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# EXISTENCE OF NONMINIMAL QUASIPERIODIC SOLUTIONS FOR SECOND ORDER EQUATIONS

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Abstract. We consider the motion of $n$ particles under the action of a potential $F$. Imposing appropriate conditions on $F$ we obtain quasiperiodic solutions using variational methods. A Diophantine condition on the frequency similar to those encountered in KAM theory allows us to establish the necessary properties of the corresponding functional. The solutions are then obtained by means of the mountain pass theorem on a suitable convex subset.

## 1. Introduction

The study of the qualitative behaviour of solutions of Hamiltonian systems goes back to Poincaré ([12]). Since this behaviour can be very complicated, as Poincaré himself realized, the study of invariant manifolds plays an important role.

Apart from equilibrium points, the simplest invariant manifolds correspond to special solutions, for instance, periodic, heteroclinic or parabolic-like solutions.

In recent years, considerable progress in this direction has been made using variational techniques, motivated to a great extent by the work iniciated by Rabinowitz ([14]). We refer to [1], [5], [6], [9] and [15] and references therein, just to provide some important works where a more complete bibliography can be found, since it would be impossible to survey it here.

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The next step in the study of invariant manifolds is trying to find invariant tori, which is motivated also by the completely integrable case and KAM theory. A particular consequence, for Hamiltonian systems which are perturbations of integrable ones, is the existence of quasiperiodic solutions. We also mention the work by Moser, who specifically addressed the problem of quasiperiodic solutions of differential equations ([7]) and also investigated the relationship with other geometric problems ([8]).

Variational methods have also been used in this case, for example by Aubry and Mather ([6]), Percival ([10]), Percival and Pomphrey ([11]) in several applications. We mention that in the work by Mather, Percival's variational principle is used in order to establish existence of quasiperiodic orbits of all frequencies for area preserving twist homeomorphisms of the annulus. Closer to our approach is a series of very original papers by Berger and Zhang ([2], [3] and [4]) where a new setting is used in order to obtain quasiperiodic solutions for some forced second order equations. Moreover, in these papers, the authors succeed in getting rid of the usual Diophatine condition which is needed in order to apply KAM theory.

In this paper we consider the system

$$\ddot{u} = -\nabla F(u). \tag{1.1}$$

and look for invariant sets as quasiperiodic solutions with a given frequency vector $\omega$. In a nondegenerate situation, one would like to conclude that such a quasiperiodic solution belongs to an invariant torus. However, there are several delicate issues concerning the actual nondegeneracy of this solution that make the analysis difficult. Therefore, we limit ourselves to establish the existence of quasiperiodic solutions. We use a variational formulation of the problem, still imposing a Diophantine condition of the type

$$|(k\cdot\omega)| \geq \frac{C}{|k|^s}, \quad \text{for all } k\in\mathbb{Z}^m\setminus\{0\}, \text{ for some } s>0. \tag{1.2}$$

We apply Struwe's critical point theory on convex sets to the energy functional

$$E(u) = \int_{T^m}\left(\frac{1}{2}|\dot{u}|^2 - F(u)\right)dx, \tag{1.3}$$

where $u=\sum_{k\in\mathbb{Z}^m} a_k e^{ik\cdot x}$, $T^m$ is the $m$-dimensional torus and

$$\dot{u}(x) = \frac{du}{dt}(x) = \sum_{i=1}^{m}\omega_i\frac{\partial u}{\partial x_i}(x),$$

i.e. the directional derivative of $u$ in the direction of $\omega$ and $\omega=(\omega_1,\dots,\omega_m)$ satisfying (1.2).

The main difficulty in obtaining critical points of $E$ is finding a suitable function space. The natural choice is the Sobolev space of square integrable functions with square integrable derivatives. However, the kinetic energy of $E$

does not provide good control on the norm in this space. This is true even if we impose condition (1.2) on $\omega$ (in this case some control is gained, although in a very weak norm).

This fact is related to the well known small divisors problem, that appears when one is looking for quasiperiodic solutions by means of formal series expansions. However, we can consider the problem restricted to a convex subset in which the term

$$\int_{T^m} \frac{1}{2}|\dot{u}|^2$$

is in fact equivalent to the usual norm in $H^1(T^m)$. Once this is done, the existence of cricital points, and therefore, of quasiperiodic solutions can be proved.

