the property*

$$\lim_{k\to\infty} \varepsilon_k = 0$$

and a bounded closed subset T_q in X such that the following three conditions are satisfied:

(i) *$\|f_k(x) - f(x)\|_Y < \varepsilon_k$ for every $x \in T_q$,*

(ii) *for any $u \in Y$ such that $\|u\|_Y \leq \varepsilon_k$ the equation*

$$f_k(x) = q + u \tag{2}$$

has exactly one solution,

(iii) *$T_k = f_k^{-1}(B(q, \varepsilon_k))) \subset T_q$.*

Then the set $S_q = f^{-1}(q) \subset T_q$ and is a compact R_δ-set.

Proof. (i) implies that $f_k(S_q) \subset B(q, \varepsilon_k)$ and hence $S_q \subset T_k \subset T_q$ for each $k = 1, 2, \dots$. As $T_q \subset X$ is a bounded closed set, by Proposition 2.2 and 2.3 in [19, pp. 20–21], (A.1) and (A.2) guarantee that the restrictions of the mappings f and f_k, $k = 1, 2, \dots$, to T_q are proper mappings. Denote these restrictions again by f and f_k, $k = 1, 2, \dots$, respectively. Then the maps $f^\star = f - q : T_q \to Y$, $f_k^\star = f_k - q : T_q \to Y$, $k = 1, 2, \dots$, are proper, too and satisfy conditions (i) and (ii), (iii) for $q = 0$, T_q being kept. Then the Górniewicz theorem (Proposition 1) implies that $S_q = (f^\star)^{-1}(0)$ is a compact R_δ-set. □

Theorem 2. *Suppose that there exists a map $f = A + B : X \to Y$ and a sequence of mappings $f_k = A_k + B_k : X \to Y$, $k = 1, 2, \dots$ such that the assumptions* (A.1), (A.2) *hold and the following assumption*

(A.4) (i) *$\lim_{k\to\infty} f_k(x) = f(x)$ uniformly on each bounded closed subset in X,*

(ii) *for any $u \in Y$, $k = 1, 2, \dots$, the equation* (1) *has at most one solution,*

(iii) *for each bounded $S \in Y$ there exists a bounded closed subset T_S in X such that $T_k = f_k^{-1}(S) \subset T_S$, $k = 1, 2, \dots$, is true.*

Then for each $q \in Y$ the set $S_q = f^{-1}(q)$ is a compact R_δ-set.

Proof. By (iii) each f_k, $k = 1, 2, \dots$ is coercive (see Definition 2.1 in [19, p. 20] and hence, in view of Proposition 2.2, [19, p. 20], it is also proper on X. Assumption (ii) implies that each f_k is locally injective and thus, by Lemma 3.1, [19, p. 23], it is locally invertible at any $x \in X$. By the Global Inversion Theorem (Proposition 2.4, [19, p. 21]), f_k is a homeomorphism of X onto Y for $k = 1, 2, \dots$ Thus (1) has exactly one solution for each $u \in Y$. Let $q \in Y$ be an arbitrary but fixed element. Choose the sequence $\varepsilon_k = 1/k$, $k = 1, 2, \dots$ of positive numbers and the set $T_{B(q,1)}$. Then (i) implies that there exists a subsequence$\{f_l\}$

of $\{f_k\}$ such that $\|f_l(x) - f(x)\| < 1/l$, $l = 1, 2, \ldots$, for every $x \in T_{B(q,1)}$. We see that assumption (A.4) implies that assumption (A.3) is fulfilled for the sequence $\{f_l\}$ and by Theorem 1 the statement of this theorem follows. □

THEOREM 3. *Suppose that there exist a map $f = A + B : X \to Y$ and a sequence of mappings $f_k = A_k + B_k : X \to Y$, $k = 1, 2, \ldots$ such that the assumptions* (A.1), (A.2) *and the following assumption holds:*

(A.5) *there exists a sequence of positive numbers ε_k having $\lim_{k\to\infty} \varepsilon_k = 0$, a positive number r and a bounded closed subset T_0 in X such that for each $k = 1, 2, \ldots$ the following three conditions are satisfied:*
- (i) $\|f_k(x) - f(x)\| < \varepsilon_k$ *for every* $x \in T_0$,
- (ii) *for any $u \in Y$ such that $\|u\| \leq r$ the set of all solutions of the equation* (1) *is connected,*
- (iii) $T_k = f_k^{-1}(B(0, r)) \subset T_0$.