The main goal of this paper is to present an instance in which Percival's variational principle can be used to rigourously establish the existence of quasiperiodic solutions using minimax methods, specifically, the mountain pass lemma. We remark that even when we use a Diophantine condition, our approach works for autonomous equations, and in this respect, it differs from the results by Berger and Zhang.

We now state the main result.

Theorem 1.1. *Let $\omega \in \mathbb{R}^m$, $m > 1$, $\omega = (\omega_1, \dots, \omega_m)$, $|\omega| = 1$ and $\omega_i > 0$, $1 \le i \le m$. Suppose further that $\omega$ satisfies the Diophantine condition* (1.2). *Assume that $F \in C^1(\mathbb{R}^n, \mathbb{R})$ satisfies*

(a) $|\nabla F(u)| = o(|u|)$ *as* $u \to 0$.

(b) *There exists $A > 0$ and $p > 1$ ($p \in (1, (m+2)/(m-2))$ in case $m > 2$) such that*

$$|\nabla F(u)| \le A(|u|^p + 1) \quad \textit{for all } u \in \mathbb{R}^n.$$

(3) *There exists $\mu > 2$, $\rho > 0$ such that, for $|u| \ge \rho$,*

$$0 < \mu F(u) \le u \cdot \nabla F(u).$$

*Then there is a positive number $s_0$ such that if $r - s \ge s_0$,* (1.1) *has at least one quasiperiodic solution with frequency $\omega$ in $M_r$ (see Section 2). That is, there exists $u \in H^1(T^m, \mathbb{R}^n)$, $T^m$ the $m$-dimensional torus, such that $u(\omega_1 t, \dots, \omega_m t)$ satisfies* (1.1).

This paper is organized as follows. In Section 2 we introduce some notation, define the convex set where the variational problem is formulated and present some estimates we need in the proof of Theorem 1.1. In Section 3 we recall the results we use from the calculus of variations on convex sets as developed by Struwe ([15]). In Section 4 we give the proof of Theorem 1.1. Section 5 is devoted to some concluding remarks related to the regularity of solutions, specific examples, applications and open questions.

## 2. Preliminary estimates

We begin by introducing some standard notation. As usual, we identify the $m$-dimensional torus with the square $[-\pi,\pi]^m$. Then functions on $T^m$ are represented by functions on $\mathbb{R}^m$ which are $2\pi$-periodic in all their arguments. It will be useful to consider for $u$ in $H^1(T^m,\mathbb{R}^n)$ its representation in Fourier series

$$u=\sum_{k\in\mathbb{Z}^m} a_k e^{ik\cdot x},\quad a_k\in \mathbf{C}^{\ n}$$

where $x=(x_1,\dots,x_m)$. Inequalities for vectors are to be understood componentwise. We then denote by $V$ the subspace of even functions. That is, those for which $a_k=a_{-k}$. For a fixed $r>0$, let $M_r$ be the set of functions in $V$ that satisfy the following conditions:

- ($\mathrm{c}_0$) $a_0=0$, that is, functions with zero mean,
- ($\mathrm{c}_k$) $a_k\neq 0$, for all $k\neq 0$ and there are constants $D_0$, $D_1$ (which may depend on $u$) and $0<\lambda\le 1$ and $d_*>0$ (both independent of $u$) such that $\lambda D_1\le D_0$, $D_0>d_*$ and

$$\frac{D_0}{|k|^r}\le a_k\le\frac{D_1}{|k|^r}.$$

Since the functions we will work with have zero mean, we can take $\|\nabla u\|_{L_2}$ as an equivalent norm in $H^1$ The estimates we need are presented in the following lemma.

Lemma 2.1. *There is a positive number $s_0$ such that if $s$, as in* (1.2)*, satisfies $s+r>s_0$, then*

- (a) $d_0|u|_{H^1}\le|\dot u|_{L^2}\le d_1|u|_{H^1}$,
- (b) $d_0'|\dot u|_{L^2}\le|u|_{L^2}\le d_1'|\dot u|_{L^2}$,

*for all $u\in M_r$; where $d_0,d_0',d_1,d_1'$ are all positive constants independent of $u$.*

Remark 2.2. Before giving the proof, we observe that in any finite dimensional subspace and for $\omega$ an irrational direction (that is, satisfying $\omega\cdot k\neq 0$ for all $k\in\mathbb{Z}^m\setminus\{0\}$), the above inequalities are immediate. In some sense, the Diophantine condition and conditions ($\mathrm{c}_0$) and ($\mathrm{c}_k$) allow us to maintain the equivalence of the norms in $M_r$.

Proof of Lemma 2.1. The right hand side inequality in (a) is immediate from Schwarz inequality:

$$|\dot u|_{L^2}^2=\sum_{k\in\mathbb{Z}^m}|(\omega\cdot k)|^2=|a_k|^2\le|\omega|^2\sum_{k\in\mathbb{Z}^m}|k|^2|a_k|^2,$$

and since $a_0=0$, the last term is less than or equal to $C=|u|_{H^1}^2$.

Now we show the right hand side inequality in (b).

$$\begin{aligned}|u|_{L^2}^2 &= \sum_{k\in\mathbb{Z}^m} |a_k|^2 \\ &\le D_1^2 \sum_{k\neq 0\in\mathbb{Z}^m} \frac{1}{|k|^{2r}} \qquad \text{by } (\mathrm{c}_k) \\ &\le \frac{1}{\lambda^2} D_0^2 \sum_{k\neq 0\in\mathbb{Z}^m} \frac{1}{|k|^{2r}} \quad \text{again by } (\mathrm{c}_k).\end{aligned}$$

The sum in the last inequality is finite provided $r > s_0$ for some $s_0$ sufficiently large. So we have

$$D_0^2 \ge C|u|_{L^2}^2, \tag{2.1}$$

with $C$ independent of $u$. On the other hand,

$$\begin{aligned}|\dot u|_{L^2}^2 &= \sum_{k\neq 0\in\mathbb{Z}^m} |(\omega\cdot k)|^2 = |a_k|^2 \\ &\ge \sum_{k\neq 0\in\mathbb{Z}^m} \frac{C}{|k|^{2s}} = |a_k|^2 \qquad \text{by the Diophantine condition} \\ &\ge C\left(\sum_{k\neq 0\in\mathbb{Z}^m} \frac{1}{|k|^{2(r+s)}}\right) D_0^2 \quad \text{by } (\mathrm{c}_k),\end{aligned}$$

this last sum being finite for $s + r \ge s_0$ with large $s_0$. Then this last inequality and (2.1) imply the result.

Before proving the left hand-side inequalities in (a) and (b), we need an auxiliary reversed Sobolev inequality:

$$|u|_{H^1} \le C|u|_{L^2}, \tag{2.2}$$

for $C$ independent of $u$. Indeed,

$$|\nabla u|_{L^2}^2 \le \sum_{k\neq 0\in\mathbb{Z}^m} |k|^2|a_k|^2 \le \sum_{k\neq 0\in\mathbb{Z}^m} |k|^2 \frac{D_1^2}{|k|^{2r}} \le CD_1^2. \tag{2.3}$$

The constant $C$ is finite for $s_0$ sufficiently big with $r > s_0$, and independent of $u$.

If we use the fact that $\lambda D_1 \le D_0$, we have

$$|u|_{L^2}^2 \ge \sum_{k\neq 0\in\mathbb{Z}^m} \frac{D_0^2}{|k|^{2r}} \ge \lambda^2 \sum_{k\neq 0\in\mathbb{Z}^m} \frac{D_1^2}{|k|^{2r}} \ge CD_1^2. \tag{2.4}$$

(2.3) and (2.4) give (2.2), since $|\nabla u| \ge C|u|_{H_1}$ (recall $a_0 = 0$). Then the left hand side of (a) follows from (2.2) and the right hand side of (b). Finally, (2.2) and the right hand side of (a) imply the left hand side of (b), completing the proof of the lemma. $\square$

REMARK 2.3. (a) Observe that in fact $M_r$ is weakly closed. Indeed, if $u_i$ is a sequence in $M_r$ converging to $u$, we may assume, by the Sobolev embedding theorem that $u_i$ also converges to $u$ in $L^q$, $q$ less than $(m+2)/(m-2)$, since by weak convergence it has to be bounded. Then $(c_0)$ follow immediatelly from this. In order to verify $(c_r)$, since $a_k \neq 0$ for all elements in the sequence (we are dropping the index $i$ for the sake of clarity) and observing that we may assume the corresponding constants $D_0^i$ and $D_1^i$ for $u_i$ to be bounded. Therefore, up to a subsequence, they converge to some $D_0$ and $D_1$ respectively. Passing to the limit we obtain the desired conclusion.