Then the set $S_0 = f^{-1}(0) \subset T_0$ and is a compact and connected set.

PROOF. Similarly as in the proof of Theorem 1 we obtain that $S_0 \subset T_k \subset T_0$ for each k, $k = 1, 2, \ldots$ and the restrictions of the mappings f and f_k, $k = 1, 2, \ldots$ to T_0 are proper mappings. We denote these restrictions again by f and f_k, $k = 1, 2, \ldots$, respectively. Then these maps satisfy conditions (i) and (ii') of the Szufla theorem on T_0 and by this theorem, S_0 is compact and connected. □

.

2. Generalized boundary value problem

Consider the generalized BVP

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x + f(t, x, \ldots, x^{(m)}) = q(t), \quad a \leq t \leq b, \tag{3}$$

$$l_i(x) = 0, \quad i = 1, \ldots, n \tag{4}$$

where $n \geqq 1$, $0 \leq m \leq n-1$, $-\infty < a < b < \infty$, $p_k, q \in C([a, b])$, $k = 1, 2, \ldots$, $f : [a, b] \times \mathbb{R}^{m+1} \to \mathbb{R}$ is continuous, $l_i : C^{n-1}([a, b]) \to \mathbb{R}$, $i = 1, \ldots, n$, are linearly independent linear continuous functionals. The topology in C^{n-1} is given by the norm $\|\cdot\|_{n-1}$ whereby $\|x\|_l = \max_{k=0,\ldots,l}\{\|x^{(k)}\|_0\}$ for each $x \in C^l = C^l([a, b])$, $l = 1, \ldots, n$ and $\|x\|_0 = \sup_{a\leq t\leq b} |x(t)|$ for each $x \in C^0 = C([a, b])$.

Let $A : D(A) \subset C^n \to C^0$ be the linear operator

$$Ax = x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x, \tag{5}$$

where

$$D(A) = \{x \in C^n : l_i(x) = 0,\ i = 1, \ldots, n\}. \tag{6}$$

As the functionals l_i are also continuous in C^n, $D(A)$ is a closed subspace of C^n and hence $X_0 = (D(A), \|\cdot\|_n)$ is a Banach space. Further $A : X_0 \to Y_0$, where $Y_0 = C^0$, is a linear bounded operator. In view of the Rudolf theorem [12, p. 56], Lemma 4.1 in [19, p. 28] implies that $\dim X_0 = \infty$ and $A : X_0 \to Y_0$ is a linear bounded operator which is Fredholm of index zero.

By Lemma 4.2, [19, p. 29], continuity of f implies that the corresponding Nemitskij operator $B : X_0 \to Y_0$ which is defined by

$$B(x) = f \circ x \quad \text{for } x \in X_0, \tag{7}$$

is completely continuous. Thus the operator

$$F = A + B : X_0 \to Y_0 \tag{8}$$

where A is defined by (5), (6) and B is defined by (7), satisfies the assumptions (A.1), (A.2).

Consider a sequence of differential equations

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x + f_k(t, x, \ldots, x^{(m)}) = q(t), \quad \text{for } a \le t \le b, \tag{3_k}$$

where $f_k : [a, b] \times \mathbb{R}^{m+1} \to \mathbb{R}$ is continuous, $k = 1, 2, \ldots$.

If the Nemitskiĭ operator $B_k : X_0 \to Y_0$ is determined by

$$B_k(x) = f_k \circ x \quad \text{for } x \in X_0,\ k = 1, 2, \ldots, \tag{7_k}$$

then the operator

$$F_k = A + B_k : X_0 \to Y_0 \quad \text{for } k = 1, 2, \ldots \tag{8_k}$$

also satisfies the assumptions (A.1) and (A.2).

Consider the corresponding homogeneous BVP to (3), (4), that is, the problem (4),

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x = 0 \quad \text{for } a \le t \le b. \tag{9}$$

By Rudolf's theorem there exists a differential equation

$$x^{(n)} + r_1(t)x^{(n-1)} + \ldots + r_n(t)x = 0 \quad \text{for } a \le t \le b \tag{10}$$

with continuous coefficients r_k, $k = 1, 2, \ldots$ in $[a, b]$ such that the BVP (10), (4) has only the trivial solution. Of course, in some cases the problem (9), (4) already has this property. Comparing the equations (9) and (10) we can come to an integer l which we will call an *admissible integer for the problem* (9), (4) and which is defined in this way: l, $0 \le l \le n-2$, is an integer such that $r_k(t) \equiv p_k(t)$ on $[a, b]$ for $k = 1, \ldots, n-l-1$, $r_{n-l}(t) \not\equiv p_{n-l}(t)$, $l = 0$ if also all $r_k(t) \equiv p_k(t)$ and $l = n-1$ if already $r_1(t) \not\equiv p_1(t)$ in $[a, b]$.

Of course, there may exist many admissible integers for a given boundary value problem. In any case there is a unique minimal admissible integer for that problem.

THEOREM 4. *Suppose that the following assumption holds:*

(H.1) (i) *for each bounded closed subset M in $\mathbb{R}^{m+1}$ it holds*

$$\lim_{k\to\infty} f_k(t,x_1,\dots,x_{m+1}) = f(t,x_1,\dots,x_{m+1}) \quad \textit{uniformly on } [a,b]\times M,$$

(ii) *the* BVP (3_k), (4) *has at most one solution for each $q \in Y_0$ and $k = 1,2,\dots,$*

(iii) *for each bounded $S \subset Y_0$ there is an $R > 0$ such that all possible solutions x of the problem (3_k), (4), $k = 1,2,\dots$ with $q \in S$ satisfy the inequality*

$$\|x\|_j \le R \quad \textit{for } j = \max(m,l)$$

where l is the minimal admissible integer for the problem (9), (4).

Then for each $q \in Y_0$ the set S_q of all solutions of the BVP (3), (4) *is a compact R_δ-set.*

PROOF. Assumptions (i), (ii) from (H.1) imply that the operators $F = f$, $F_k = f_k$, $k = 1,2,\dots$ clearly satisfy (i), (ii) in (A.4). From the proof of Lemma 4.3 [19, p. 30] it follows that if (iii) in (H.1) is satisfied, then $F_k^{-1}(S)$ is bounded not only in the norm $\|\cdot\|_j$, but also in the norm $\|\cdot\|_{n-1}$ and hence in X_0. Also, the bounding constants do not depend on k. Thus (A.4), (iii) is fulfilled and by Theorem 2 this theorem follows. □

By Theorem 3 the following theorem is true.

THEOREM 5. *Suppose that the assumption is fulfilled:*

(H.2) (i) *For each bounded closed subset M in $\mathbb{R}^{m+1}$ it holds*

$$\lim_{k\to\infty} f_k(t,x_1,\dots,x_{m+1}) = f(t,x_1,\dots,x_{m+1}) \quad \textit{uniformly on } [a,b]\times M,$$

(ii) *the* BVP (3_k), (4) *has a connected set of solutions for each $q \in Y_0$ and each $k = 1,2,\dots,$*

(iii) *for each $u \in Y_0$ there exist positive constants r_u and R_u such that all possible solutions x of the problem (3_k), (4), $k = 1,2,\dots$ with $q \in B(u,r_u)$ satisfy the inequality*

$$\|x\|_j \le R_u \quad \textit{for } j = \max(m,l),$$

where l is the minimal admissible integer for the problem (9), (4).