(b) Notice also that the right inequality in (a) of the previous lemma holds for any $u \in H^1$.

## 3. Lusternik–Schnirelman theory on convex sets

Here we recall the variational facts from critical point theory as applied to functionals defined on convex sets of Banach spaces. This theory was systematically developed by Struwe, and we refer the reader to [15] for the proofs and further details.

Suppose that $M$ is a closed convex subset of a Banach space $V$, and assume further that $E\colon M \to \mathbb{R}$ possesses an extension $E \in C^1(V,\mathbb{R})$ to $V$. For $u \in M$ define

$$g(u) = \sup_{\substack{v\in M \\ \|u-v\|<1}} (u-v, DE(u)).$$

Then, if $E \in C^1(V)$, the function $g$ is continuous in $M$ (see [15, Lemma 11.1, Chapter II]).

DEFINITION 3.1. A *point is critical* if $g(u)=0$, otherwise $u$ is *regular*. If $E(u)=\beta$ for some critical point $u \in M$ of $E$, the value $\beta$ *is critical*, otherwise $\beta$ *is regular*.

We use the conventional notation:

$$\begin{aligned} M_\beta &= \{u \in M : E(u) < \beta\}, \\ K_\beta &= \{u \in M : E(u) = \beta,\ g(u) = 0\}, \\ N_{\beta,\delta} &= \{u \in M : |E(u)-\beta| < \delta, g(u) < \delta\}, \\ U_{\beta,\rho} &= \{u \in M : \text{exists } v \in K_\beta \text{ such that } \|u-v\| < \rho\}. \end{aligned}$$

DEFINITION 3.2. $E$ satisfies the Palais–Smale condition on $M$ if the following is true.

$(\mathrm{PS})_M$ Any sequence in $M$ such that $|E(u_m)| \le c$ uniformly, and $g(u_m) \to 0$ $(m \to \infty)$, is relatively compact.

DEFINITION 3.3. A locally Lipschitz vector field $v: \widetilde{M} \to V$, with

$$\widetilde{M} = \{u \in M; g(u) \neq 0\},$$

is a *pseudogradient vector field for $E$ on $M$* if there exists $c > 0$ such that

(a) $u + v(u) \in M$,
(b) $\|v(u)\| < \min\{1, g(u)\}$,
(b) $(v(u), DE(u)) < -c \min\{1, g(u)\} g(u)$, for all $u \in \widetilde{M}$.

Then we have the following lemma.

LEMMA 3.4. *There exists a pseudogradient vector field $v: \widetilde{M} \to V$, satisfying* (c) *with $c = 1/2$. Moreover, $v$ extends to a locally Lipschitz continuous vector field on $V \setminus K$, $K = \{u \in M : g(u) = 0\}$.*

Using this result one can prove a deformation lemma ([15, Theorem 11.7]).

THEOREM 3.5. *Suppose $M \subset V$ is closed and convex, $E \in C^1(V)$ and it satisfies the* (PS) *condition on $M$. Let $\beta \in \mathbb{R}$, $\bar{\varepsilon} > 0$ be given. Then for any neighbourhood $N$ of $K_\beta$ there exists $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous deformation $\Phi: M \times [0,1] \to M$ such that*

(a) *$\Phi(u,t) = u$ if $g(0) = 0$, or if $t = 0$, or if $|E(u) - \beta| \geq \bar{\varepsilon}$,*
(b) *$E(\Phi(u,t))$ is nondecreasing in $t$, for any $u \in M$,*
(c) *$\Phi(M_{\beta+\varepsilon}, 1) \subset M_{\beta-\varepsilon} \cup N$, respectively $\Phi(M_{\beta+\varepsilon} \setminus N, 1) \subset M_{\beta-\varepsilon}$.*

A version of the mountain pass theorem on convex sets is then proved in the same way as the usual result.