Then for each $q \in Y_0$ *the set* S_q *of all solutions of the* BVP (3), (4) *is a compact connected set.*

Proof. Putting the function q into the left-hand side of (3) and (3_k), $k = 1, 2, \dots$, we see that it is sufficient to consider the special case $q = 0$. Similarly as in the proof of Theorem 4, (iii) implies that there exists an $R_1 \geqq R_0$ such that $\|x\|_n \le R_1$ is true for all solutions of (3_k), (4), $k = 1, 2, \dots$ with $q \in B(0, r_0)$. Hence in assumption (A.5) (iii) is satisfied with $T_0 = \{x \in X_0 : \|x\|_n \le R_1\}$. Clearly (i) and (ii) from (A.5) are fulfilled, too, and by Theorem 3 this theorem follows. □

3. Second-order boundary value problem

Now we apply Theorem 4 to the boundary value problem

$$x'' + f(t, x, x') = q(t), \quad x(a) = x(b) = 0. \tag{11}$$

J. W. Bebernes in [4] and L. K. Jackson in [8] have proved a sufficient condition for the existence of a unique solution to (11) as well as an apriori estimate for this solution. Their result (see also [16, p. 231]) is given here as

Proposition 3. *Suppose that* $a < b$ *are two real numbers,* q *is continuous on* $[a, b]$*,* $f = f(t, x, y)$ *is continuous on* $E = [a, b] \times \mathbb{R}^2$ *and is such that*

- (p) $f(t, \cdot, y)$ *is nonincreasing in* $\mathbb{R}$ *for each* $(t, y) \in [a, b] \times \mathbb{R}$*,*
- (q) *there is a constant* $k > 0$ *such that* $|f(t, 0, y) - f(t, 0, 0)| \le k|y|$ *on* $[a, b]$ *for all* y*,*
- (r) f *satisfies a Lipschitz condition with respect to* y *on each compact subset of* E*.*

Then the boundary value problem (11) *has a unique solution* $x \in C^2([a, b])$*. Furthermore, on* $[a, b]$

$$|x(t)| \le \frac{M}{k^2}\left[\exp k(b - a) - \exp \frac{1}{2}k(b - a) - \frac{1}{2}k(b - a)\right]$$

and

$$|x'(t)| \le \frac{M}{k}[\exp k(b - a) - 1]$$

where $M = \max_{t \in [a,b]} |f(t, 0, 0) - q(t)|$.

The condition (r) can be dropped out and the uniqueness of the solution to (11) will be replaced by the statement that the set of all solutions to (11) is a compact R_δ-set. To that aim we shall need the Stone theorem in the following formulation [17, p. 184].

THEOREM (Stone's theorem). *Let M be a compact set in a metric space, $f \in C^0(M)$ and let A be a lattice of continuous functions on M with the following property:*

(a) *For every pair x, y, $x \neq y$ of points of M, there exists a function $g \in A$ such that $g(x) = f(x)$, $g(y) = f(y)$.*

Then there exists a sequence $\{f_k\}$ of functions $f_k \in A$ which uniformly converges to f on M.

By means of the Stone theorem the following proposition has been proved ([16, p. 232]).

PROPOSITION 4. *Suppose that $a < b$ are two real numbers, $f = f(t, x, y)$ is continuous on $E = [a, b] \times \mathbb{R}^2$ and satisfies the conditions* (p) *and* (q) *of Proposition* 3. *Then there exists a sequence $\{f_k\}$ of functions $f_k \in C^0(E)$ satisfying the conditions* (p)–(r) *of that proposition which uniformly converges to f on each compact subset of E.*

By Theorem 4 we obtain from Propositions 3 and 4 the following

THEOREM 6. *Suppose that $a < b$ are two real numbers, $f = f(t, x, y)$ is continuous on $E = [a, b] \times \mathbb{R}^2$ and satisfies the conditions* (p) *and* (q) *of Proposition* 3. *Then for each $q \in C([a, b])$ the set S_q of all solutions of the* BVP (11) *is a compact R_δ-set.*

REFERENCES

[1] J. ANDRES, G. GABOR AND L. GÓRNIEWICZ, *Boundary value problems on infinite intervals*, Trans. Amer. Math. Soc. **351** (2000), 4861–4903.

[2] ———, *Topological structure of solution sets to multi-valued asymptotic problems*, Z. Anal. Anwendungen **19** (2000), 35–60.

[3] N. ARONSZAJN, *Le correspondant topologique de l'unicité dans la théorie des équations différentielles*, Ann. of Math. **43** (1942), 730–738.

[4] J. W. BEBERNES, *A subfunction approach to boundary value problems for ordinary differential equations*, Pacific J. Math. **13** (1963), 1053–1066.

[5] K. BORSUK, *Theory of Retracts*, PWN – Polish Scientific Publishers, Warszawa, 1967.

[6] F. E. BROWDER AND CH. P. GUPTA, *Topological degree and nonlinear mappings of analytic type in Banach spaces*, J. Math. Anal. Appl. **26** (1969), 390–402.

[7] L. GÓRNIEWICZ, *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer Acad. Publ., Dordrecht–Boston–London, 1999.

[8] L. K. JACKSON, *Subfunctions and second-order ordinary differential inequalities*, Adv. in Math. **2** (1968), 307–363.

[9] Z. KUBÁČEK, *A generalization of N. Aronszajn's theorem on connectedness of the fixed point set of a compact mapping*, Czechoslovak Math. J. **37 (112)** (1987), 415–423.

[10] ———, *On the structure of the fixed point sets of some compact maps in the Fréchet space*, Math. Bohem. **118** (1993), 343–358.

[11] ———, *On the structure of the solution set of a functional differential system on an unbounded interval*, Arch. Math. (Brno) **35** (1999), 215–228.

[12] B. Rudolf, *The generalized boundary value problem is a Fredholm mapping of index zero*, Arch. Math. (Brno) **31** (1995), 55–58.

[13] ———, *A multiplicity result for a periodic boundary value problem*, Nonlinear Anal. **28** (1997), 137–144.

[14] S. Szufla, *Solutions sets of nonlinear equations*, Bull. Polish Acad. Sci. Math. **21** (1973), 971–976.

[15] ———, *Sets of fixed points of nonlinear mappings in function spaces*, Funkcialaj Ekvacioj **22** (1979), 121–126.

[16] V. Šeda, *On a boundary value problem for a nonlinear second-order differential equation*, Mat. Čas. **22** (1972), 231–234.

[17] ———, *On an application of the Stone theorem in the theory of differential equations*, Čas. Pěst. Mat. **97** (1972), 183–189.

[18] ———, *On some non-linear boundary value problems for ordinary differential equations*, Arch. Math. (Brno) **25** (1989), 207–222.

[19] ———, *Fredholm mappings and the generalized boundary value problem*, Differential Integral Equations **8** (1995), 19–40.

[20] ———, *Generalized boundary value problems and Fredholm mappings*, Proc. 2nd World Congress of Nonlinear Analysts, Nonlinear Anal. **30** (1997), 1607–1616.

[21] ———, *Generalized boundary value problems with linear growth*, Math. Bohem. **123** (1998), 385–404.

[22] G. Vidossich, *On the structure of the set of solutions of nonlinear equations*, J. Math. Anal. Appl. **34** (1971), 602–617.

Manuscript received August 14, 2000

Valter Šeda
Department of Mathematical Analysis
Faculty of Mathematics, Physics and Informatics
Comenius University
Mlynská Dolina
842 15 Bratislava, SLOVAK REPUBLIC

E-mail address: seda@fmph.uniba.sk