THEOREM 3.6. *Suppose that $M$ is a closed, convex subset of a Banach space $V$, containing* 0, *$E \in C^1(V)$ satisfies* $(\mathrm{PS})_M$. *Assume further that*

(a) *$E(0) = 0$,*
(b) *there exists $\rho > 0$, $\alpha > 0$ such that, if $\|u\| = \rho$ for $u \in M$ implies $E(u) \geq \alpha$,*
(c) *there exists $u_1 \in M$, $\|u_1\| > \rho$ with $E(u_1) < \alpha$.*

*Define*

$$\Gamma = \{p \in C^0([0,1]; M) : p(0) = 0,\ p(1) = u_1\}.$$

*Then $\beta = \inf_{p \in \Gamma} \sup_{t \in [0,1]} E(p(t))$ is a critical value of $E$ in $M$.*

## 4. Variational formulation

In this section we formulate problem (1.0) in a variational way and prove Theorem 1.1.

We first consider the functional (1.3) in $H^1(T^m, \mathbb{R}^n)$ and establish some differentiability properties.

Lemma 4.1. *Let $F \in C(\mathbb{R}^n, \mathbb{R})$ satisfy assumption* (b) *of Theorem* 1.1*. Then $E \in C^1(H^1(T^m, \mathbb{R}^n))$ and*

$$E'(u)\phi = \int_{T^m} (\dot{u} \cdot \dot{\phi} - \nabla F(u)\phi)\, dx \quad \textit{for all } \phi \in H^1.$$

Proof. The proof that $\int F(u)$ is differentiable is given in [13] as part of Proposition B.10.

Now, since $|\dot{u}|_{L^2} \leq C|u|_{H^1}$ (see Remark 2.2(b)), the first term is differentiable.

As a result, it is a direct computation to verify that the Euler–Lagrange equations of $E$ correspond to (1.1) and that critical points $u \in H^1(T^m, \mathbb{R}^n)$ restricted to the line $x = \omega t$, are indeed quasiperiodic weak solutions of this system.

In order to be able to apply the results of the previous section on Lusternik–Schnirelman theory on convex sets, we need to check that a critical point of $E$ in $M_r$ is also a critical point of $E$ in $V$. We also have to verify that the $(\mathrm{PS})_{M_r}$ condition holds. This is the content of the next lemmas. □

Lemma 4.2. *With the same hypotheses as for Theorem* 1.1*, let $\overline{u}$ be a critical point of $E$ in $M_r$. Then it is also a critical point in $V$.*

Proof. Consider the function

$$v_k^+(x) = \overline{u} + \varepsilon Re(a_k e^{ik\cdot x}).$$

We claim that for $\varepsilon > 0$ sufficiently small, $\overline{u} - v_k^{\pm}$ is in $M_r$. First, recall that since $(c_r)$ is satisfied, $a_k \neq 0$, for all $k \in \mathbb{Z}^m$ different from zero. Let $D_0$, $D_1$ the numbers defined in section 2 for $\overline{u}$. For $v_k^+ = \overline{u} + \varepsilon a_k Re(e^{ik\cdot x})$, we have that for the Fourier coefficient corresponding to $k$

$$\frac{D_0'}{|k|^r} = \frac{D_0(1+\varepsilon)}{|k|^r} \leq a_k(1+\varepsilon) \leq \frac{D_1(1+\varepsilon)}{|k|^r} = \frac{D_1'}{|k|^r},$$

with $D_0' = D_0(1+\varepsilon)$, $D_1' = D_1(1+\varepsilon)$. Since $\lambda D_1 \leq D_0$

$$\lambda D_1' \leq D_0'.$$

We also have $D_0' > D_0 \geq d_*$ Similarly, define

$$v_k^- = \frac{1}{1-\varepsilon} u_k - \frac{\varepsilon Re(a_k e^{ik\cdot x})}{1-\varepsilon}$$

and compute the corresponding Fourier coefficient

$$\frac{D_0}{|k|^r} \leq a_k \leq \frac{D_1}{|k|^r}.$$

The other coefficients are the same as for $\overline{u}$, except for a factor bigger than one. Therefore, $v_k$ is also in $M_r$. We then have, according to the definition of a critical point on $M_r$,

$$(DE(\overline{u}), \overline{u} - v_k^{\pm}) \leq 0,$$

which implies

$$(DE(\overline{u}), Re(e^{ik\cdot x})) = 0 \quad \text{for all } k \in \mathbb{Z}^m.$$

Thus $DE(\overline{u}) = 0$ in $V$. This completes the proof. □

Finally we have

LEMMA 4.3. *With the same hypotheses as in Theorem* 1.1*, $E$ satisfies the* $(\mathrm{PS})_{M_r}$ *condition provided $r - s$ is sufficiently large.*

PROOF. We use the same argument as in [13, Appendix B]. We present the details for the sake of completeness. First notice that, formally, the inverse operator of $d/dt$ is given by

$$\left(\frac{d}{dt}\right)^{-1} u = \sum_{k\in\mathbb{Z}^m} \frac{a_k}{i(\omega\cdot k)} e^{ik\cdot x}.$$

Besides, it is actually well defined in $M_r$ provided $r - s$ is sufficiently large. Indeed, using the Diophantine condition (1.2) and the fact that $u$ is in $M_r$

$$\left|\left(\frac{d}{dt}\right)^{-1} u\right|^2 = \sum_{k\in\mathbb{Z}^m} \frac{|a_k|^2}{(\omega\cdot k)^2} \leq D_1^2 C \sum \frac{|k|^{2s}}{|k|^{2r}} < \infty,$$

provided $r - s$ is sufficiently large. It is also immediate from the expression for $(d/dt)^{-1}$ that it is continuous in $M_r$.

Now we show that if $(u_i)$ is a bounded sequence in $M_r$ and $g(u_i) \to 0$ (see Section 3) as $i \to \infty$, then it admits a convergent subsequence.

Since the $k$-th Fourier coefficient of the $i$-th element of such a sequence satisfies

$$a_k^i \neq 0 \quad \text{for all } i \in \mathbb{N} \text{ and for all } k \neq 0 \in \mathbb{Z}^m, \tag{4.1}$$

using the same functions $v_k^{\pm}$ as in the proof of Lemma 4.2, we have

$$(DE(u_i), e^{ik\cdot x}) \geq \varepsilon_i, \quad (DE(u_i), e^{ik\cdot x}) \leq \varepsilon_i,$$

with $\varepsilon_i \to 0$ with $i$. From the previous inequalities we conclude

$$\|DE(u_i)\| \to 0 \tag{4.2}$$

as $i \to \infty$. Thus $(u_i)$ is a (PS) sequence in $V$ in the usual sense.

Denoting by $K\colon V \to V^*$ the map from $V$ to its dual given by

$$(Ku)\phi = \int_{T^m} \dot{u}\cdot\dot{\phi}\,dx,$$

we have

$$K^{-1}E'(u) = u - K^{-1}J'(u), \tag{4.3}$$

where $J(u) = \int_{T^m} F(u)\,dx$. $K^{-1}E'(u)$ is of the form $\mathrm{Id} - L$, so the conclusion follows if we can show that $J'(u_m)$ admits a convergent subsequence. This is due to the continuity of $K^{-1}$ (recall the observation at the beginning of the proof), since from (4.2) and (4.3) we obtain

$$u_m = K^{-1}E'(u_m) + K^{-1}J'(u_m) \to K^{-1}J'(u_m).$$

But by our preliminary assumption $u_m$ is bounded and $J'$ is indeed compact (see Proposition B.10 in [13]), so $J'(u_m)$ has a convergent subsequence.

It remains to show that a $(\mathrm{PS})_{M_r}$ sequence is bounded. Using Lemma 2.1, the proof is the same as for the functional

$$I(u) = \frac{1}{2}\int_{T^m} (|\nabla u|^2 - F(u))\,dx,$$

which is presented in [13, p. 11].

Now the proof of Theorem 1.1 is a direct consequence of Theorem 3.6 and Lemma 4.2. In fact, we only have to verify that conditions (a)–(c) in Theorem 3.2 hold. (a) is trivial and (b) and (c) follow in the standard way again using (a) in Lemma 2.1. □

## 5. Concluding remarks

Once the existence of solutions has been established, their regularity follows in the standard way, since the estimates provided in Lemma 2.1 allow us to use elliptic regularity.

The extension of these ideas to the Hamiltonian case as well as to some other problems like those involving singular potentials or to semilinear wave equations is under current investigation.

Observe that in case some additional properties of the solutions obtained here can be proved, these would provide more interesting invariant regions (e.g. when the solution $u$ is an embedding, then we would obtain an invariant torus).

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